Some aspects
of
consistency and stability
for
lattice Boltzmann methods

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Sufficient condition for stability of LB schemes

Stability-structure for collision operator: → [*Junk & Yong, 07*] and [*MRh, 08*]

$$\left.\begin{array}{ll} \text{i)} & BJ = -\mathrm{diag}(\lambda_1, ..., \lambda_q)B \\ \text{ii)} & B^\top B = \mathrm{diag}(b_1, ..., b_q) \\ \text{iii)} & \lambda_k \in [0, 2] \quad \text{for all } k \in \{1, ..., q\} \end{array}\right\} \quad B \text{ invertible}$$

Question: BGK *collision operators* with *stability-structure*

$\longrightarrow$ MRT *collision operators* with *stability-structure*?

$$J_{\mathrm{BGK}} = \frac{1}{\omega}(G - I) \qquad \leftrightarrow \qquad J_{\mathrm{MRT}} = A(G - I)$$

Properties of A:

$\ker A^\top = \ker(G^\top - I) = \ker J_{\mathrm{BGK}}^\top$ ($\Rightarrow$ J_{MRT} has same conserved moments as J_{BGK})

symmetric ($A = A^\top$) and *positive semidefinite*

J admits stability structure $\quad\Leftrightarrow\quad JD = DJ^\top$ (D diagonal & positive definite)

Construction: BGK $\to$ MRT [*MRh, 08*]

1) $G = WS$ (positive definite & diagonal *weight* matrix W, symmetric S)

2) no specific assumptions: $J_{\mathrm{BGK}}\tilde{D} = \tilde{D}J_{\mathrm{BGK}}^\top$

$(G-I)\tilde{D} = \tilde{D}(G-I)^\top \qquad \underbrace{A(G-I)}_{J_{\mathrm{MRT}}} D = D \underbrace{(G-I)^\top A}_{J_{\mathrm{MRT}^\top}} \qquad A, D = ?$

$$\left.\begin{array}{c} J_{\mathrm{BGK}}\tilde{D} \text{ symmetric} \\ D := \tilde{D} \end{array}\right\} \Rightarrow [A, J_{\mathrm{BGK}}\tilde{D}] = 0$$

A symmetric $\quad\Rightarrow\quad$ eigenspaces of A = eigenspaces of $J_{\mathrm{BGK}}\tilde{D}$

$\ker A^\top = \ker\big(J_{\mathrm{BGK}}\tilde{D}\big)^\top = \ker\big(DJ_{\mathrm{BGK}}^\top\big) \quad\Rightarrow\quad \ker A^\top = \ker J_{\mathrm{BGK}}^\top$

Observe: $\mathrm{spec}(J_{\mathrm{MRT}}) = \mathrm{spec}(A)$

- Advantage of MRT w.r.t. BGK:

  more parameters $\leftrightarrow$ more flexibility

  $\Rightarrow$ Hope to improve stability bahavior.

- However, modification $\rightarrow$ *consistency check*.

  Does the algorithm still what it is intended to do?

- Example to be studied exemplarily:

  D1Q3 lattice Boltzmann algorithm discretizing
  advection diffusion equation.

  $\rightarrow$ admits *advection velocity* as additional parameter *affecting stability*.

$$\boldsymbol{e} := (1,1,1)^\top \qquad \boldsymbol{c} := (-1,1,0)^\top$$

LB equation: $\hat{f}_k(t+h^2, x+\boldsymbol{c}_k h) = \hat{f}_k(t,x) + \Big[J\hat{\boldsymbol{f}}(t,x)\Big]_k$

Collision and equilibrium operator:

$$\begin{cases} J = \underbrace{A}_{\text{relaxation}}\Big(\underbrace{G}_{\text{equilibrium}} - \underbrace{I}_{\text{identity}}\Big) & J = J_d + hAG_a \\ G\boldsymbol{f} := G_d\boldsymbol{f} + hG_a\boldsymbol{f} := \langle \boldsymbol{e}, \boldsymbol{f}\rangle \boldsymbol{w} + ah\theta\langle \boldsymbol{e}, \boldsymbol{f}\rangle \boldsymbol{c}\boldsymbol{w} & \text{for } \boldsymbol{f} \in \mathbb{R}^3 \end{cases}$$

Regular expansion: $\underbrace{\hat{\boldsymbol{f}}}_{\text{discrete arguments}} = \underbrace{\boldsymbol{f}^{(0)} + h\boldsymbol{f}^{(1)} + h^2\boldsymbol{f}^{(2)} + h^3\boldsymbol{f}^{(3)} + \ldots}_{\text{continuously varying arguments}}$

Taylor expansion: $\Big[f_k^{(j)}(t+h^2, x+\boldsymbol{c}_k h)\Big]_{k\in\{1,2,3\}} = \sum_\alpha h^\alpha D_\alpha(\partial_t, \boldsymbol{c}\partial_x)\boldsymbol{f}^{(j)}$

$$D_\alpha(\vartheta, \varsigma) := \sum_{2\beta+\gamma=\alpha} \frac{\vartheta^\beta \varsigma^\gamma}{\beta!\gamma!}$$

$$D_0(\partial_t, \boldsymbol{c}\partial_x) = 1, \quad D_1(\partial_t, \boldsymbol{c}\partial_x) = \boldsymbol{c}\partial_x, \quad D_2(\partial_t, \boldsymbol{c}\partial_x) = \partial_t + \tfrac{1}{2}\boldsymbol{c}^2\partial_x^2,$$

$$D_3(\partial_t, \boldsymbol{c}\partial_x) = \boldsymbol{c}\partial_x\partial_t + \tfrac{1}{6}\boldsymbol{c}^3\partial_x^3, \quad D_4(\partial_t, \boldsymbol{c}\partial_x) = \tfrac{1}{2}\partial_t^2 + \tfrac{1}{2}\boldsymbol{c}^2\partial_x^2\partial_t + \tfrac{1}{24}\boldsymbol{c}^4\partial_x^4$$

Equating terms of order h^ℓ:

$$h^\ell : \quad J_d \boldsymbol{f}^{(\ell)} = -AG_a \boldsymbol{f}^{(\ell-1)} + \sum_{j=1}^{\ell} D_j(\partial_t, \boldsymbol{c}\partial_x)\boldsymbol{f}^{(\ell-j)}$$

$$h^0 : \quad J_d \boldsymbol{f}^{(0)} = 0$$

$$h^1 : \quad J_d \boldsymbol{f}^{(1)} = -AG_a \boldsymbol{f}^{(0)} + \boldsymbol{c}\partial_x \boldsymbol{f}^{(0)}$$

$$h^2 : \quad J_d \boldsymbol{f}^{(2)} = -AG_a \boldsymbol{f}^{(1)} + \boldsymbol{c}\partial_x \boldsymbol{f}^{(1)} + \partial_t \boldsymbol{f}^{(0)} + \tfrac{1}{2}\boldsymbol{c}^2\partial_x^2 \boldsymbol{f}^{(0)}$$

- Discussion of eqn. (h^0)

$$\boldsymbol{f}^{(0)} \in \ker J_d = \ker A(G_d - I) \quad \Rightarrow \quad \boldsymbol{f}^{(0)} = \rho^{(0)}\boldsymbol{w} \quad \text{with } \rho^{(0)}(t,x) = ?$$

- Discussion of eqn. (h^1)

$$\text{solution exists} \quad \Leftrightarrow \quad \text{RHS} \in \operatorname{im} J_d \quad \Leftrightarrow \quad \text{RHS} \in (\ker J_d^\top)^\perp = \operatorname{span}(\boldsymbol{e})^\perp$$

$$0 = \langle \boldsymbol{e}, -AG_a \boldsymbol{f}^{(0)} + \boldsymbol{c}\partial_x \boldsymbol{f}^{(0)} \rangle \quad \Leftarrow \quad \boldsymbol{f}^{(0)} = \rho^{(0)}\boldsymbol{w}.$$

Formal solution:
$$\begin{cases} \boldsymbol{f}^{(1)} & = \rho^{(1)}\boldsymbol{w} + J_d^\dagger\Big(- AG_a \boldsymbol{f}^{(0)} + (\boldsymbol{c}\partial_x \boldsymbol{f}^{(0)})\Big) \\ & = \rho^{(1)}\boldsymbol{w} - a\theta\rho^{(0)} J_d^\dagger A(\boldsymbol{c}\boldsymbol{w}) + \partial_x \rho^{(0)} J_d^\dagger(\boldsymbol{c}\boldsymbol{w}) \end{cases}$$

$J_d^\dagger$ = pseudo-inverse of J_d.

$J : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ not invertible (neither *injective* nor *surjective*)

$J\boldsymbol{x} = \boldsymbol{y}$ difficulty: $\begin{cases} \boldsymbol{y} \in \mathrm{im}J \Rightarrow & \text{not unique solution} \\ \boldsymbol{y} \notin \mathrm{im}J \Rightarrow & \text{no solution} \end{cases}$ $\cancel{\boldsymbol{x} = J^{-1}\boldsymbol{y}}$

Best *approximate* solution of *minimal* norm:

$$\overline{\boldsymbol{x}} \in \left\{\boldsymbol{a} : \ \|J\boldsymbol{a} - \boldsymbol{y}\| = \min_{\boldsymbol{z}} \|J\boldsymbol{z} - \boldsymbol{y}\|\right\} =: \mathcal{A}, \qquad \|\overline{\boldsymbol{x}}\| \leq \|\boldsymbol{a}\| \text{ for all } \boldsymbol{a} \in \mathcal{A}$$

Pseudo-inverse: $J^\dagger \boldsymbol{y} := \overline{\boldsymbol{x}}$ (well-defined & linear)

Explicit construction: $J|_{(\mathrm{ker}J)^\perp} =: \tilde{J} : (\mathrm{ker}J)^\perp \longrightarrow \mathrm{im}J$ (invertible restriction)

$$\boxed{J^\dagger = \tilde{E}_{(\mathrm{ker}J)^\perp} \ \tilde{J}^{-1} \ \tilde{P}_{\mathrm{im}J}}$$

$J = \tilde{E}_{\mathrm{im}J} \tilde{J} \tilde{P}_{(\mathrm{ker}J)^\perp}$

Some properties:

$\boldsymbol{y} \in \mathrm{im}J \ \Rightarrow$ exact solution $J^\dagger \boldsymbol{y} + \mathrm{kerJ}$

$J^\dagger J = P_{(\mathrm{ker}J)^\top} \in \mathrm{End}(\mathbb{R}^q)$ but $JJ^\dagger = P_{\mathrm{im}J} \in \mathrm{End}(\mathbb{R}^q)$, $(AB)^\dagger \neq B^\dagger A^\dagger$

Subspace $V \subset \mathbb{R}^q$, orthogonal projector $\tilde{P}_V : \mathbb{R}^q \rightarrow V$, embedding $\tilde{E}_V : V \rightarrow \mathbb{R}^q$.

- J symmetric: $J = J^\top$ (orthogonal eigenspaces)

$$\ker J = \text{eig}(J,0) \perp \text{im}J \quad \Leftrightarrow \quad \ker(J)^\perp = \text{im}J \qquad (\Rightarrow \quad J^\dagger J = JJ^\dagger = P_{\text{im}J})$$

$$J = \quad M\ \text{diag}\big(0,...,0,\lambda_1,...,\lambda_{\dim(\text{im }J)}\big)\ M^{-1}$$

$$\Rightarrow \qquad J^\dagger = M\ \text{diag}\big(0,...,0,\lambda_1^{-1},...,\lambda_{\dim(\text{im }J)}^{-1}\big)\ M^{-1} \qquad (M^{-1} = M^\top)$$

$\Rightarrow \quad J^\dagger$ symmetric

- J projection: $J = J^2$ (BGK: $J_d = \frac{1}{\omega}(G_d - I)$)

  Two possibilities for solving $J\boldsymbol{x} = \boldsymbol{y} \in \text{im}J$

  i) $\boldsymbol{x} = \boldsymbol{y} + \ker J$ $\quad(\boldsymbol{y} \in \text{im}J \ \Leftrightarrow\ \boldsymbol{y} = J\boldsymbol{z} \quad \Rightarrow \quad J\boldsymbol{y} = J^2\boldsymbol{z} = J\boldsymbol{z} = \boldsymbol{y})$

  ii) $\boldsymbol{x} = J^\dagger\boldsymbol{y} + \ker J$

$$\boldsymbol{y} - J^\dagger\boldsymbol{y} \in \ker J \qquad \Leftrightarrow \qquad \underbrace{J\boldsymbol{y}}_{=\boldsymbol{y}} - \underbrace{JJ^\dagger\boldsymbol{y}}_{=\boldsymbol{y}} = \boldsymbol{0}$$

$$J_d \boldsymbol{f}^{(2)} = -AG_a \boldsymbol{f}^{(1)} + \boldsymbol{c}\partial_x \boldsymbol{f}^{(1)} + \partial_t \boldsymbol{f}^{(0)} + \tfrac{1}{2}\boldsymbol{c}^2 \partial_x^2 \boldsymbol{f}^{(0)}$$

- Discussion of eqn. (h^2)

  Solvability condition for $\boldsymbol{f}^{(2)}$: $\text{RHS} \in \text{im} J_d \;\Leftrightarrow\;$ RHS $\in (\ker J_d^\top)^\perp = \text{span}(\boldsymbol{e})^\perp$

$$0 = \left\langle \boldsymbol{e}, -AG_a \boldsymbol{f}^{(1)} + \boldsymbol{c}\partial_x \boldsymbol{f}^{(1)} + \partial_t \boldsymbol{f}^{(0)} + \tfrac{1}{2}\boldsymbol{c}^2 \partial_x^2 \boldsymbol{f}^{(0)} \right\rangle = \left\langle \boldsymbol{e}, \boldsymbol{c}\partial_x \boldsymbol{f}^{(1)} + \partial_t \boldsymbol{f}^{(0)} + \tfrac{1}{2}\boldsymbol{c}^2 \partial_x^2 \boldsymbol{f}^{(0)} \right\rangle$$

$$\Big\downarrow \quad \begin{aligned} \boldsymbol{f}^{(0)} &= \rho^{(0)}\boldsymbol{w} \\ \boldsymbol{f}^{(1)} &= \rho^{(1)}\boldsymbol{w} - a\theta\rho^{(0)} J_d^\dagger A(\boldsymbol{c}\boldsymbol{w}) + \partial_x \rho^{(0)} J_d^\dagger(\boldsymbol{c}\boldsymbol{w}) \end{aligned}$$

$$0 \;=\; \partial_t \rho^{(0)} \underbrace{- a\theta \langle \boldsymbol{c}, J_d^\dagger A \boldsymbol{c}\boldsymbol{w} \rangle}_{\text{advection velocity}} \partial_x \rho^{(0)} + \underbrace{\left(\langle \boldsymbol{c}, J_d^\dagger \boldsymbol{c}\boldsymbol{w} \rangle + \frac{1}{2\theta} \right)}_{-\text{diffusivity}} \partial_x^2 \rho^{(0)}$$

  $\Rightarrow \quad \rho^{(0)}$ has to satisfy a drift-diffusion equation.

- Turns out:

  Transport coefficients (almost) independent of specific A and thus of J_d.

- $A = \begin{pmatrix} \ddots & & \\ & \omega & \\ & & \omega \end{pmatrix}$ or $A = \omega I \;\Rightarrow\; J = \frac{1}{\omega}(G - I)$ **(BGK)**

$$\langle \boldsymbol{c}, J_d^\dagger A \boldsymbol{c}\boldsymbol{w} \rangle = -\langle \boldsymbol{c}, \boldsymbol{c}\boldsymbol{w} \rangle = -\tfrac{1}{\theta}, \qquad \langle \boldsymbol{c}, J_d^\dagger \boldsymbol{c}\boldsymbol{w} \rangle = -\tfrac{1}{\omega}\langle \boldsymbol{c}, \boldsymbol{c}\boldsymbol{w} \rangle = -\tfrac{1}{\theta\omega}$$

- $A = M^{-1} \begin{pmatrix} \ddots & & \\ & \omega & \\ & & \lambda \end{pmatrix} M \qquad M := \begin{pmatrix} \boldsymbol{e}^\top \\ \boldsymbol{c}^\top \\ (\boldsymbol{c}^2 - \frac{1}{\theta}\boldsymbol{e})^\top \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1-1/\theta & 1-1/\theta & -1/\theta \end{pmatrix}$

M has W-orthogonal rows $\;\Rightarrow\; J_d^\dagger = -A^\dagger$

$$\langle \boldsymbol{c}, J_d^\dagger A \boldsymbol{c}\boldsymbol{w} \rangle = -\langle \boldsymbol{c}, A^\dagger A \boldsymbol{c}\boldsymbol{w} \rangle = -\tfrac{1}{\theta}, \qquad \langle \boldsymbol{c}, J_d^\dagger \boldsymbol{c}\boldsymbol{w} \rangle = -\langle \boldsymbol{c}, A^\dagger \boldsymbol{c}\boldsymbol{w} \rangle = -\tfrac{1}{\theta\omega}$$

- $A = M^{-1} \begin{pmatrix} \ddots & & \\ & \omega & \\ & & \lambda \end{pmatrix} M \qquad M := \begin{pmatrix} \boldsymbol{e}^\top/\sqrt{3} \\ \boldsymbol{c}^\top/\sqrt{2} \\ (3\boldsymbol{c}^2 - 2\boldsymbol{e})^\top/\sqrt{6} \end{pmatrix} = \begin{pmatrix} (\;1 & 1 & 1\;)/\sqrt{3} \\ (-1 & 1 & 0\;)/\sqrt{2} \\ (\;1 & 1 & -2)/\sqrt{6} \end{pmatrix}$

A is symmetric $\;\Leftarrow\;$ M is orthonormal, $\quad J_d^\dagger \neq -A^\dagger$ but

$$\langle \boldsymbol{c}, J_d^\dagger A \boldsymbol{c}\boldsymbol{w} \rangle = -\langle \boldsymbol{c}, A^\dagger A \boldsymbol{c}\boldsymbol{w} \rangle = -\tfrac{1}{\theta}, \qquad \langle \boldsymbol{c}, J_d^\dagger \boldsymbol{c}\boldsymbol{w} \rangle = -\langle \boldsymbol{c}, A^\dagger \boldsymbol{c}\boldsymbol{w} \rangle = -\tfrac{1}{\theta\omega}$$

$$\partial_t \rho^{(0)} \underbrace{- a\theta\left\langle \boldsymbol{c}, J_d^\dagger A \boldsymbol{c}\boldsymbol{w}\right\rangle}_{\text{advection velocity}} \partial_x \rho^{(0)} + \underbrace{\left(\left\langle \boldsymbol{c}, J_d^\dagger \boldsymbol{c}\boldsymbol{w}\right\rangle + \frac{1}{2\theta}\right)}_{-\text{diffusivity}} \partial_x^2 \rho^{(0)} = 0$$

$$\partial_t \rho^{(0)} + a\partial_x \rho^{(0)} - \underbrace{\frac{1}{\theta}\left(\frac{1}{\omega} - \frac{1}{2}\right)}_{=\nu} \partial_x^2 \rho^{(0)} = 0$$

- $A \sim \begin{pmatrix} \vdots & \dot{\omega} & \vdots \\ \cdot & \cdot & \lambda \end{pmatrix}$, $\quad A = A^\top, \quad [A, GD] = 0 \qquad (GD) = (GD)^\top = DG^\top$

A symmetric, $\quad J_d^\dagger \neq -A^\dagger$ still

$$\langle \boldsymbol{c}, J_d^\dagger A \boldsymbol{c}\boldsymbol{w}\rangle = -\langle \boldsymbol{c}, A^\dagger A \boldsymbol{c}\boldsymbol{w}\rangle = -\frac{1}{\theta} \qquad \langle \boldsymbol{c}, J_d^\dagger \boldsymbol{c}\boldsymbol{w}\rangle = -\langle \boldsymbol{c}, A^\dagger \boldsymbol{c}\boldsymbol{w}\rangle$$

but $\langle \boldsymbol{c}, A^\dagger \boldsymbol{c}\boldsymbol{w}\rangle = -\nu(ah, \theta, \omega, \lambda)$

i.e. diffusivity $\rightarrow$ (complicated) function of all algorithmic parameters

A has same eigenspaces as GD depending on ah

- Evolution of $\rho^{(\ell)}$:

  Drift-diffusion equation $\leftarrow$ driven by derivatives of lower orders

- Unlike BGK, in general: $\rho^{(\ell)} \neq \langle \boldsymbol{e}, \boldsymbol{f}^{(\ell)} \rangle \qquad \ell > 0$

$$J_d \boldsymbol{f}^{(\ell)} = \underbrace{-AG_a \boldsymbol{f}^{(\ell-1)} + \textstyle\sum_{j=1}^{\ell} D_j(\partial_t, \boldsymbol{c}\partial_x) \boldsymbol{f}^{(\ell-j)}}_{\mathrm{RHS}^{(\ell)}}$$

$$\mathrm{RHS}^{(\ell)} \in \mathrm{im} J_d \quad \Leftrightarrow \quad \mathrm{RHS}^{(\ell)} \perp \ker J_d^\top \qquad \Rightarrow \quad \langle \boldsymbol{e}, \mathrm{RHS}^{(\ell)} \rangle = 0$$

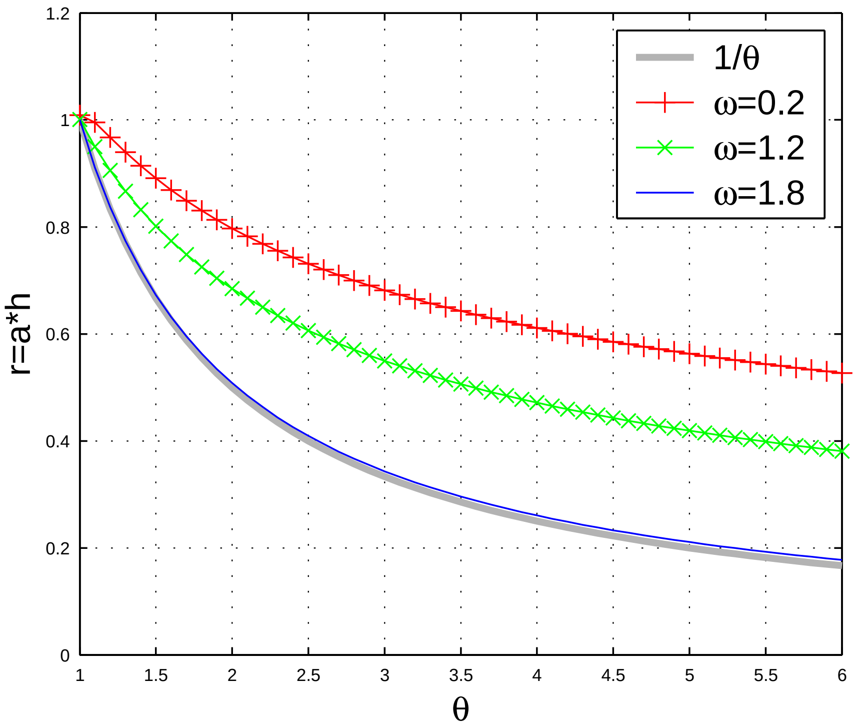
$$\boldsymbol{f}^{(\ell)} = \rho^{(\ell)} \boldsymbol{w} + J_d^\dagger \, \mathrm{RHS}^{(\ell)} \qquad \Rightarrow \quad \langle \boldsymbol{e}, \boldsymbol{f}^{(\ell)} \rangle = \rho^{(\ell)} + \underbrace{\left\langle (J^\top)^\dagger \boldsymbol{e}, \, \mathrm{RHS}^{(\ell)} \right\rangle}_{=0 \text{ if } J_d \text{ sym.}}$$

$$J^\dagger = \tilde{E}_{(\ker J)^\perp} \; \tilde{J}^{-1} \; \tilde{P}_{\mathrm{im} J} \qquad \Rightarrow \begin{cases} \ker J^\dagger = (\mathrm{im} J)^\perp = \ker J^\top & \Rightarrow \ \ker (J^\top)^\dagger = (\mathrm{im} J^\top)^\perp = \ker J \\ \mathrm{im} J^\dagger = (\ker J)^\perp = \mathrm{im} J^\top & \Rightarrow \ \mathrm{im} (J^\top)^\dagger = (\ker J^\top)^\perp = \mathrm{im} J \end{cases}$$

- Avoid this $\rightarrow$ successive solution: $J_d \boldsymbol{f}^\ell = A(G - I) \boldsymbol{f}^{(\ell)} = \mathrm{RHS}^{(\ell)}$ (A symmetric)

$$\langle \boldsymbol{e}, \mathrm{RHS}^{(\ell)} \rangle = 0 \quad \Rightarrow \quad (G - I) \boldsymbol{f}^{(\ell)} = A^\dagger \, \mathrm{RHS}^{(\ell)}$$ (not general solution)

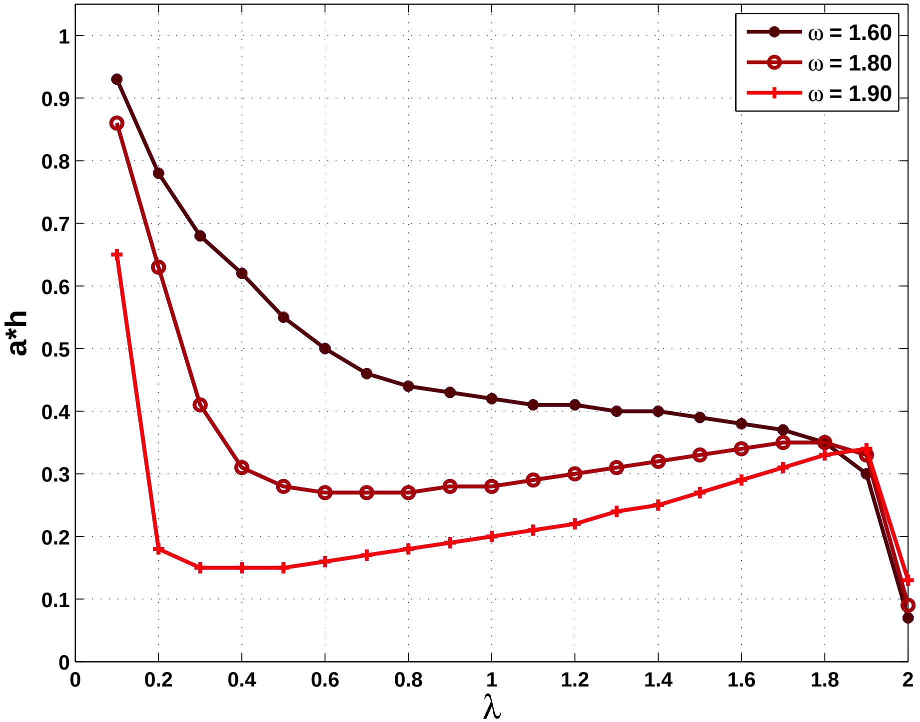
$$\underbrace{\langle \boldsymbol{e}, A^\dagger \, \mathrm{RHS}^{(\ell)} \rangle = 0}_{\text{automatically satisfied}} \quad \Rightarrow \quad \boldsymbol{f}^{(\ell)} = \rho^{(\ell)} \boldsymbol{w} - A^\dagger \, \mathrm{RHS}^{(\ell)}$$ ($G - I$ negative projector)

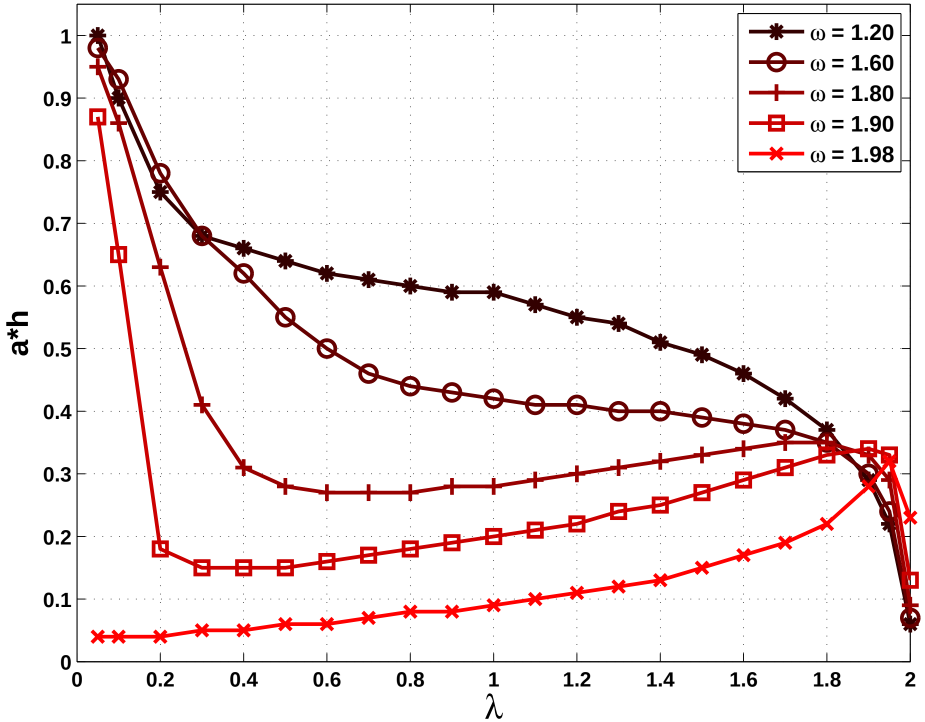


Stability-structure: guarantees stability for $|ah| < \frac{1}{\theta}$.

W-orthogonal moment generating vectors:

$$M := \begin{pmatrix} \boldsymbol{e}^\top \\ \boldsymbol{c}^\top \\ (\boldsymbol{c}^2 - \frac{1}{\theta}\boldsymbol{e})^\top \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1-\frac{1}{\theta} & 1-\frac{1}{\theta} & -\frac{1}{\theta} \end{pmatrix} \qquad A = M^{-1} \begin{pmatrix} 0 & \cdot & \cdot \\ \cdot & \omega & \cdot \\ \cdot & \cdot & \lambda \end{pmatrix} M$$

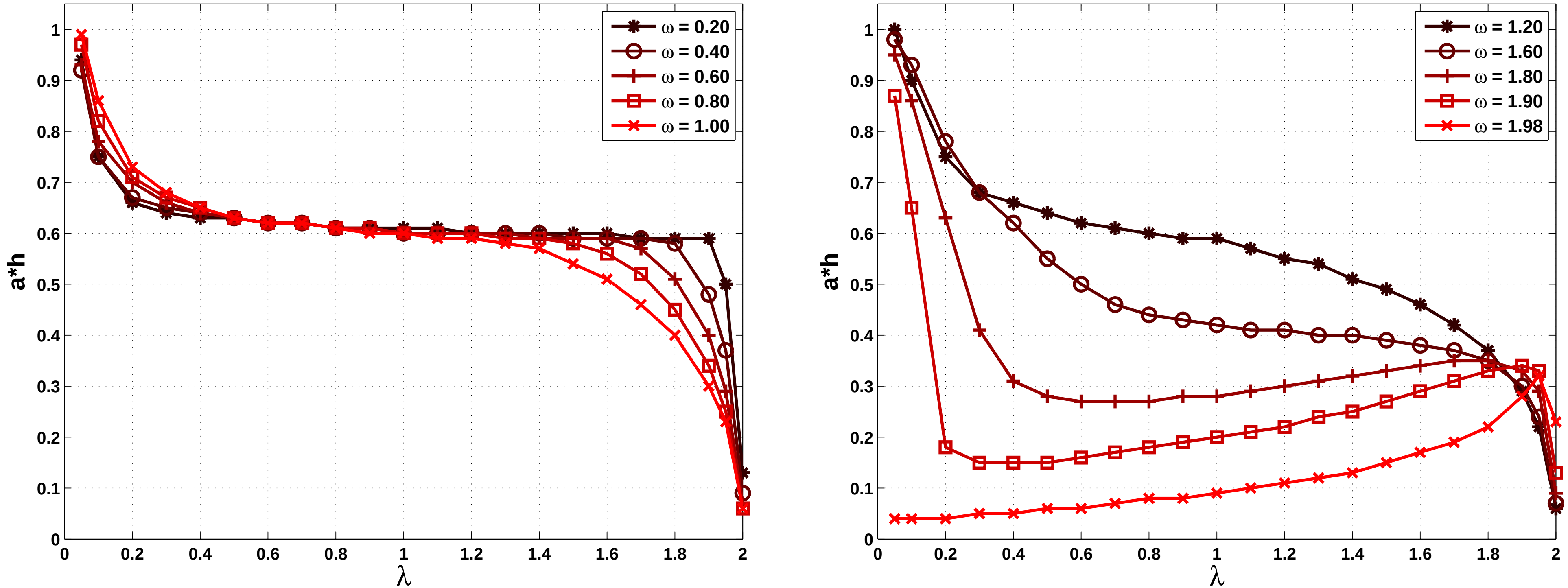




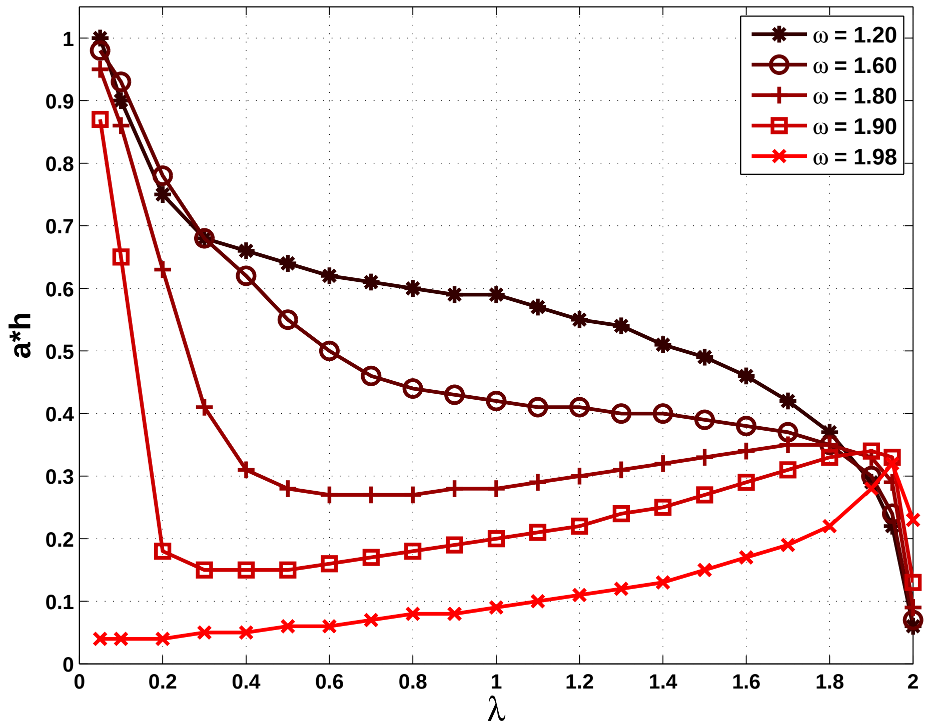
Observation:

Orthonormal moment generating vectors:

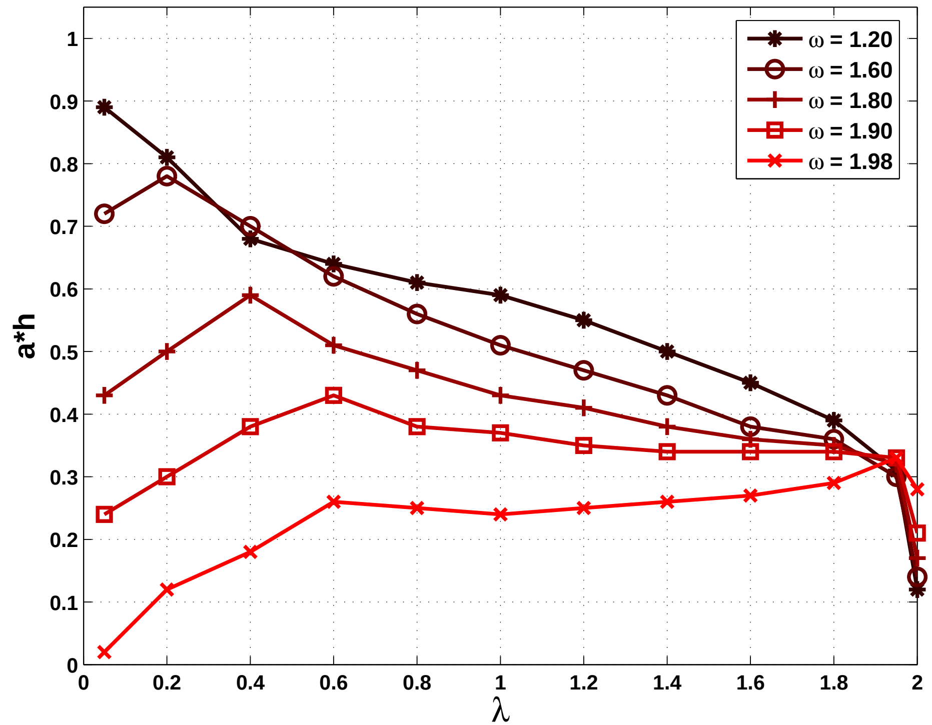
$$M := \begin{pmatrix} \boldsymbol{e}^\top/\sqrt{3} \\ \boldsymbol{c}^\top/\sqrt{2} \\ (3\boldsymbol{c}^2 - 2\boldsymbol{e})^\top/\sqrt{6} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \qquad A = M^{-1} \begin{pmatrix} 0 & \cdot & \cdot \\ \cdot & \omega & \cdot \\ \cdot & \cdot & \lambda \end{pmatrix} M$$



Observation:



standard MRT



cascaded MRT ([Geier],[Asinari])