New constructions of perfectly matched layers for the linearized Euler equations

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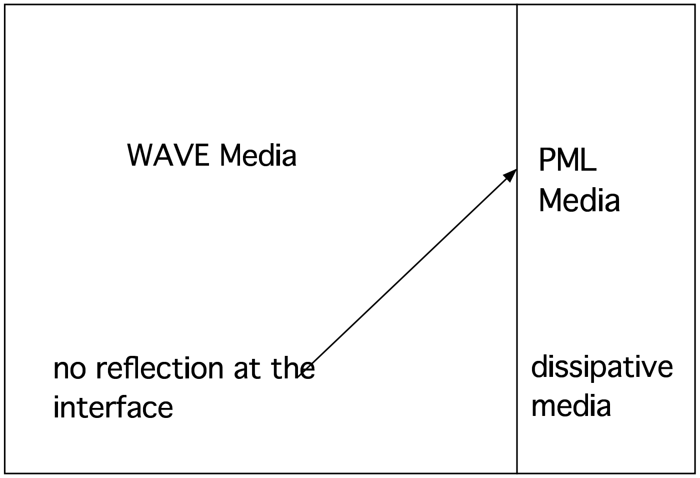
Outline of the talk

1. Introduction
2. The Smith factorization
3. Perfectly matched layers for the compressible Euler equations (JCP, 2006)
4. Conclusion and perspectives

Perfectly Matched Layers (Berenger, 94)

The PML layer

- is dissipative
- creates no reflection at the interface with the "wave" media



For the wave equation, the PML media is defined by a change of variable in the complex plane so that:

$$\mathcal{L}_{pml} = \partial_{tt} - c^2\partial_{yy} - c^2(\partial_x^{pml})^2$$

where

$$\partial_x^{pml} := \frac{i\omega}{i\omega + c\sigma}\partial_x$$

and σ is a damping parameter and ω is the Fourier variable for the Fourier transform in time. The operator has a simple implementation: $\partial_x^{pml}(u) = \phi$ where $\phi_t + c\sigma\phi = u_{tx}$

Perfectly Matched Layers for the compressible Euler equations

Linearized Euler equations around a constant state:

$$\begin{pmatrix} \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & \bar{\rho}\bar{c}^2\partial_x & \bar{\rho}\bar{c}^2\partial_y \\ \frac{1}{\bar{\rho}}\partial_x & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & 0 \\ \frac{1}{\bar{\rho}}\partial_y & 0 & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix} = \begin{pmatrix} f_p \\ f_u \\ f_v \end{pmatrix}$$

Hu (1996, 2001), Hestaven (1998), Tam, Auriault & Cambuli (1998), Hagstrom & Nazarov (2002), Rahmouni (2004).

Challenge: Stable and non reflective PML for the Euler system. Hu (2001) $->$ Flow normal to the interface

Goal: Stable PML for oblique (as well as normal) flow see also Hagstrom (2006)

Difficulties

Manipulations on the equations reveal two scalar operators:

- The advective wave operator

  $$\mathcal{L} = \partial_{tt} + 2\bar{u}\bar{v}\partial_{xy} + 2\partial_t(\bar{u}\partial_x + \bar{v}\partial_y) - (\bar{c}^2 - \bar{v}^2)\partial_{yy} - (\bar{c}^2 - \bar{u}^2)\partial_{xx}$$

  The pressure p satisfies $\mathcal{L}(p) = \ldots$. Related solutions to Euler are called "pressure waves"

- The first order transport operator

  $$\mathcal{G} = \partial_t + \bar{u}\partial_x + \bar{v}\partial_y$$

  The vorticity ω satisfies $\mathcal{G}(\omega) = \ldots$. Related solutions are called "vorticity waves".

Difficulties

$$\begin{pmatrix} \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & \bar{\rho}\bar{c}^2\partial_x & \bar{\rho}\bar{c}^2\partial_y \\ \frac{1}{\bar{\rho}}\partial_x & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & 0 \\ \frac{1}{\bar{\rho}}\partial_y & 0 & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix} = \begin{pmatrix} f_p \\ f_u \\ f_v \end{pmatrix}$$

A proper change of variable w.r.t. the advective wave equation ($\mathcal{L}$)

$$\partial_x \longrightarrow \partial_x^{pml}$$

might destabilize the first order transport operator $\mathcal{G}$. Moreover, the operator $\mathcal{G}$ does not need to be "PMLized".

STRATEGY Modify in the Euler system only the ∂_x that correspond to the operator $\mathcal{L}$.
At first glance, this seems impossible.

Tool: The Smith factorization makes it possible

The Smith Factorization, (Smith, $\simeq$1860)

Polynomial version

Theorem 1 *Let n be an integer and A an invertible $n \times n$ matrix with polynomial entries in one variable λ : $A = (a_{ij}(\lambda))_{1 \leq i,j \leq n}$. Then, there exist three matrices with polynomial entries E, D and F with the following properties:*

- *det(E) and det(F) are constant polynomials.*
- *D is a diagonal matrix.*
- *$A = EDF$.*

Morevoer, D is uniquely defined up to a reordering and multiplication of each entry by a constant.

Suppose $A = (a_{ij}(\partial_x))_{1 \leq i,j \leq n}$ is 1D system of PDEs. Solving $A(U) = G$ amounts to solving decoupled scalar equations:

$$D(V) = E^{-1}G \quad \text{and} \quad U = F^{-1}(V)$$

Computing a Smith factorization

The diagonal matrix D is given by a formula defined as follows. Let $1 \leq k \leq n$,

- S_k is the set of all the submatrices of order $k \times k$ extracted from A.
- $Det_k = \{\mathrm{Det}(B_k) \backslash B_k \in S_k\}$
- LD_k is the greatest common divisor of the set of polynomials Det_k.

Then,

$$D_{kk}(\lambda) = \frac{LD_k(\lambda)}{LD_{k-1}(\lambda)}, 1 \leq k \leq n$$

(by convention, $LD_0 = 1$).

- The factorization can be computed by "hand" easily in a similar fashion to a Gauss factorization
- There is a Maple routine called `smith`

Example: Stokes and the stream function formulation

Consider the 2D Stokes system:

$$\begin{pmatrix} -\nu\Delta & 0 & \partial_x \\ 0 & -\nu\Delta & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} f_u \\ f_v \\ 0 \end{pmatrix}.$$

We particularize the x direction since for the PML (compressible Euler) we shall truncate in the x direction.

We perform the factorization of the Stokes system ($\mathcal{A}_{Stokes}$) by considering it as a matrix with polynomial in ∂_x entries. The coefficients of the polynomes are pseudo-differential operators in the y direction.

Stream function formulations via Smith

Or, we can take the Fourier transform in the other variables

$$\hat{\mathcal{A}}_{Stokes} := \begin{pmatrix} -\nu(\partial_{xx} - k^2) & 0 & \partial_x \\ 0 & -\nu(\partial_{xx} - k^2) & ik \\ \partial_x & ik & 0 \end{pmatrix}.$$

and apply the Smith factorization to the above matrix.

Stream function formulations via Smith

We have

$$\mathcal{A}_{Stokes} = EDF \quad (1)$$

where

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\nu\triangle^2 \end{pmatrix}$$

One should note that a stream function formulation gives the same differential equation for the stream function.

In the same way, the three-dimensional case can be characterized.

In this case, the diagonal matrix D_{3D} is a four by four matrix whose entries are: $D_{3D,11} = D_{3D,22} = 1$, $D_{3D,33} = -\nu\Delta$ and $D_{3D,44} = -\nu\Delta^2$. We have two scalar equations.

In some sense, this is consistent with the well-known fact that in 2D scalar stream function formulations are possible but not in 3D where they have to be vectorial.

Modes analysis

We look for non trivial solutions of

$$A(U) = 0$$

in the form $U = W\, exp(\lambda(k)x + iky)$.
Let $A = EDF$ and $V = F(U)$ and suppose $D_{11} = 1$ and $D_{22} = D_{33} = -\Delta$. Then,

$$U = F^{-1} \begin{pmatrix} 0 \\ e^{\pm|k|x+iky} \\ 0 \end{pmatrix} \quad \text{or} \quad U = F^{-1} \begin{pmatrix} 0 \\ 0 \\ e^{\pm|k|x+iky} \end{pmatrix}$$

- No need to diagonalize a matrix
- Reveals a structure in the modes

Application to the compressible Euler equations

Linearized Euler equations around a constant state:

$$\begin{pmatrix} \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & \bar{\rho}\bar{c}^2\partial_x & \bar{\rho}\bar{c}^2\partial_y \\ \frac{1}{\bar{\rho}}\partial_x & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y & 0 \\ \frac{1}{\bar{\rho}}\partial_y & 0 & \partial_t + \bar{u}\partial_x + \bar{v}\partial_y \end{pmatrix} \begin{pmatrix} p \\ u \\ v \end{pmatrix} = \begin{pmatrix} f_p \\ f_u \\ f_v \end{pmatrix}$$

The Smith factorization yields: $D_{11} = D_{22} = 1$ and $D_{33} = \mathcal{L}\mathcal{G}$ with

$$\mathcal{G} = \partial_t + \bar{u}\partial_x + \bar{v}\partial_y$$

is a first order transport operator and

$$\mathcal{L} = \partial_{tt} + 2\bar{u}\bar{v}\partial_{xy} + 2\partial_t(\bar{u}\partial_x + \bar{v}\partial_y) - (\bar{c}^2 - \bar{v}^2)\partial_{yy} - (\bar{c}^2 - \bar{u}^2)\partial_{xx}$$

is the advective wave operator.

The rationale for the PML is that only the advective wave operator needs a “PML” procedure.

PML for the advective wave operator

σ is a damping parameter

(Re)call: For the wave equation,

$$\partial_x \qquad \longrightarrow \qquad \partial_x^{pml} := \frac{i\omega}{i\omega + \sigma}\partial_x$$

Following Dubois-Duceau-Maréchal-Terrasse (2000), Bécache -Bonnet-Ben Dhia -Legendre (2004), Hu (2001) and Hagstrom-Nazarov (2002), we write for the advective wave equation

$$\mathcal{L}_{pml} = \partial_{tt} + 2\bar{u}\bar{v}\partial_y(\partial_x^{pml}) + 2\partial_t(\bar{u}\partial_x^{pml} + \bar{v}\partial_y) - (\bar{c}^2 - \bar{v}^2)\partial_{yy} - (\bar{c}^2 - \bar{u}^2)(\partial_x^{pml})^2$$

where

$$\partial_x^{pml} := \alpha(x)[\partial_x - \frac{\bar{u}}{\bar{c}^2 - \bar{u}^2}(\partial_t + \bar{v}\partial_y)] + \frac{\bar{u}}{\bar{c}^2 - \bar{u}^2}(\partial_t + \bar{v}\partial_y)$$

where the operator $\alpha(x)$ is the operator:

$$\alpha(x)(\phi) = \mathcal{F}^{-1}\left(\frac{\bar{c}(i\omega + ik\bar{v})}{\bar{c}(i\omega + ik\bar{v}) + (\bar{c}^2 - \bar{u}^2)\sigma(\omega, x, k)}\ \hat{\hat{\phi}}\right)$$

k is the variable for the Fourier transform in the y direction.

A first PML for the Euler equation

Substitute $\mathcal{L}$ with $\mathcal{L}^{pml}$ in matrix D. In matrices E and F and in the operator $\mathcal{G}$, the x derivatives are not modified. Modifying only the advective wave operator avoids instability problems with the vorticity wave. We thus define:

$$A^{pml1}_{Euler} = ED^{pml}F \tag{2}$$

where

$$D^{pml} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathcal{G}\mathcal{L}_{pml} \end{pmatrix} \tag{3}$$

A first PML for the Euler equation

A direct computation yields:

$$A_{Euler}^{pml1} = A_{Euler} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_1 & C_2 & 0 \end{pmatrix} \qquad (4)$$

where

$$\hat{\hat{C}}_1 = \frac{(\partial_x - \partial_x^{pml})\hat{\hat{\mathcal{G}}}[(\bar{u}^2 - \bar{c}^2)(\partial_x + \partial_x^{pml}) + 2\bar{u}(i\omega + i\bar{v}k)]}{i\bar{\rho}\bar{c}^2 k(i\omega + ik\bar{v})} \text{ and } C_2 = \frac{C_1}{\bar{\rho}\bar{u}}$$

The difference with the Euler system concerns only the last equation on the variable v, but :

1. The formula is complex and involves third order derivatives on both the pressure p and the normal velocity u.
2. The formula implies a division by $i\bar{\rho}\bar{c}^2 k(i\omega + ik\bar{v})$ which can be zero. Possible cure: $\sigma(\omega, x, k) := \tilde{\sigma}(x)\left(\bar{\rho}\bar{c}^2 k(\omega + k\bar{v})\right)^2$

Conclusion on the first PML for the Euler equation

- Very complex
- No damping where $\sigma(\omega, x, k) = 0$

BUT deserves interest since:

- the procedure is general. Other systems of PDEs can be addressed:

  water free surface equations (St Venant equations)

  ...

- a simplification might be possible since E and F are not unique

A second PML for the Euler equation

The rationale for this model is that the pressure p satisfies an advective wave equation which is the only equation that demands a PML. Indeed, apply the matrix El to the Euler system:

$$El = \begin{pmatrix} \mathcal{G} & -\bar{\rho}\bar{c}^2\partial_x & -\bar{\rho}\bar{c}^2\partial_y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We get:

$$El\, A_{Euler} = \begin{pmatrix} \mathcal{L} & 0 & 0 \\ \frac{1}{\bar{\rho}}\partial_x & \mathcal{G} & 0 \\ \frac{1}{\bar{\rho}}\partial_y & 0 & \mathcal{G} \end{pmatrix} \tag{5}$$

A second PML for the Euler equation

We substitute $\mathcal{L}$ with $\mathcal{L}^{pml}$ and apply

$$El^{-1} = \begin{pmatrix} \mathcal{G}^{-1} & -\bar{\rho}\bar{c}^2\partial_x\mathcal{G}^{-1} & -\bar{\rho}\bar{c}^2\partial_y\mathcal{G}^{-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and we are thus led to define:

$$A_{Euler}^{pml2} = \begin{pmatrix} \mathcal{G}^{-1}(\mathcal{L}^{pml} + \bar{c}^2(\partial_{xx} + \partial_{yy})) & \bar{\rho}\bar{c}^2\partial_x & \bar{\rho}\bar{c}^2\partial_y \\ \frac{1}{\bar{\rho}}\partial_x & \mathcal{G} & 0 \\ \frac{1}{\bar{\rho}}\partial_y & 0 & \mathcal{G} \end{pmatrix}$$

A direct computation yields:

$$A_{Euler}^{pml2} = A_{Euler} + \begin{pmatrix} (\mathcal{L}^{pml} - \mathcal{L})\mathcal{G}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A second PML for the Euler equation

In order to get rid of the operator $\mathcal{G}^{-1}$, we introduce a new variable $\mathcal{P}$ such that $\mathcal{G}(\mathcal{P}) = p$ so that the enlarged PML system we consider reads:

$$\mathcal{A}^{pml2}_{Euler}\begin{pmatrix}\mathcal{P}\\ p\\ u\\ v\end{pmatrix} = \begin{pmatrix}\mathcal{G} & -1 & 0 & 0\\ \mathcal{L}^{pml}-\mathcal{L} & \mathcal{G} & \bar{\rho}\bar{c}^2\partial_x & \bar{\rho}\bar{c}^2\partial_y\\ 0 & \dfrac{1}{\bar{\rho}}\partial_x & \mathcal{G} & 0\\ 0 & \dfrac{1}{\bar{\rho}}\partial_y & 0 & \mathcal{G}\end{pmatrix}\begin{pmatrix}\mathcal{P}\\ p\\ u\\ v\end{pmatrix} = 0$$

with the following interface conditions between the Euler media and the PML

$$\mathcal{P} = 0,\ p \text{ and } u \text{ are continuous}, \partial_x(p_{Euler}) = \partial_x^{pml}(p_{pml})$$

A plane-wave analysis shows that the PML is dissipative and that there is no reflection at the interface between the Euler media and the PML media.

Extensions to the three-dimensional case

The Smith form of the 3D compressible Euler equations is

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mathcal{G} & 0 \\ 0 & 0 & 0 & \mathcal{G}\mathcal{L} \end{pmatrix}$$

As in the 2D case, only the advective wave operator $\mathcal{L}$ needs a "pml" procedure. For instance, the second model reads:

$$\begin{pmatrix} \mathcal{G} & -1 & 0 & 0 & 0 \\ \mathcal{L}^{pml} - \mathcal{L} & \mathcal{G} & \bar{\rho}\bar{c}^2\partial_x & \bar{\rho}\bar{c}^2\partial_y & \bar{\rho}\bar{c}^2\partial_z \\ 0 & \frac{1}{\bar{\rho}}\partial_x & \mathcal{G} & 0 & 0 \\ 0 & \frac{1}{\bar{\rho}}\partial_y & 0 & \mathcal{G} & 0 \\ 0 & \frac{1}{\bar{\rho}}\partial_z & 0 & 0 & \mathcal{G} \end{pmatrix} \begin{pmatrix} \mathcal{P} \\ p \\ u \\ v \\ w \end{pmatrix} = 0$$

All models could be derived and used with variable coefficients.

Numerical results for the second PML

2D numerical results on a staggered grid with constant coefficients

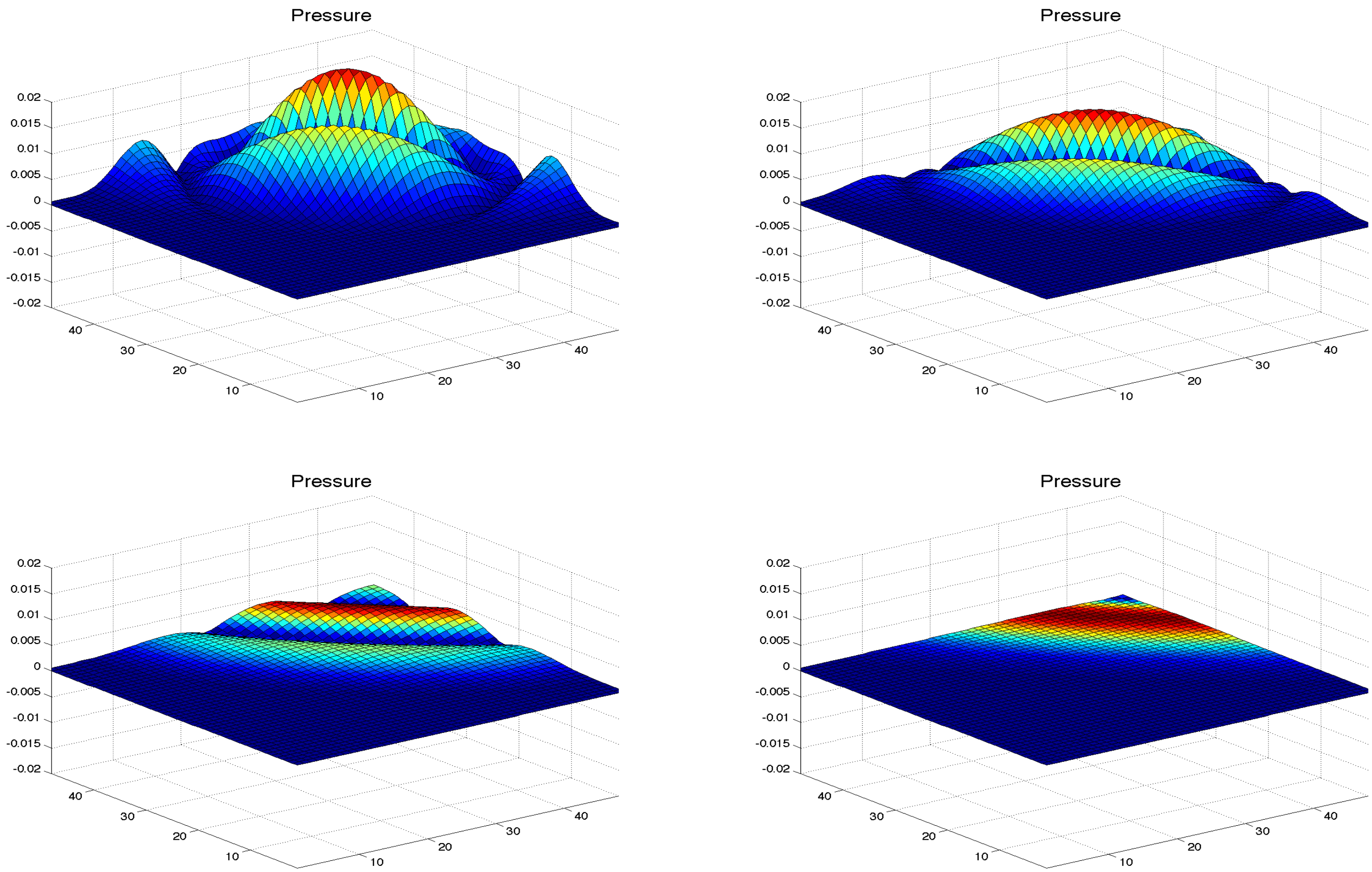


Figure 1: Pressure – oblique velocity $M = 0.9$ at successive times

Numerical results for the second PML

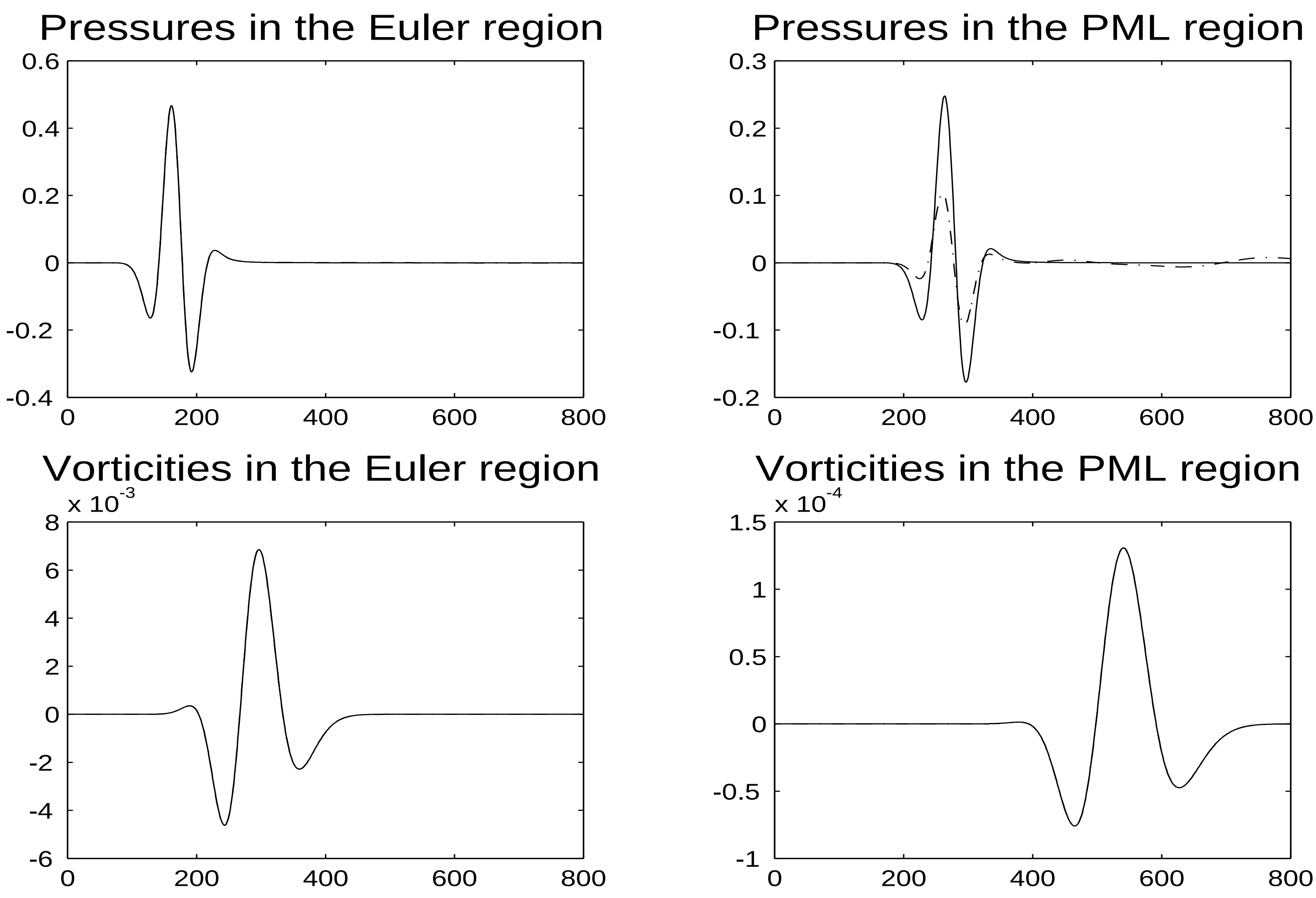


Figure 2: Reference and "PML" solutions in the Euler and PML regions vs. time steps ($\bar{u} = 200$, $\bar{v} = 100$)

Numerical results for the second PML

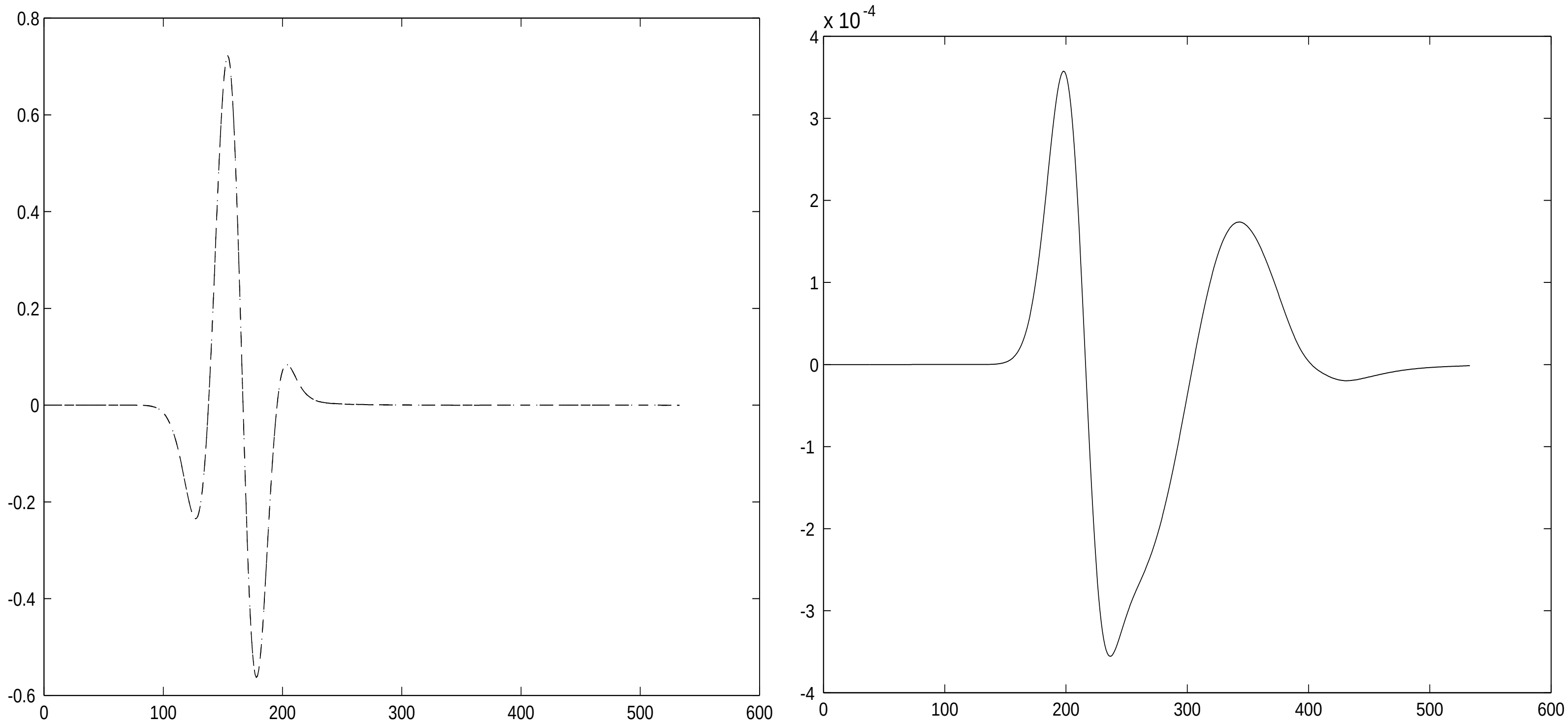


Figure 3: Pressure field (left) and error on the pressure (right) near the upperleft corner for a horizontal flow $M = 0.33$ vs. time steps

Stability was assessed by computing over time intervals much longer than those used to generate the figures.

Conclusion

The Smith factorization gives an insight in systems of PDEs.
It has been used for

- Desiging two PML models for the compressible Euler equations
- Designing new domain decomposition methods for systems of PDEs: Stokes, Oseen and compressible Euler (not shown here)

It was used by Wloka, Rowley and Lawruk (1995) for studying the regularity up to the boundary of partial differential systems with variable coefficients.

PMLs for other systems of PDE's could be addressed:

- Water free surface equations (St Venant equations)
- ...

Thanks !