Interactions entre théories algébriques
et calcul scientifique

Conservatoire National des Arts et Métiers
Paris, 18 septembre 2007

Integer matrix factorization and computation of homology groups for three dimensional meshes

François Dubois

CNAM Paris and University Paris Sud, Orsay

Survey of the lecture

1) A mesh $\mathcal{T}$ as a simplicial complex

2) Discrete vector fields

3) Chains, borders and incidence matrices

4) Homology groups $H_p(\mathcal{T})$

5) A first numerical algorithm for computing $H_1(\mathcal{T})$

6) A Smith algorithm for computing $H_1(\mathcal{T})$

7) Conclusion

Consider $s_0, s_1, \ldots, s_p$ $p+1$ points of $\mathbb{R}^3$
that are affinely independent

$(s_0, s_1, \ldots, s_p)$ p-simplex generated by $s_0, s_1, \ldots, s_p$:
convex hull of the $p+1$ previous points

Definition of a "conforming mesh" $\mathcal{T}_h$
of a three-dimensional domain Ω composed by tetrahedra
(P.G. Ciarlet, 1978)

an "element" K is a nondegenerated closed tetrahedron

$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$

if K and L belong to $\mathcal{T}_h$, $K \cap L$ is either void,
or is a vertex of K **and** L,
or is an edge of K **and** L,
or is a face of K **and** L,
or $K = L$.

A mesh is defined through its "elements".

"Abstract simplicial complex" (Σ, Φ)

definition proposed by H. Cartan (1948)

A set Σ.

A family Φ of **finite** parts of Σ such that

if $s \in \Sigma$, then $\{s\} \in \Phi$

if $S \in \Phi$ and $T \subset S$, then $T \in \Phi$.

"Simplicial complex" for defining a conforming simplicial mesh $\mathcal{T}$

a set Σ of "vertices"

set $\mathcal{T}^0 = \bigcup_{s\in\Sigma} \{s\} \simeq \Sigma$ of vertices s "sommet"

set $\mathcal{T}^1$ of edges a "arête"

set $\mathcal{T}^2$ of (triangular) faces f "face"

set $\mathcal{T}^3$ of tetrahedra (elements) t "tétraèdre"

$$\mathcal{T} = \Big(\Sigma, \, \Phi \equiv \bigcup_{p=0}^{3} \mathcal{T}^p \Big) \simeq \bigcup_{p=0}^{3} \mathcal{T}^p$$

is an abstract simplicial complex in the sense of H. Cartan.

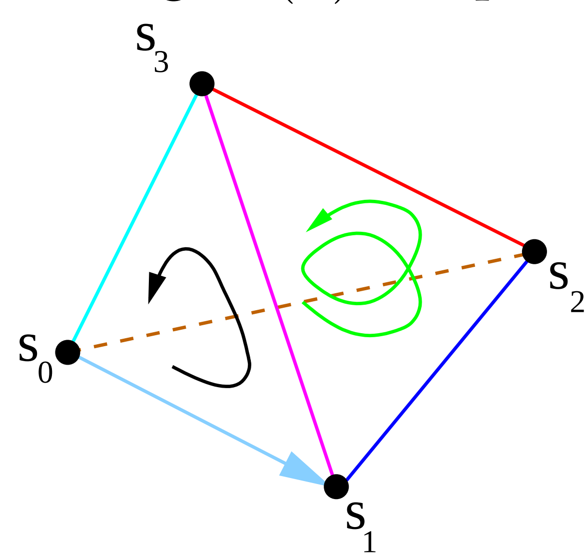
Rigorous definitions (with quotient sets) proposed by J.P. Serre (1948)

a vertex has no orientation

for **each** $p \geq 1$ and **each** p-simplex $(s_0, s_1, \ldots, s_p)$,
make a choice of an "orientation"

for a permutation σ of $\{0, 1, \ldots, p\}$
operating on the set $\{s_0, s_1, \ldots, s_p\}$,
$(s_0, s_1, \ldots, s_p) \simeq (s_{\sigma(0)}, s_{\sigma(1)}, \ldots, s_{\sigma(p)})$
if and only if the sign $\epsilon(\sigma)$ is equal to $+1$.

edge $a = (s_0, s_1)$
face $f = (s_0, s_1, s_3)$
tetrahedron $t = (s_0, s_1, s_2, s_3)$



Notations: $s \in \mathcal{T}^0$ a vertex of the simplicial mesh $\mathcal{T}$
$a \in \mathcal{T}^1$ an edge
$f \in \mathcal{T}^2$ a face
$t \in \mathcal{T}^3$ a tetrahedron

Basis functions of discrete spaces

$\varphi_s^0 \in H^1_{\mathcal{T}}(\Omega)$ scalar valued, affine in each tetrahedron

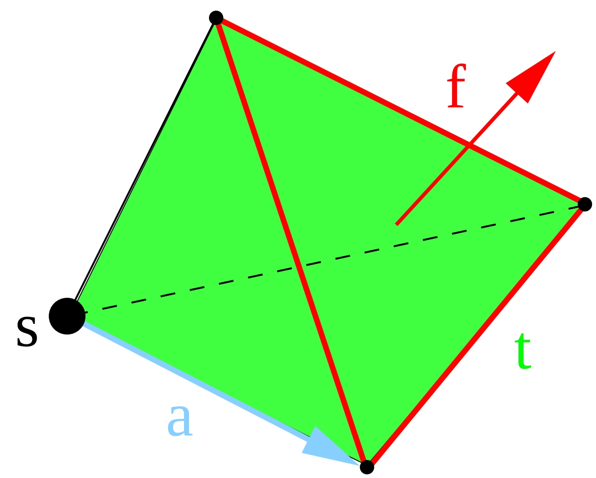
$\varphi_a^1 \in H_{\mathcal{T}}(\mathrm{curl},\,\Omega)$ vector valued,
in each tetrahedron, $\varphi_a^1 \in \mathrm{NR}$ Nédélec-Rao
$\mathrm{NR} \equiv \{\, \mathbb{R}^3 \ni x \longmapsto \alpha + \beta \times x \in \mathbb{R}^3 \,\}$, $\alpha,\, \beta \in \mathbb{R}^3$

$\varphi_f^2 \in H_{\mathcal{T}}(\mathrm{div},\,\Omega)$ vector valued,
in each tetrahedron, $\varphi_f^2 \in \mathrm{RTN}$ Raviart-Thomas-Nédélec
$\mathrm{RTN} \equiv \{\, \mathbb{R}^3 \ni x \longmapsto \alpha + \beta\, x \in \mathbb{R}^3 \,\}$, $\alpha \in \mathbb{R}^3$, $\beta \in \mathbb{R}$

$\varphi_t^3 \in L^2_{\mathcal{T}}(\Omega)$ scalar valued, constant in each tetrahedron

Degrees of freedom

nodal value	$\varphi_s^0(\sigma) = \delta_{s,\sigma}\,,$	$\forall\, s,\, \sigma \in \mathcal{T}^0$
circulation	$\displaystyle\int_\alpha \varphi_a^1 \bullet \tau_\alpha \,\mathrm{d}\gamma = \delta_{a,\alpha}\,,$	$\forall\, a,\, \alpha \in \mathcal{T}^1$
flux	$\displaystyle\int_g \varphi_f^2 \bullet n_g \,\mathrm{d}\sigma = \delta_{f,g}\,,$	$\forall\, f,\, g \in \mathcal{T}^2$
mean value	$\displaystyle\int_K \varphi_t^3 \,\mathrm{d}x = \delta_{t,K}\,,$	$\forall\, t,\, K \in \mathcal{T}^3$



p-chain: a formal sum of the type $\quad \gamma = \sum_{\alpha \in \mathcal{T}^p} n_\alpha \, \alpha \,, \quad n_\alpha \in \mathbb{Z}$

$C_p(\mathcal{T})$: space of p-chains $= < \mathcal{T}^p >$

Border of a simplex $(s_0, \, s_1, \, \ldots \, , \, s_p)$, $p \geq 1$.

$$\partial \, (s_0, \, s_1, \, \ldots \, , \, s_p) \, = \, \sum_{j=0}^{p} (-1)^j \, (s_0, \, s_1, \, \ldots s_{j-1}, \, s_{j+1}, \, \ldots \, , \, s_p)$$

Examples

$$\partial \, s \, = \, 0 \,, \qquad s \in \mathcal{T}^0$$
$$\partial \, (s_0, \, s_1) \, = \, -s_0 + s_1 \,, \qquad (s_0, \, s_1) \in \mathcal{T}^1$$
$$\partial \, (s_0, \, s_1, \, s_2) \, = \, (s_1, \, s_2) - (s_0, \, s_2) + (s_0, \, s_1) \,, \qquad (s_0, \, s_1, \, s_2) \in \mathcal{T}^2$$
$$\partial \, (s_0, \, s_1, \, s_2, \, s_3) \, = \, (s_1, \, s_2, \, s_3) - (s_0, \, s_2, \, s_3) + (s_0, \, s_1, \, s_3) - (s_0, \, s_1, \, s_2) \,, \qquad (s_0, \, s_1, \, s_2, \, s_3) \in \mathcal{T}^3$$

By linearity, the border defines a linear operator

$$\partial_p \, : \, C_p(\mathcal{T}) \longrightarrow C_{p-1}(\mathcal{T})$$

Write the operator ∂ in the basis of simplicies.

$s \in \mathcal{T}^0$ a vertex of the mesh $\quad \partial_0 \, s = 0.$

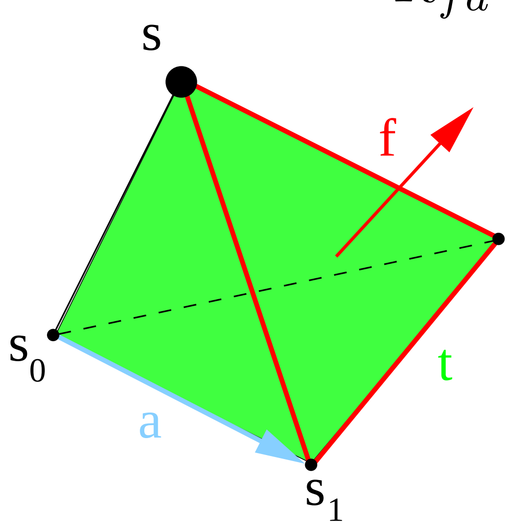
$a \in \mathcal{T}^1$ an edge of the mesh $\quad \partial_1 \, a \equiv \sum_{s \in \mathcal{T}^0} G_{as} \, s \quad \in C_0(\mathcal{T})$

if $a \equiv (s_0, \, s_1)$, $G_{as_1} = +1$, $G_{as_0} = -1$

and $G_{as} = 0$ for $s \neq s_0, \, s_1$

$f \in \mathcal{T}^2$ a face of the mesh $\quad \partial_2 \, f \equiv \sum_{a \in \mathcal{T}^1} R_{fa} \, a \quad \in C_1(\mathcal{T})$

R_{fa} is not null only for the three edges that compose ∂f



$t \in \mathcal{T}^3$ a tetrahedron of the mesh

$\partial_3 \, t \equiv \sum_{f \in \mathcal{T}^1} D_{tf} \, f \quad \in C_2(\mathcal{T})$

D_{tf} is not null only for the four faces that compose ∂t

The matrix of ∂_1 in the basis of simplices is equal to G^{t}
∂_2 R^{t}
∂_3 D^{t}

Beautiful property (A. Bossavit, 1986).

$$\nabla \varphi_s^0 = \sum_{a \in \mathcal{T}^1} G_{as}\, \varphi_a^1$$

$$\mathrm{curl} \varphi_s^1 = \sum_{f \in \mathcal{T}^2} R_{fa}\, \varphi_f^2$$

$$\mathrm{div} \varphi_f^2 = \sum_{t \in \mathcal{T}^3} D_{tf}\, \varphi_t^3$$

The matrix of ∇ operator relatively to the φ_α^p basis is equal to G
for curl operator, we recover matrix R
for div operator, we obtain matrix D.

The derivation is the adjoint of the border operator ∂

The derivation is the adjoint of the border operator ∂: $\quad \mathrm{d} = \partial^*$

$$
\begin{array}{ccccccccccc}
 & 0 & & D^{\mathrm{t}} & & R^{\mathrm{t}} & & G^{\mathrm{t}} & & 0 & \\
 & \partial_4 & & \partial_3 & & \partial_2 & & \partial_1 & & \partial_0 & \\
0 & \longrightarrow & C_3(\mathcal{T}) & \longrightarrow & C_2(\mathcal{T}) & \longrightarrow & C_1(\mathcal{T}) & \longrightarrow & C_0(\mathcal{T}) & \longrightarrow & 0 \\
\mathcal{T}^4 & & \mathcal{T}^3 & & \mathcal{T}^2 & & \mathcal{T}^1 & & \mathcal{T}^0 & & \mathcal{T}^{-1} \\
0 & \longleftarrow & L^2(\mathcal{T}) & \longleftarrow & H_{\mathcal{T}}(\mathrm{div},\,\Omega) & \longleftarrow & H_{\mathcal{T}}(\mathrm{curl},\,\Omega) & \longleftarrow & H^1_{\mathcal{T}}(\Omega) & \longleftarrow & 0 \\
 & 0 & & \mathrm{div} & & \mathrm{curl} & & \nabla & & 0 & \\
 & 0 & & D & & R & & G & & 0 &
\end{array}
$$

Fondamental property: $\partial_p \circ \partial_{p+1} \equiv 0\,.$

proof by linearity; exercice for a simplex.

Well known fact: $\mathrm{div} \circ \mathrm{curl} \equiv 0\,, \quad \mathrm{curl} \circ \nabla \equiv 0.$

Classical spaces:

$Z_p(\mathcal{T})$ space of closed p-chains,

id est p-chains γ such that $\partial\gamma = 0$

$Z_p(\mathcal{T}) = \ker \partial_p$

$B_p(\mathcal{T})$ space of border p-chains,

p-chains γ such that $\exists\, \beta \in C_{p+1}(\mathcal{T}), \quad \gamma = \partial\beta$

$B_p(\mathcal{T}) = \mathrm{Im}\, \partial_{p+1}$

Of course, $B_p(\mathcal{T}) \subset Z_p(\mathcal{T})$

Define the p^{o} homology group $H_p(\mathcal{T})$

as the quotient of $Z_p(\mathcal{T})$ *modulo* $B_p(\mathcal{T})$:

$H_p(\mathcal{T}) \equiv Z_p(\mathcal{T}) \,/\, B_p(\mathcal{T})$.

$H_0(\mathcal{T}) \simeq \mathbb{Z}^{\text{number of connected components of } \Omega}$

$H_1(\mathcal{T}) \simeq \mathbb{Z}^{\text{number of nontrivial circuits in } \Omega}$

$H_2(\mathcal{T}) \simeq \mathbb{Z}^{\text{number of connected components of } \partial\,\Omega}$

classical!

Matricial point of view:

we have $R\,G = 0\,.$ Then $G^{\mathrm{t}}\,R^{\mathrm{t}} = 0\,.$

and $\mathrm{Im}\,R^{\mathrm{t}} \subset \ker G^{\mathrm{t}}$

we search a decomposuiton of the space $\ker G^{\mathrm{t}}$ under the form

$$\ker G^{\mathrm{t}} = \mathrm{Im}\,R^{\mathrm{t}} \oplus \widetilde{H}_1(\mathcal{T})$$

Idea proposed by F. Rapetti, FD, A. Bossavit (2002):

try to factorize the matrix R^{t} with a "QR like" algorithm

id est find three matrices

Q (invertible over $\mathbb{Z}$)

U (upper triangular with integer coefficients)

P (invertible over $\mathbb{Z}$)

such that $R^{\mathrm{t}} = Q\,U\,P\,.$

Multiply $R^{\rm t}$ on the left by simple “two by two like” matrices
Q_j in order to force a triangular form.

idea of Givens rotations for factorization of a general matrix A
under the form $A = Q\,U$
with Q orthogonal and R upper triangular

replace the condition of orthogonality for Q
by the fact that Q is invertible over $\mathbb{Z}$.

better: suppose $\det Q = 1$ (for this lecture ...)

Use $\rho_{ij}(\epsilon) \equiv$

$$\begin{pmatrix} 1 & 0 & \dots & & & & \dots & 0 \\ 0 & 1 & 0 & \dots & & & \dots & 0 \\ \vdots & \dots & 1 & \dots & & & & \vdots \\ & & \dots & 0 & \dots & \epsilon & \dots & \\ & & \dots & 0 & 1 & 0 & \dots & \\ & & \dots & -\epsilon & \dots & 0 & \dots & \vdots \\ \vdots & \dots & & & \dots & 0 & 1 & 0 \\ 0 & \dots & & & & \dots & 0 & 1 \end{pmatrix}$$

with $\epsilon^2 \equiv 1$

id est $\rho_{ij}(\epsilon) \equiv \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}$, for lines $i < j$ and identity elsewhere

to "exchange" lines number i and j: $\rho_{ij}(\epsilon) \bullet \begin{pmatrix} 0 \\ \epsilon \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

local Givens rotation of angle $-\epsilon \, \dfrac{\pi}{2}$.

Use $\theta_{ij}(\epsilon, \varphi) \equiv \begin{pmatrix} 1 & 0 & \dots & & & & \dots & 0 \\ 0 & 1 & 0 & \dots & & & \dots & 0 \\ \vdots & \dots & 1 & \dots & & & & \vdots \\ & & \dots & \epsilon & \dots & 0 & \dots & \\ & & \dots & 0 & 1 & 0 & \dots & \\ & & \dots & -\varphi & \dots & \epsilon & \dots & \vdots \\ \vdots & \dots & & & \dots & 0 & 1 & 0 \\ 0 & \dots & & & & \dots & 0 & 1 \end{pmatrix}$

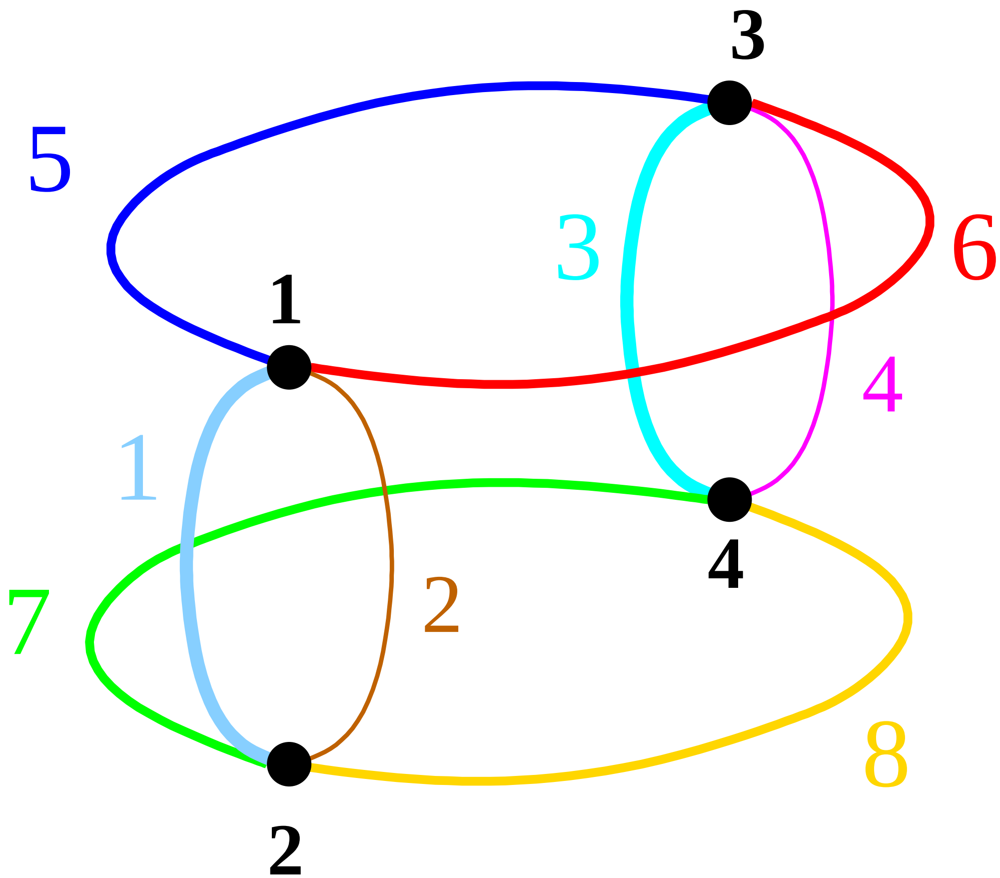
with $\epsilon^2 \equiv \varphi^2 \equiv 1$

id est $\theta_{ij}(\epsilon, \varphi) \equiv \begin{pmatrix} \epsilon & 0 \\ -\varphi & \epsilon \end{pmatrix}$, identity elsewhere

to "kill" non null values at column i and line $j > i$:

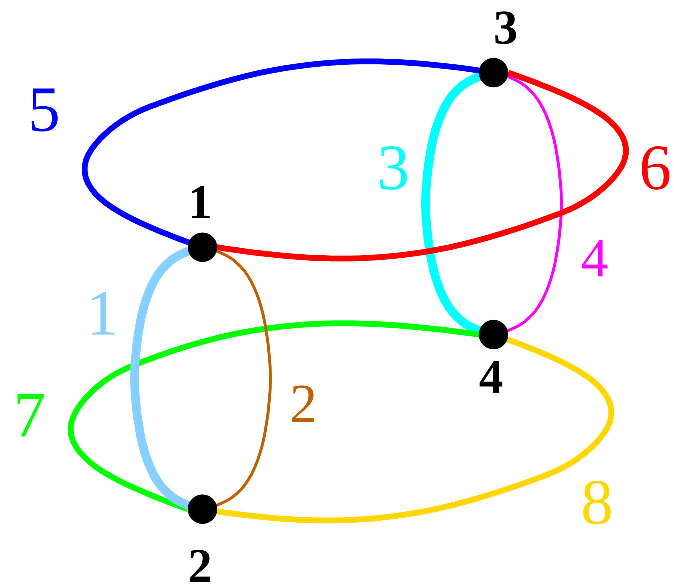
$$\theta_{ij}(\epsilon, \varphi) \bullet \begin{pmatrix} \epsilon \\ \varphi \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

generalized transvection



3
5
3
6
1
4
1
4
7
2
8
2

$$G = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$



face 1: edges 1, 7, −4, −5, face 2: edges 2, 7, −3, −5,
face 3: edges 1, 8, −4, −6, face 4: edges 2, 8, −3, −6 .

$$R = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

$$Q_1 \bullet R^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_1^{\mathrm{t}}$$

$$Q_2 \bullet R_1^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_2^{\mathrm{t}}$$

$$Q_3 \bullet R_2^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_3^{\mathrm{t}}$$

$$Q_4 \bullet R_3^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_4^{\mathrm{t}}$$

$$Q_5 \bullet R_4^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_5^{\mathrm{t}}$$

$$Q_6 \bullet R_5^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_6^{\mathrm{t}}$$

$$Q_7 \bullet R_6^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_7^{\mathrm{t}}$$

$$Q_8 \bullet R_7^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_8^{\mathrm{t}}$$

$$Q_9 \bullet R_8^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_9^{\mathrm{t}}$$

$$Q_{10} \bullet R_9^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv U$$

$$Q \equiv Q_{10} \bullet Q_9 \bullet Q_8 \bullet Q_7 \bullet Q_6 \bullet Q_5 \bullet Q_4 \bullet Q_3 \bullet Q_2 \bullet Q_1$$

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & -0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

$$G^{\mathrm{t}}\, R^{\mathrm{t}} = G^{\mathrm{t}} \bullet Q^{-1} \bullet U \equiv G_1^{\mathrm{t}} \bullet U$$

$$G_1^{\mathrm{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{pmatrix}$$

$$G_1^{\mathrm{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{pmatrix}$$

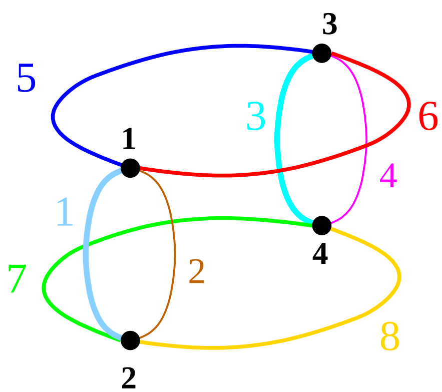
Find the other vectors in $\ker G_1^{\mathrm{t}}$ that are **not** among the three firsts:

$$c_1 = (0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0)^{\mathrm{t}}$$

and $c_2 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1)^{\mathrm{t}}$, clear for this case ...

Then $\nu_j = Q^{-1} c_j$ satisfy $G^{\mathrm{t}} \bullet \nu_j = 0$

and are **not** is the range of R^{t}.



$$\nu_1 = (0 \quad 0 \quad -1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)^{\mathrm{t}}$$

$$\nu_2 = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1)^{\mathrm{t}}$$

could be worse ...

$\Delta = Q \bullet R^{\mathrm{t}} \bullet P$

Q invertible over $\mathbb{Z}$ and $\det Q = 1$
Δ diagonal with integer coefficients
P invertible over $\mathbb{Z}$ and $\det P = 1$

Act on lines by left multiplication
and on columns by right multiplication.

For the previous example of “mini-torus”:

$$R^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$Q_1 \bullet R^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_1^{\mathrm{t}}$$

$$R_1^{\mathrm{t}} \bullet P_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_2^{\mathrm{t}}$$

$$Q_2 \bullet R_2^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_3^{\mathrm{t}}$$

$$R_3^{\mathrm{t}} \bullet P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \equiv R_4^{\mathrm{t}}$$

$$Q_3 \bullet R_4^{\mathrm{t}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv R_5^{\mathrm{t}}$$

$$R_5^{\mathrm{t}} \bullet P_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \equiv \Delta$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \bullet \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\equiv Q \bullet R^{\mathrm{t}} \bullet P = \Delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then $G^{\mathrm{t}} \bullet R^{\mathrm{t}} = G^{\mathrm{t}} \bullet Q^{-1} \bullet \Delta \bullet P^{-1}$

$$G^{\mathrm{t}} \bullet Q^{-1} \equiv G_1^{\mathrm{t}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{pmatrix}$$

Make a "Smith ascent" instead of a (classical!) "Smith descent"
in order to put the diagonal bloc at bottom right

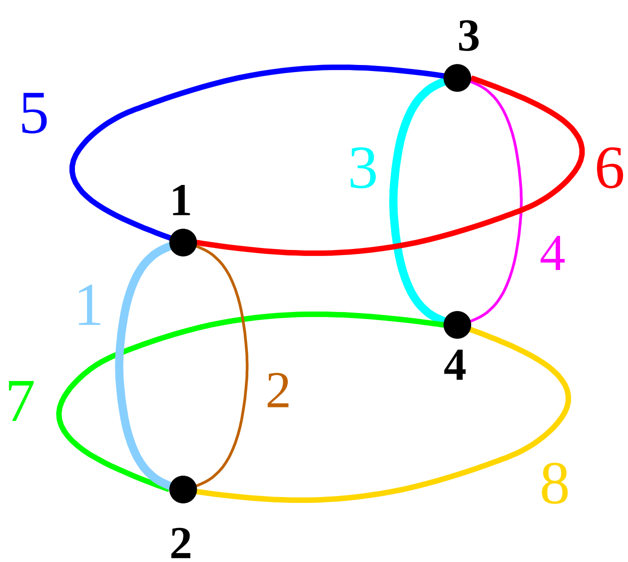
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \bullet G_1^{\mathrm{t}} \bullet \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix} \equiv S \bullet G_1^{\mathrm{t}} \bullet T = V$$

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The columns with label 4 and 5

correspond to a basis of the space $H_1(\mathcal{T})$

we have $G^{\mathrm{t}} \bullet R^{\mathrm{t}} = S^{-1} \bullet V \bullet T^{-1} \bullet \Delta \bullet P^{-1}$, then



$$\nu_1 = Q^{-1} \bullet T \bullet e_4$$
$$= (0 \quad 0 \quad -1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0)^{\mathrm{t}}$$
$$\nu_2 = Q^{-1} \bullet T \bullet e_5$$
$$= (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 1)^{\mathrm{t}}$$

much easier to determine!

Importance of the notion of simplicial complex to understand
what is inside the notion of "mesh" from a topological view point.

Fundamental link between topological objects
vertices $\mathcal{T}^0$, edges $\mathcal{T}^1$, faces $\mathcal{T}^2$, tetrahedra $\mathcal{T}^3$
including the associated incidence matrices
and the discretization of vector and scalar fields

"$Q\,R$" type factorization of integer matrices
to compute the first homology group of a simplicial mesh $\mathcal{T}$

Other approaches for big matrices: see J.G. Dumas (Grenoble).

Question: why when computing the Smith decomposition
of an incidence matrix, all terms in the diagonal are equal to 1 or 0 ?
Due to orientation of the mesh ?