INVOLUTIVITY AND (NUMERICAL SOLUTION OF) OVERDETERMINED PDEs

Jukka Tuomela

University of Joensuu
Finland

joint work with

- Katya Krupchyk (Joensuu, Finlande) and
- Bijan Mohammadi (Montpellier)

K. Krupchyk, J. Tuomela :
Completion of overdetermined parabolic PDEs,
to appear in J. Symb. Comp.

B. Mohammadi, J. Tuomela, *Involutive Methods for Navier-Stokes Equations* ,
to appear in Numerical Analysis and Scientific Computing for PDEs and their Challenging Applications.

1. GENERAL SYSTEMS $\longrightarrow$ INVOLUTIVE SYSTEMS
   - OVERDETERMINED ?
   - FORMAL THEORY OF PDEs
   - WHAT ARE PDEs, REALLY ?
2. ELLIPTIC SYSTEMS
   - MODULES AND SYMBOLS
   - A NEW KIND OF COMPLETION
3. NUMERICAL ISSUES
   - AUGMENTED SYSTEM
   - STOKES AND INF SUP CONDITION

GENERAL $\longrightarrow$ INVOLUTIVE

M. Janet : Leçons sur les systèmes d'équations aux dérivées partielles, Gauthier-Villars, 1929.

J.-F. Pommaret : Systems of Partial Differential Equations and Lie Pseudogroups, Gordon & Breach, 1978.

W. Seiler : Involution — The Formal Theory of Differential Equations and its Applications in Computer Algebra and Numerical Analysis, Habilitation thesis, Universität Mannheim, 2001.

What is in fact over/underdetermined ?

EXAMPLE

$$\nabla \times y = f$$

– square (determined ?) system

– underdetermined : y solution $\quad \Rightarrow \quad y + \nabla g$ also solution (infinite dimensional kernel)

– overdetermined : compatibility condition $\quad \nabla \cdot f = 0$ (infinite dimensional cokernel)

C. Riquier, *Les systèmes d'EDP*, 1910 :

En 1892, je réussis à opérer la réduction d'un système quelconque à une forme complètement intégrable.

A. Tresse, Acta Math., 1894 :

Etant donné un système quelconque d'équations aux dérivées partielles, on peut, après un nombre limité de différentiations et d'éliminations, ou bien montrer qu'il est incompatible, ou bien le mettre sous forme d'un système complètement intégrable.

So in the study of general systems of PDEs :

(1) find the canonical/involutive/complètement intégrable form of the system

(2) then study the existence of the solution in a suitable sense

According to Spencer, the first step is the

formal theory of PDEs

D. Spencer, *Overdetermined systems of linear partial differential equations*, Bull. Am. Math. Soc, vol. 75, (1969), 179–239.

other approaches :

– differential algebra (Ritt, Kolchin)
– exterior differential systems (É. Cartan)

Lead to similar completion procedures.

However, formal theory most convenient when studying boundary value problems.

PDEs are submanifolds of jet spaces

Notations

independent variables : $x = (x_1, \ldots, x_n)$

dependent variables : $y = (y^1, \ldots, y^m)$

derivatives :

$$\frac{\partial^{|\mu|} y}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}} = \frac{\partial^{|\mu|} y}{\partial x^\mu} = \partial^\mu y = y_\mu$$

Let $\Omega \subset \mathbb{R}^n$, $\mathcal{E} = \Omega \times \mathbb{R}^m$.

Then qth order jet space/bundle is :

$$J_q(\mathcal{E}) \simeq \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn_1} \times \cdots \times \mathbb{R}^{mn_q}$$

$\dim\big(J_q(\mathcal{E})\big) = n + md_q$ where

$$n_q = \binom{n+q-1}{q} \qquad \text{and} \qquad d_q = \binom{n+q}{q}$$

$$\begin{tikzcd}
J_q(\mathcal{E}) \arrow[r, "\pi^q_{q-1}"] \arrow[rrrr, bend right, "\pi^q"'] & J_{q-1}(\mathcal{E}) \arrow[r] & \cdots \arrow[r] & J_1(\mathcal{E}) \arrow[r, "\pi^1_0"] & J_0(\mathcal{E}) \simeq \mathcal{E} \arrow[d] \\
 & & & & \Omega
\end{tikzcd}$$

projection π_q^{q+r} is a “forgetful” map :
we “forget” the derivatives of order $q+1\ ,\dots,q+r$.

canonical form ?

Let

$$\mathcal{R}_q \subset J_q(\mathcal{E}) \quad : \quad f(x, y, \ldots, y_{\mu^i}, \ldots) = 0$$

prolongation/differentiation

$$\mathcal{R}_{q+1} \subset J_{q+1}(\mathcal{E}) \quad : \quad \begin{cases} \frac{\partial}{\partial x_1} f(x, y, \ldots) = 0 \\ \qquad \vdots \\ \frac{\partial}{\partial x_n} f(x, y, \ldots) = 0 \\ f(x, y, \ldots) = 0 \end{cases}$$

Prolongation : algebraically easy, geometrically difficult.

projection/elimination :

$$\pi_q^{q+r} \; : \; J_{q+r}(\mathcal{E}) \rightarrow J_q(\mathcal{E}) \quad \Rightarrow \quad \text{by restriction :}$$

$$\pi_q^{q+r} \; : \; \mathcal{R}_{q+r} \rightarrow \mathcal{R}_q \quad , \quad \mathcal{R}_q^{(r)} := \pi_q^{q+r}\big(\mathcal{R}_{q+r}\big)$$

Projection : geometrically easy, algebraically difficult.

If $\mathcal{R}_q^{(r)} \neq \mathcal{R}_q$, we have found integrability conditions : new algebraically independent equations.

EXAMPLE

$$\mathcal{R}_1 \quad : \quad \nabla \times y + y = 0$$

$$\mathcal{R}_1^{(1)} \quad : \quad \begin{cases} \nabla \times y + y = 0 \\ \nabla \cdot y = 0 \end{cases}$$

Involutive :

all integrability conditions have been found
(+ some technical conditions)

Theorem

(under some reasonable hypothesis)
the involutive form can always be constructed.

So Riquier, Tresse and others were right !

PACKAGES :

DETools in MuPAD (formal theory)

by Seiler et al.

diffalg and rif in maple (differential algebra)

by Mansfield, Reid, Wittkopf, Hubert et al.

Elliptic systems

Let a_μ be $k \times m$ matrices, $k \geq m$.

$$Ay = \sum_{|\mu| \leq q} a_\mu(x) \partial^\mu y = f$$

(principal) *symbol* of A :

$$\sigma A = \sum_{|\mu| = q} a_\mu(x) \xi^\mu$$

A is *elliptic*, if σA is injective for all $\xi \neq 0$.

EXAMPLE

$$Ay = \nabla \times y + y = 0 \qquad \sigma A = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

A is not elliptic.

$$A'y = \begin{cases} \nabla \times y + y = 0 \\ \nabla \cdot y = 0 \end{cases} \qquad \sigma A' = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \\ \xi_1 & \xi_2 & \xi_3 \end{pmatrix}$$

A' is elliptic and involutive.

Algebraically, it is convenient to regard the symbol in 2 different ways.

Let $\mathbb{A} = \mathbb{K}[\xi_1, \ldots, \xi_n]$; then the symbol σA is
– a homomorphism of modules :

$$\sigma A \,:\, \mathbb{A}^m \to \mathbb{A}^k$$

– and the submodule generated by rows :

$$\sigma A \subset \mathbb{A}^m$$

Example

$$\sigma A = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} \subset \mathbb{A}^3$$

Note that $\det(\sigma A) = 0 \quad \Longrightarrow \quad \mathrm{rank}(\sigma A) = 2\,;$
however, the module cannot be generated by 2 elements.

This is algebraic way of seeing that operator $\nabla\times$ is “bad”.

Generalisation due to Douglis & Nirenberg (1955). However :

Theorem (Krupchyk, Seiler, Tuomela)

DN–elliptic systems become elliptic when completed.

– there are systems which are not even DN–elliptic, but become elliptic when completed
– similar result for parabolic systems (Krupchyk, Tuomela)

Consider the symbol :

$$\sigma A \,:\, \mathbb{A}^m \to \mathbb{A}^k$$

There is an exact complex

$$0 \longrightarrow \mathbb{A}^{k_r} \xrightarrow{S_r} \cdots \xrightarrow{S_2} \mathbb{A}^{k_1} \xrightarrow{S_1} \mathbb{A}^k \xrightarrow{(\sigma A)^T} \mathbb{A}^m$$

free resolution of $(\sigma A)^T$

idea of construction

- compute S_1 of $(\sigma A)^T$
- consider the operator $A^{(1)} = (A, \hat{S}_1^T A)$
- in this way the symbol is “filled” until it becomes elliptic/parabolic
- if necessary the system can further be completed to involutive form (ellipticity is preserved)

S_1 can actually be computed using Gröbner bases.

INDEPENDENT COMPLETION PROCEDURE ADAPTED TO ELLIPTIC/PARABOLIC PROBLEMS

(not equivalent to formal theory, differential algebra, EDS)

EXAMPLE

$$\begin{cases} -y^1_{20} + y^1_{10} + y^2 = 0, \\ -y^1_{02} - y^2 = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} -y^1_{20} + y^1_{10} + y^2 = 0, \\ -y^1_{02} - y^2 = 0 \\ y^1_{12} + \Delta y^2 = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} -y^1_{20} + y^1_{10} + y^2 = 0, \\ -y^1_{02} - y^2 = 0 \\ y^1_{12} + \Delta y^2 = 0 \\ -\Delta y^2 + y^2_{10} = 0 \\ y^1_{12} + y^2_{30} + y^2_{12} + y^2_{02} = 0 \end{cases}$$

$$\Downarrow$$

$$Ay = \begin{cases} -y^1_{20} + y^1_{10} + y^2 = 0, \\ -y^1_{02} - y^2 = 0 \\ -\Delta y^2 + y^2_{10} = 0 \end{cases} \qquad \sigma A = \begin{pmatrix} \xi_1^2 & 0 \\ \xi_2^2 & 0 \\ 0 & |\xi|^2 \end{pmatrix}$$

NUMERICAL ISSUES

Involutive/complete systems usually have more equations than unknowns.
How to handle them numerically ?

Consider an elliptic problem

$$A_0 y = f$$

with appropriate boundary conditions $B_0 y = g$.

one possibility

Now let us suppose that we have an exact (or Fredholm) complex :

$$0 \longrightarrow V_0 \xrightarrow{A_0} V_1 \xrightarrow{A_1} V_2 \longrightarrow 0$$

A_1 is the *compatibility operator* for A_0.

This suggests that we can decompose V_1 as follows :

$$\text{image}\big(A_0\big) \oplus \text{image}\big(A_1^T\big) \simeq V_1$$

hence we define the augmented system

$$A_0 y + A_1^T z = f$$

– for reasonable spaces the operator (A_0, A_1^T) should be bijective (or Fredholm).
– z is artificial variable ; however, it can be useful in error control.
– square system $\rightarrow$ standard software available
– augmented system is also elliptic.

EXAMPLE

Stokes system

$$A_0 y = \begin{cases} u_t - \Delta u + \nabla p = 0 \\ \nabla \cdot u = 0 \\ -\Delta p = 0 \end{cases}$$

– equation $\Delta p = 0$ often not written explicitly

– often numerical codes use it anyway.

compatibility operator :

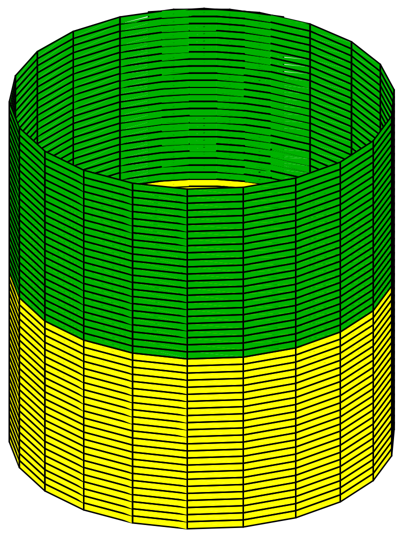
$$A_1 = \Big(\nabla\cdot\ ,\ 1\ ,\ \partial_t - \Delta\Big)$$

augmented system

$$A_0 y + A_1^T z = \begin{cases} u_t - \Delta u + \nabla p - \nabla z = 0 \\ -\Delta p + z = 0 \\ z_t - \nabla \cdot u - \Delta z = 0 \end{cases}$$

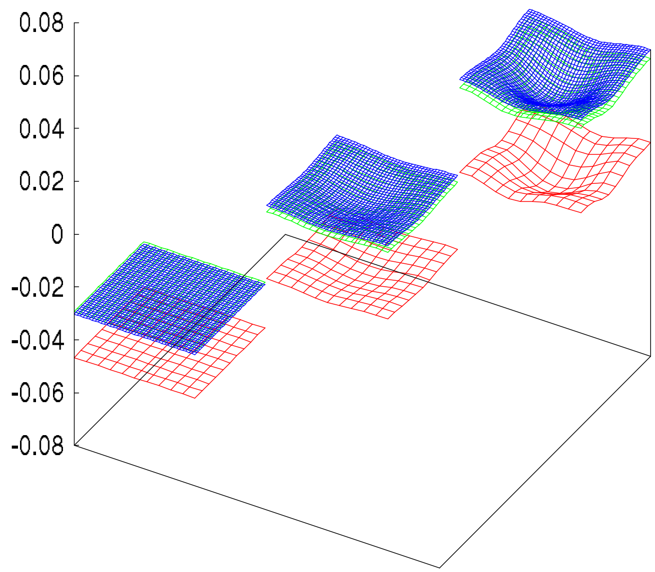
this is of parabolic-elliptic type
((u, z) parabolic variables, p elliptic variable)

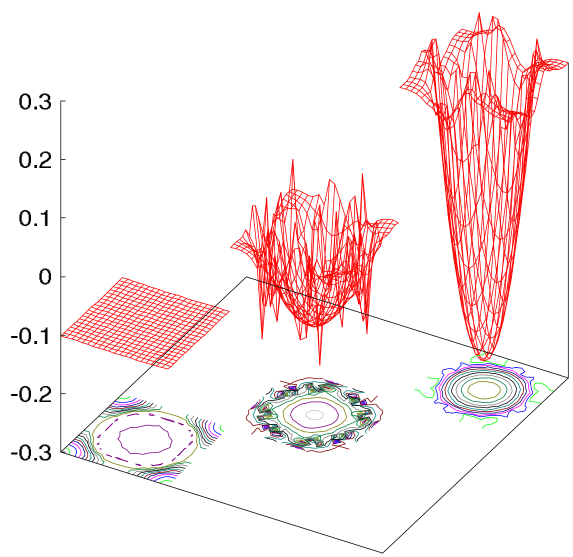
Cylinder with rotating upper part.

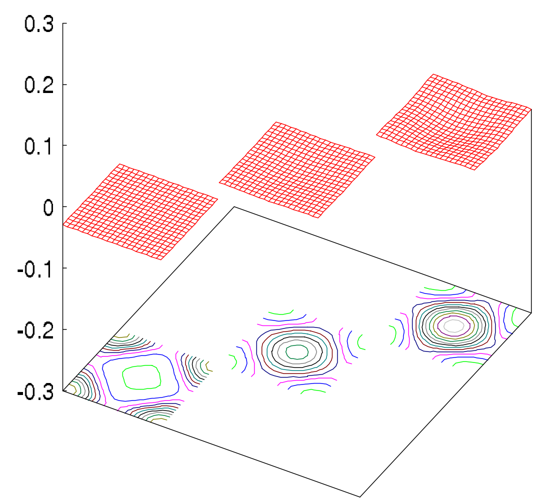


One can compute the exact (stationary) solution ;
in fact the pressure is constant.

Discontinuity in boundary conditions $\rightarrow$
numerical difficulties.


0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08


0.3
0.2
0.1
0
-0.1
-0.2
-0.3


0.3
0.2
0.1
0
-0.1
-0.2
-0.3

no inf-sup condition

In our formulation one can use P_1 – elements both for velocity and pressure.

In fact almost any choice of elements is fine.

Apparently this happens in general :
– DN–elliptic problems require special finite element spaces
– for elliptic problems generic methods suffice.

Perspectives & problems

– extension to hyperbolic problems
– currently studying microfluidic system :
  Stokes $+$ electric field $+$ several charged species ;
  overdetermined because of charge neutrality ;
  also of parabolic-elliptic type
– other ways of treating nonsquare systems ?
  (least squares usually badly conditioned)
– boundary conditions...
– formal theory & time dependent problems ?