Euler-characteristic boundary conditions for lattice Boltzmann methods

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Outline

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Motivation: open boundaries

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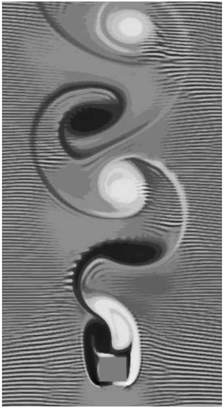
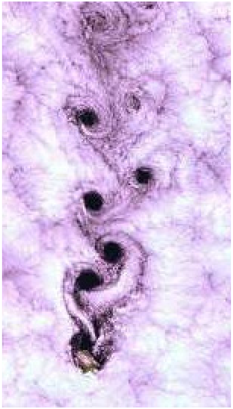
LB LES



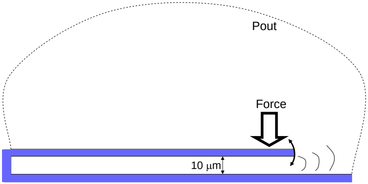


LANDSAT 7
on 9/15/1999

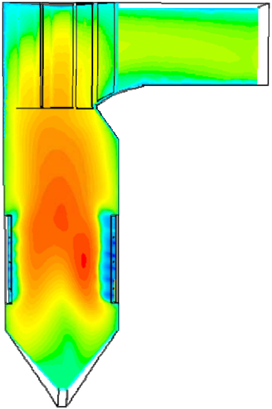
Alejandro Selkirk Island

- Open boundary conditions (constant pressure, far-field environment)
- Non-reflecting open boundaries

Motivation: examples



Microcantilever



- **Microcantilever**: fluid-structure interaction where small pressure perturbations modify the behavior.
- **Industrial boiler**: flames generate pressure waves which have influence on the combustion.

Review of solutions

- Zero-gradient boundary conditions at outlet (Incompressible!?)
  - [Yu et al. (2005)]
  - [Yang's Thesis (2007)]

- Non-reflecting boundary conditions (thermodynamical consistency?)
  - Absorbing layers [Ricot et al. (2008), Tekitek et al. (2008), da Silva (2007)]
  - Characteristic BC [Kam et al. (2007), Dehee (2008)] (without details of the implementation)

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Lattice Boltzmann method

LBM with **MRT** collision operator

$$\mathbf{f}(x_i + \mathbf{e}_i \delta t, t + \delta t) - \mathbf{f}(x_i, t) = -\mathbf{M}^{-1}\mathbf{S}[\mathbf{m}(x_i, t) - \mathbf{m}^{eq}(x_i, t)]$$

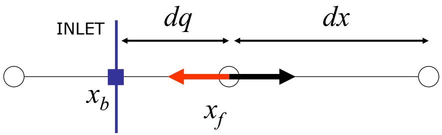
Transformation matrix $D2Q9$

$$\begin{pmatrix} \rho \\ e \\ \epsilon \\ j_x \\ q_x \\ j_y \\ q_y \\ p_{xx} \\ p_{xy} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -4 & -1 & -1 & -1 & -1 & 2 & 2 & 2 & 2 \\ 4 & -2 & -2 & -2 & -2 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & -1 & -1 & 1 \\ 0 & -2 & 0 & 2 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 & 1 & -1 & -1 \\ 0 & 0 & -2 & 0 & 2 & 1 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{pmatrix}$$

Relaxation matrix

- MRT: $\mathbf{S} = diag(0, s_e, s_\epsilon, 0, s_q, 0, s_q, s_\nu, s_\nu) \rightarrow$ related to transport properties
- SRT: $\tau = 1/s_\nu \rightarrow$ viscosity (and stability)
- TRT: $s_\nu = s_e = s_\epsilon$, s_q

LB boundary conditions: velocity



UBB – velocity bounce-back

APPROACH: reflection rule + Dirichlet BC (+ correction)

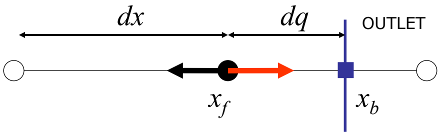
$$f_{\bar{\alpha}}(\mathbf{x}_f, t+1) \quad = \quad \tilde{f}_{\alpha}(\mathbf{x}_f, t) - 2 f_{\alpha}^{eq-}(\mathbf{x}_b, \hat{t})$$

where:

$$f_{\alpha}^{eq-} = \omega_{\alpha} \rho_0 c_s^{-2} (\mathbf{e}_{\alpha} \cdot \mathbf{u})$$

is the anti-symmetric part of the f_{α}^{eq}

LB boundary conditions: pressure



PAB – pressure anti-bounce-back

APPROACH: reflection rule + Dirichlet BC + correction

$$\begin{aligned} f_{\bar{\alpha}}(\mathbf{x}_f, t+1) &= -\tilde{f}_\alpha(\mathbf{x}_f, t) \\ &+ 2 f_\alpha^{eq+}(\mathbf{x}_b, \hat{t}) \\ &+ (2 - s_\nu)\left(f_\alpha^+(\mathbf{x}_f, t) - f_\alpha^{eq+}(\mathbf{x}_f, t)\right) \end{aligned}$$

where:

$$f_\alpha^{eq+} = \omega_\alpha \rho + \frac{1}{2}\omega_\alpha \rho_0 c_s^{-4}\left[(\mathbf{e}_\alpha \cdot \mathbf{u})^2 - c_s^2(\mathbf{u} \cdot \mathbf{u})\right]$$

$$f_\alpha^+ = \frac{1}{2}(f_\alpha + f_{\bar{\alpha}})$$

are the symmetric part of f_α^{eq} and f_α

LODI – Local One-Dimensional Inviscid equations

Objective

To find the ρ or u_i to set the Dirichlet boundary condition in the open boundary, extracting the pressure-wave reflection component. Based on [Poinsot and Lelle (1992)]

Solving **in the boundary** the 2D Euler equations in x-direction:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial\left[u(\rho E + p)\right]}{\partial x} = 0$$

Eigenvalues!

LODI with wave amplitudes

Wave amplitudes: (isothermal! $\rightarrow p = c_s^2 \rho$)

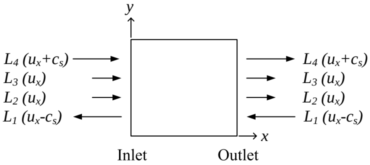
$$\mathbf{L} = \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} = \begin{Bmatrix} (u - c_s)\left(\frac{\partial p}{\partial x} - \rho c_s \frac{\partial u}{\partial x}\right) \\ u\frac{\partial v}{\partial x} \\ u\left(c_s^2 \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\right) \\ (u + c_s)\left(\frac{\partial p}{\partial x} + \rho c_s \frac{\partial u}{\partial x}\right) \end{Bmatrix}$$

LODI equation using L (without the energy equation):

$$\frac{\partial \rho}{\partial t} + \frac{1}{2c_s^2}(L_4 + L_1) + \frac{1}{c_s^2} L_3 = 0$$

$$\frac{\partial u}{\partial t} + \frac{1}{2\rho c_s}(L_4 - L_1) = 0$$

$$\frac{\partial v}{\partial t} + L_2 = 0$$



Equilibrium distribution functions

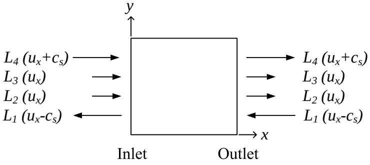
Modified m_α^{eq} at the boundary: ($\kappa \rightarrow$ heat capacity ratio)

$$
\begin{aligned}
e^{eq} &= -2(2-\kappa)\rho + \rho_0(u^2+v^2) \\
\epsilon^{eq} &= \rho + \rho_0(u^2+v^2) \\
q_x^{eq} &= -\rho_0 u \\
q_y^{eq} &= -\rho_0 v \\
p_{xx}^{eq} &= \rho_0(u^2-v^2) \\
p_{xy}^{eq} &= \rho_0 uv
\end{aligned}
$$

In the continuum limit:

- Speed of sound $\rightarrow c_s = \sqrt{\kappa RT} = \sqrt{\kappa \frac{p}{\rho}}$
- Viscosity $\rightarrow \nu = \frac{1}{3}\left(\frac{1}{s_\nu} - \frac{1}{2}\right)$
- Bulk viscosity $\rightarrow \zeta = \frac{2-\kappa}{6}\left(\frac{1}{s_e} - \frac{1}{2}\right)$

Implementation



Approach: UBB or PAB with computed ρ and u_i from LODI

LODI discretization (OUTLET–PAB)

$$\rho(\hat{t}) \approx \rho(\hat{t}-1) - \frac{\delta t}{2c_s^2}\left(L_4(\hat{t}-1) + L_1(\hat{t}-1)\right) - \frac{1}{c_s^2} L_3(\hat{t}-1)$$

$$u(\hat{t}) \approx u(\hat{t}-1) - \frac{\delta t}{2\rho c_s}\left(L_4(\hat{t}-1) - L_1(\hat{t}-1)\right)$$

$$v(\hat{t}) \approx v(\hat{t}-1) - \delta t L_2(\hat{t}-1)$$

Models for L_{in}

$$L_1(\mathbf{x}_b, \hat{t}-1) = k_1(p(\mathbf{x}_b, \hat{t}-1) - p_b)$$

where:

$$k_1 = \sigma_1(1 - Ma^2)\frac{c_s}{L}$$

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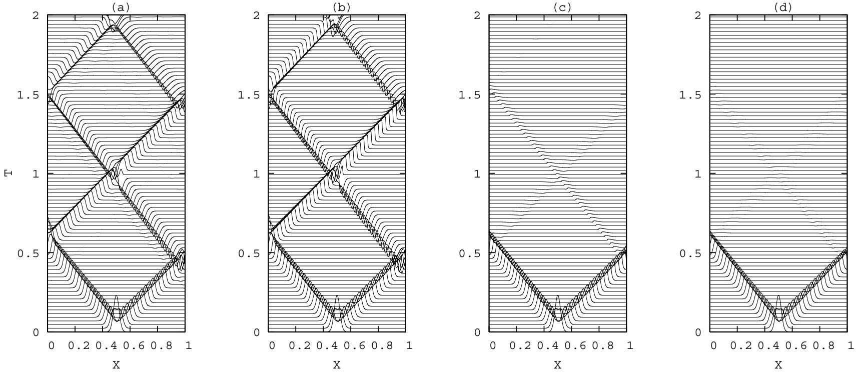
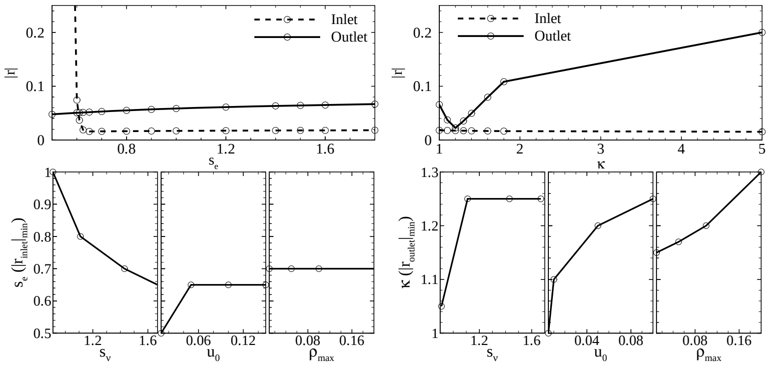
Results I: 1D wave



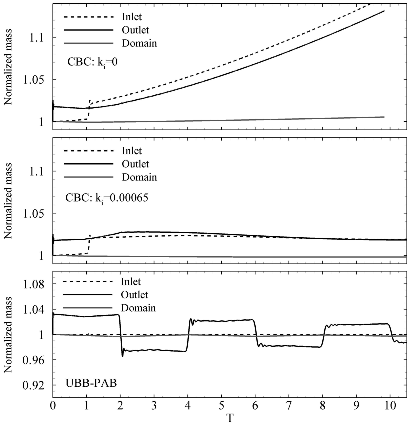
- (a) equilibrium distribution functions
- (b) inlet: UBB; outlet:PAB
- (c) characteristic boundary conditions (CBC)
- (d) CBC with corrections

Results II: reflection ratio



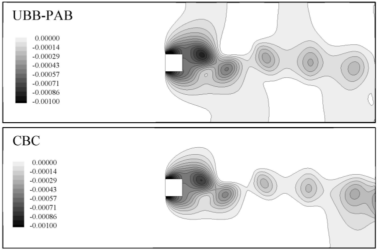
- Evaluate performance of two parameters:
  - bulk viscosity in the fluid domain (s_e)
  - heat capacity ratio in the boundary (κ)

Results III: mass balance



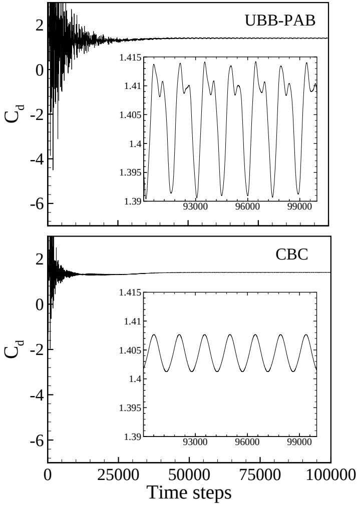
- Laminar channel
- CBC: (i) avoids pressure reflection + (ii) allows mass conservation (well-posedness of BC)

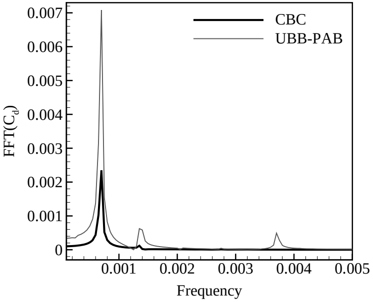
Results IV: unsteady simulation



- Flow around a square cylinder
- UBB-PAB: resonance has effects on the solution
- CBC is the solution

Results IV: unsteady simulation





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Conclusions

- No previous implementation description of CBC for LB (isothermal)
- Characteristic boundary conditions:
    - Presented for 2D open boundaries with Dirichlet conditions
    - Direct application for: 3D, walls and Neumann conditions
    - Reduction of the interaction up to 99%
- Key points:
    1. NSCBC by Poinsot and Lele (1992)
    2. Multireflection boundary conditions by Ginzburg et al.(2008)

Bibliography

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