From the Spectral Stokes solvers
· · · to the Stokes eigenmodes in square/cube,
· · · until new questions in Fluid Dynamics.

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Plan

Introduction

¿ Why to pay a particular attention
to the
Unsteady Stokes Problem (USP) ?

- NS spectral numerical solutions are in fact USP solutions
- even for DNS of turbulence (considered as reliable)
- necessity of consistent and cheap USP pseudo-spectral solvers

Continuous Unsteady Stokes Problem

Let $(\vec{\mathbf{v}},\ p)$ be solutions of

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} - \vec{\nabla}^2 \vec{\mathbf{v}} + \vec{\nabla} p = \vec{\mathbf{f}} \quad , \quad in\ \Omega \times t > 0,$$

$$\vec{\nabla} \cdot \vec{\mathbf{v}} = 0 \qquad\qquad , \quad in\ \bar{\Omega} = \Omega \cup \partial\Omega,$$

$$\vec{\mathbf{v}} = \vec{\mathbf{V}} \ \ (\mathrm{or}\ \frac{\partial \vec{\mathbf{v}}}{\partial n} = \cdots) \qquad\qquad , \quad on\ \partial\Omega,$$

equivalent to

$$(1) \quad \vec{\nabla}^2 p = \vec{\nabla} \cdot \vec{\mathbf{f}}, \tag{1}$$

$$(2) \quad \left(\frac{\partial}{\partial t} - \vec{\nabla}^2 \right) \vec{\nabla}^2 \vec{\mathbf{v}} = \vec{\nabla} \times \vec{\nabla} \times \vec{\mathbf{f}} \ , \ \vec{\nabla} \cdot \vec{\mathbf{v}} = 0. \tag{2}$$

time-discretized Stokes Problem (1)

Let us define $\bullet^{(n)} \equiv \bullet(t = n\,\delta t)$. The USP high-order ($J_i$) in time discretized formulation (KIO, JCP 1991) is

$$\frac{\gamma_0\,\vec{\mathbf{v}}^{(n+1)} - \sum_{q=0}^{J_i-1}\alpha_q\,\vec{\mathbf{v}}^{(n-q)}}{\delta t} - \vec{\nabla}^2\vec{\mathbf{v}}^{(n+1)} + \vec{\nabla}p = \vec{\mathbf{f}}^{(n+1)} \quad , \quad in\ \Omega,$$

$$\vec{\nabla}\cdot\vec{\mathbf{v}}^{(n+1)} = 0 \qquad , \quad in\ \bar{\Omega} = \Omega \cup \partial\Omega,$$

$$\vec{\mathbf{v}}^{(n+1)} = \vec{\mathbf{V}}^{(n+1)} \qquad , \quad on\ \partial\Omega,$$

γ_0, α_q given in Table I, p. 1390, of [E.Leriche and G.Labrosse, "High-order direct Stokes solvers with or without temporal splitting : numerical investigations of their comparative properties". **SIAM J. Scient. Computing**, 22(4) (2000), pp. 1386-1410].

time-discretized Stokes Problem (2)

$$\overbrace{\left(\frac{\gamma_0}{\delta t} - \vec{\nabla}^2\right)} \vec{\mathbf{v}}^{(n+1)} + \vec{\nabla} p = \overbrace{\vec{\mathbf{f}}^{(n+1)} + \frac{\sum_{q=0}^{J_i-1} \alpha_q \, \vec{\mathbf{v}}^{(n-q)}}{\delta t}} \quad , \quad in \ \Omega$$

$$\Downarrow$$

$$H \vec{\mathbf{v}} + \vec{\nabla} p = \vec{\mathbf{S}} \quad , \quad in \ \Omega, \tag{3}$$

$$\vec{\nabla} \cdot \vec{\mathbf{v}} = 0 \qquad , \quad in \ \bar{\Omega} = \Omega \cup \partial\Omega, \tag{4}$$

$$\vec{\mathbf{v}} = \vec{\mathbf{V}} \quad (\text{or} \ \ \frac{\partial \vec{\mathbf{v}}}{\partial n} = \cdots) \qquad , \quad on \ \partial\Omega. \tag{5}$$

¿ How to uncouple $\vec{\mathbf{v}}$ from p, and enforce (more or less) $\vec{\nabla} \cdot \vec{\mathbf{v}} = 0$?

Stokes Solvers Families and Properties

$(\vec{\mathbf{v}}, p)$ Uncoupling Option	Consistency	Cost	¿ $\vec{\nabla} \cdot \vec{\mathbf{v}} = 0$?
UZAWA ('68)	YES	EXP.	YES
-			
GREEN or Influence Matrix (Kleiser, Schumann,'80)	YES	EXP.	YES
-			
Time Splitting (Chorin, '68, Temam, '69)	NO	CHEAP	at spectral convergence
-			
Projection-Diffusion (PrDi) (Batoul et al., '95)	YES	CHEAP	at spectral convergence

CHEAP = POISSON + VECTORIAL HELMHOLTZ to be solved

SAME SPACE-TIME ACCURACY

UZAWA (1)

The system (3-5) is space discretized,

$$(H\,\vec{\mathbf{v}})_{Int} + \left(\vec{\mathbf{D}}\,p\right)_{Int} = \vec{\mathbf{S}}_{Int},$$

$$\vec{\mathbf{D}} \cdot \vec{\mathbf{v}} = 0,$$

$$\vec{\mathbf{v}} = \vec{\mathbf{V}} \quad (\text{or} \quad (\vec{\mathbf{D}} \cdot \vec{\mathbf{n}})\vec{\mathbf{v}} = \cdots) \quad \text{at the boundary nodes.}$$

with henceforth

- $\vec{\mathbf{v}} = (u, v, w)$ and p are column vectors of (velocity, pressure) nodal values,
- $\bullet_{Int}$, column vectors of internal nodal values of $\bullet$,
- H, Helmholtz discrete operator,
- $\vec{\mathbf{D}}$, $\vec{\nabla}$ discrete operator.

UZAWA (2)

Eliminating the boundary nodal $\vec{\mathbf{v}}$ values through the BC, the USP reads

$$\begin{pmatrix} \mathcal{H}_u \, u_{Int} \\ \mathcal{H}_v \, v_{Int} \\ \mathcal{H}_w \, w_{Int} \end{pmatrix} +\text{"}\ \vec{\mathbf{D}}\ \text{"}\ p =\text{"}\ \vec{\mathbf{S}}\text{"}_{Int}, \tag{6}$$

$$\vec{\mathbf{D}} \cdot \vec{\mathbf{v}} = 0, \tag{7}$$

- $(\mathcal{H}_u, \mathcal{H}_v, \mathcal{H}_w) \ \leftarrow \ H$, square matrix with the BC on $\vec{\mathbf{v}}$ plugged in,
- $\text{"}\vec{\mathbf{D}}\text{"} \ \leftarrow \ \vec{\mathbf{D}}$, rectangular matrix.

$(\mathcal{H}_u, \mathcal{H}_v, \mathcal{H}_w)$ are invertible, then ...

UZAWA (3)

$$\vec{\mathbf{v}}_{\color{red}{Int}} = -\vec{\mathcal{H}}^{-1}\cdot\left(''\,\vec{\mathbf{D}}\,''\;p + ''\;\vec{\mathbf{S}}''_{\color{red}{Int}}\right),$$

and completing $\vec{\mathbf{v}}_{\color{red}{Int}}$ with the $\vec{\mathbf{v}}$ boundary values leads to $\vec{\mathbf{v}}$ written in term of p, and, by (7), one gets a pressure equation to be solved …

Example with $\vec{\mathbf{v}}|_{\partial\Omega} = 0$:

$$\vec{\mathbf{D}}\cdot\vec{\mathbf{v}} =_{,,}\vec{\mathbf{D}}_{,,}\cdot\vec{\mathbf{v}}_{\color{red}{Int}},$$

$$\Rightarrow \underbrace{\color{red}{\left({}_{,,}\vec{\mathbf{D}}_{,,}\cdot\vec{\mathcal{H}}^{-1}\cdot''\;\vec{\mathbf{D}}\,''\right)}}\;p = -{}_{,,}\vec{\mathbf{D}}_{,,}\cdot\vec{\mathcal{H}}^{-1}\cdot''\;\vec{\mathbf{S}}''_{\color{red}{Int}}.$$

UZAWA operator, full 2D/3D matrix, with a kernel (spurious pressure modes), only iteratively solved …

GREEN, or Influence Matrix (1)

This method is based on

$$\left(\frac{\partial}{\partial t} - \vec{\nabla}^2\right) \vec{\mathbf{v}} + \vec{\nabla} p = \vec{\mathbf{f}}\ ,\ \ \vec{\nabla} \cdot \vec{\mathbf{v}} = 0,$$

$$\Downarrow$$

$$\left(\frac{\partial}{\partial t} - \vec{\nabla}^2\right) \left(\vec{\nabla} \cdot \vec{\mathbf{v}}\right) = 0,$$

$$\Rightarrow \text{if } \vec{\nabla} \cdot \vec{\mathbf{v}} = 0 \text{ at } t = 0 \ \underline{\text{and}} \text{ on } \partial\Omega,$$
$$\Rightarrow \text{then } \vec{\nabla} \cdot \vec{\mathbf{v}} = 0 \text{ everywhere, } \forall t > 0.$$

GREEN, or Influence Matrix (2)

Let us introduce

(a) N_b boundary nodes, $\underline{x}_i = (x_i, y_i, z_i)$, $i = 1, \cdots, N_b$;

(b) N_b pressure fields $q_i\left(\underline{x}\right)$ verifying

$$\vec{\nabla}^2 q_i\left(\underline{x}\right) = 0 \ , \ \ q_i\left(\underline{x}_j\right) = \delta_{ij}$$

(c) N_b fields $\vec{\mathbf{v}}_i\left(\underline{x}\right)$ verifying

$$\vec{\nabla}^2 \vec{\mathbf{v}}_i\left(\underline{x}\right) - \vec{\nabla}\, q_i\left(\underline{x}\right) = 0 \ , \ \ \vec{\mathbf{v}}_i\left(\underline{x}_j\right) = 0,$$

(d) the $N_b \times N_b$ matrix $\mathcal{D}^{-1}$, from

$$\mathcal{D}_{ji} = \left(\vec{\nabla} \cdot \vec{\mathbf{v}}_i\right)|_{\underline{x}_j}.$$

GREEN, or Influence Matrix (3)

Then, at each time step,

(1) Evaluate $\vec{\mathbf{v}}_0$ and p_0 such that

$$\vec{\nabla}^2 p_0\left(\underline{x}\right) = \vec{\nabla}\cdot\vec{\mathbf{f}}\ ,\ \ p_0\left(\underline{x}_j\right) = 0,$$

$$\left(\frac{\partial}{\partial t} - \vec{\nabla}^2\right)\vec{\mathbf{v}}_0\left(\underline{x}\right) + \vec{\nabla}\,p_0\left(\underline{x}\right) = \vec{\mathbf{f}}\ ,\ \ \vec{\mathbf{v}}_0\left(\underline{x}_j\right) = 0,$$

(2) Compute $Q_j = \left(\vec{\nabla}\cdot\vec{\mathbf{v}}_0\right)|_{\underline{x}_j}$ and $\alpha = -\mathcal{D}^{-1}\,Q$,

(3) the solution is

$$p = p_0 + \sum_{i=1}^{N_b}\alpha_i\,q_i\quad,\quad \vec{\mathbf{v}} = \vec{\mathbf{v}}_0 + \sum_{i=1}^{N_b}\alpha_i\,\vec{\mathbf{v}}_i$$

GREEN, or Influence Matrix (4)

LIMITATIONS of this method : $\mathcal{D}^{-1}$

- in 2D with $(L+1)(M+1)$ nodes,

$$N_b = 2(L+M-2)$$

- in 3D with $(L+1)(M+1)(N+1)$ nodes,

$$N_b = 2(1+LM+MN+NL)$$

... SO ...

Time-Splitting (1)

The initial problem

$$\frac{\gamma_0\,\vec{\mathbf{v}}^{(n+1)}-\sum_{q=0}^{J_i-1}\alpha_q\,\vec{\mathbf{v}}^{(n-q)}}{\delta t}-\vec{\nabla}^2\vec{\mathbf{v}}^{(n+1)}+\vec{\nabla}p=\vec{\mathbf{f}}^{(n+1)}\ ,\ \vec{\nabla}\cdot\vec{\mathbf{v}}^{(n+1)}=0,$$

$$\vec{\mathbf{v}}^{(n+1)}\big|_{\ \partial\Omega}=\vec{\mathbf{V}}^{(n+1)},$$

is replaced by

$$(1)\quad\frac{\hat{\mathbf{v}}-\sum_{q=0}^{J_i-1}\alpha_q\,\vec{\mathbf{v}}^{(n-q)}}{\delta t}+\vec{\nabla}p=\vec{\mathbf{f}}^{(n+1)}\ ,\ \vec{\nabla}\cdot\hat{\mathbf{v}}=0,$$

$$(2)\quad\frac{\gamma_0\,\vec{\mathbf{v}}^{(n+1)}-\hat{\mathbf{v}}}{\delta t}=\vec{\nabla}^2\vec{\mathbf{v}}^{(n+1)}\quad,\quad\vec{\mathbf{v}}^{(n+1)}\big|_{\ \partial\Omega}=\vec{\mathbf{V}}^{(n+1)}.$$

Time-Splitting (2)

This scheme is not consistent with (1, 2)

$$\vec{\nabla}^2 p = \vec{\nabla}\cdot\vec{\mathbf{f}}\ ,\ \ \left(\frac{\partial}{\partial t} - \vec{\nabla}^2\right)\vec{\nabla}^2\,\vec{\mathbf{v}} = \vec{\nabla}\times\vec{\nabla}\times\vec{\mathbf{f}}\ ,\ \ \vec{\nabla}\cdot\vec{\mathbf{v}} = 0.$$

This is demonstrated by a normal mode analysis,

$$(\vec{\mathbf{v}}_\eta(\underline{x},t), p_\eta(\underline{x},t)) = \left(\vec{\mathbf{v}}^{(0)}(\underline{x}),\ p^{(0)}(\underline{x})\right)e^{\lambda t},$$

and considering that

$$-\sum_{q=0}^{J_i-1}\alpha_q\,\vec{\mathbf{v}}_\eta^{(n-q)} = \eta\left(\frac{\partial\vec{\mathbf{v}}_\eta}{\partial t}\right)^{(n+1)}\ ,\ \frac{\gamma_0\,\vec{\mathbf{v}}_\eta^{(n+1)}}{\delta t} = (1-\eta)\left(\frac{\partial\vec{\mathbf{v}}_\eta}{\partial t}\right)^{(n+1)}.$$

Time-Splitting (3)

One gets the equivalent continuous formulation of the time splitting scheme,

$$\vec{\mathbf{a}} + \eta \frac{\partial \vec{\mathbf{v}}_\eta}{\partial t} = -\vec{\nabla} p_\eta,$$

$$\vec{\nabla} \cdot \vec{\mathbf{a}} = 0,$$

$$\left((1-\eta) \frac{\partial}{\partial t} - \vec{\nabla}^2 \right) \vec{\mathbf{v}}_\eta = \vec{\mathbf{a}}.$$

$$\boxed{\eta = 0 \quad \Rightarrow\Rightarrow \quad PrDi}$$

Time-Splitting (4)

It can be shown to be equivalent to ($\vec{\mathbf{f}} = 0$)

$$\underbrace{\left((1-\eta)\frac{\partial}{\partial t} - \vec{\nabla}^2\right)}_{!!!!!!!!!}\left\{\vec{\nabla}^2 p_\eta \;,\; \left(\frac{\partial}{\partial t} - \vec{\nabla}^2\right)\vec{\nabla}^2\,\vec{\mathbf{v}}_\eta\right\} = 0,$$

with

$$\frac{1-\eta}{\eta} = -\frac{\gamma_0\;\kappa}{\sum_{q=0}^{J_i-1}\alpha_q\kappa^{-q}} \quad , \quad \kappa = e^{\lambda\delta t}.$$

Projection-Diffusion (1)

No temporal scheme required for writing the 2-steps $(\vec{\mathbf{v}}, p)$ uncoupling :

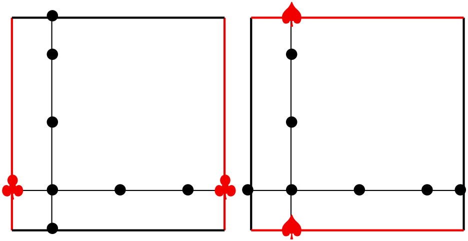
$$(1) \left\{ \begin{array}{ccc} \vec{\mathbf{a}} & = & -\vec{\nabla} p, \\ \vec{\nabla} \cdot \vec{\mathbf{a}} & = & 0, \\ (\vec{\mathbf{a}} \cdot \hat{\mathbf{n}}) \,|_{\partial\Omega} & = & \left(\left(\frac{\partial \vec{\mathbf{V}}}{\partial t} - \vec{\nabla}^2 \vec{\mathbf{v}} \right) \cdot \hat{\mathbf{n}} \right) |_{\partial\Omega} \ , \end{array} \right.$$

$$(2) \quad \left(\frac{\partial}{\partial t} - \vec{\nabla}^2 \right) \vec{\mathbf{v}} = \vec{\mathbf{a}} \quad , \quad \vec{\mathbf{v}}|_{\partial\Omega} = \vec{\mathbf{V}}.$$

1^{st} step : Projection, a Darcy problem
2^{nd} step : Diffusion

Projection-Diffusion (2) : 1^{st} step ⇒ Pressure Operator

$\vec{\nabla} \cdot \vec{\mathbf{a}} = 0$ is exactly imposed with $(\vec{\mathbf{v}}|_{\partial\Omega} = \vec{\mathbf{V}} = (U, W))$



$$(\vec{\mathbf{a}} \cdot \hat{\mathbf{n}})\,|_{\clubsuit} = \left(\frac{\partial U}{\partial t} - \vec{\nabla}^2 u\right)|_{x=\pm 1}$$

$$(\vec{\mathbf{a}} \cdot \hat{\mathbf{n}})\,|_{\spadesuit} = \left(\frac{\partial W}{\partial t} - \vec{\nabla}^2 w\right)|_{z=\pm 1}$$

$$\Downarrow$$

$$\vec{\mathbf{a}}_B = ((\vec{\mathbf{a}} \cdot \hat{\mathbf{n}})\,|_{\clubsuit}, (\vec{\mathbf{a}} \cdot \hat{\mathbf{n}})\,|_{\spadesuit})$$

and $\left(\vec{\mathbf{a}} = -\vec{\nabla} p\right)|_{\bullet} \Rightarrow \ \vec{\mathbf{a}}_{Int}$

Then : $\vec{\nabla} \cdot \vec{\mathbf{a}} \equiv \vec{\nabla}_{Int} \cdot \vec{\mathbf{a}}_{Int} + \vec{\nabla}_B \cdot \vec{\mathbf{a}}_B = 0$

Projection-Diffusion (2) : 1^{st} step ⇒ Pressure Operator

$$\Downarrow$$

$$"\vec{\nabla}^2" p = -\vec{\nabla}_B \cdot \vec{\mathbf{a}}_B$$

$$\boxed{quasi - Poisson \;\; : \;\; no \;\; BC \;\; on \;\; p}$$

A.Batoul, H.Khallouf, G.Labrosse, "Une méthode de résolution directe (pseudo-spectrale) du problème de Stokes 2D/3D instationnaire. Application à la cavité entrainée carrée", **C.R. Acad. Sc. Paris**, Tome 319, Série II (1994), pp. 1455-1461.

Projection-Diffusion (3): Time Discretization

- 2^{nd} step : $\frac{\gamma_0\, \vec{\mathbf{v}}^{(n+1)} - \sum_{q=0}^{J_i-1} \alpha_q\, \vec{\mathbf{v}}^{(n-q)}}{\delta t} - \vec{\nabla}^2 \vec{\mathbf{v}}^{(n+1)} + \vec{\nabla} p = \vec{\mathbf{f}}^{(n+1)}$
- 1^{st} step : ¿ how to evaluate $\left(\left(\frac{\partial \vec{\mathbf{V}}}{\partial t} - \vec{\nabla}^2 \vec{\mathbf{v}} \right)^{(n+1)} \cdot \hat{\mathbf{n}} \right) |_{\partial\Omega}$?

By writing

$$\begin{aligned} &\left(\vec{\nabla}^2 \vec{\mathbf{v}}\right)^{(n+1)} \equiv \underbrace{\left(\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathbf{v}}\right)\right)^{(n+1)}}_{} - \underbrace{\left(\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathbf{v}}\right)\right)^{(n+1)}}_{}, \\ &\text{with} \qquad\qquad\qquad 0 \qquad\qquad \sum_{q=0}^{J_e-1} \beta_q\, \vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathbf{v}}\right)^{n-q}. \end{aligned}$$

γ_0, α_q and β_q in Leriche & Labrosse, SIAM 2000

Stokes eigenmodes Problem

Let $(\vec{\mathbf{v}},\, p)$ be solutions of

$$\lambda\vec{\mathbf{v}} - \vec{\nabla}^2\vec{\mathbf{v}} = -\vec{\nabla}p \quad , \quad in\ \Omega \times t > 0,$$

$$\vec{\nabla}\cdot\vec{\mathbf{v}} = 0 \qquad , \quad in\ \bar{\Omega} = \Omega \cup \partial\Omega,$$

$$\vec{\mathbf{v}} = 0 \qquad , \quad on\ \partial\Omega,$$

equivalent to

$$(1) \quad \vec{\nabla}^2 p = 0, \tag{8}$$

$$(2) \quad \left(\lambda - \vec{\nabla}^2\right)\, \vec{\nabla}^2\,\vec{\mathbf{v}} = 0\ ,\ \vec{\nabla}\cdot\vec{\mathbf{v}} = 0. \tag{9}$$

$$\mathrm{PrDi} \quad \Rightarrow \quad \mathcal{L}\vec{\mathbf{V}} \equiv (\mathcal{A}_D + \mathcal{B})\,\vec{\mathbf{V}} = \lambda\,\vec{\mathbf{V}},$$

$\vec{\mathbf{V}}$ being the column vector of the unknown nodal values of $\vec{\mathbf{v}}$.

Some references

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- E.LERICHE and G.LABROSSE, "Fundamental Stokes eigenmodes in the square : which expansion is more accurate, Chebyshev or Reid-Harris ?" **Numerical Algorithms**, 38 (2005), pp. 1-21.
- E. LERICHE and G. LABROSSE, "Vector potential - vorticity relationship for the Stokes flows : application to the Stokes eigenmodes in 2D/3D closed domain". **Theoretical and Computational Fluid Dynamics**, 21 (2007), pp. 1-13.
- E.LERICHE, P. LALLEMAND and G. LABROSSE, "Stokes eigenmodes in cubic domain: primitive variable and Lattice Boltzmann formulations". **App. Num. Math**, ?? (2007), pp. ???.

2D Stokes eigenmodes (1)

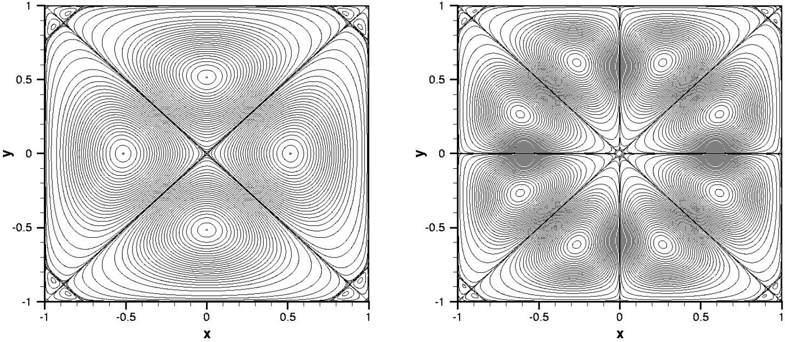


Figure: $\psi(x, y)$ contour plots, from $N = 96$ PrDi solver.

2D Stokes eigenmodes (2)

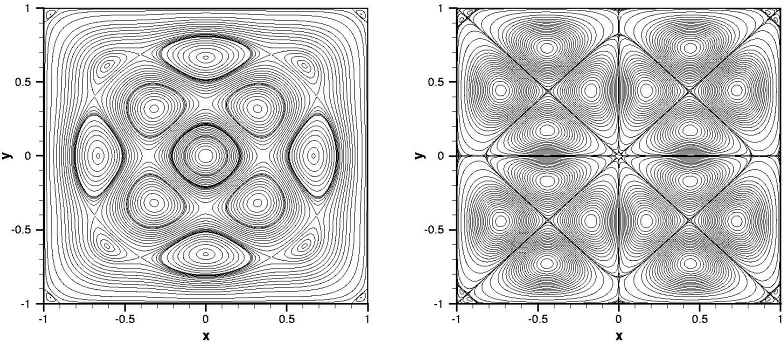


Figure: $\psi(x, y)$ contour plots, from $N = 96$ PrDi solver.

2D Stokes eigenmodes (3)

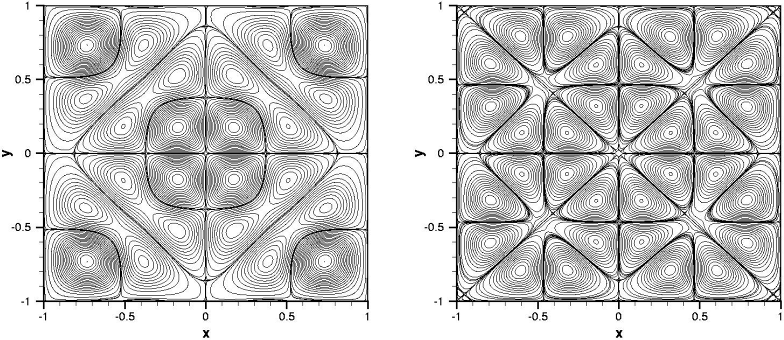


Figure: $\psi(x, y)$ contour plots, from $N = 96$ PrDi solver.

2D Stokes eigenmodes (4)

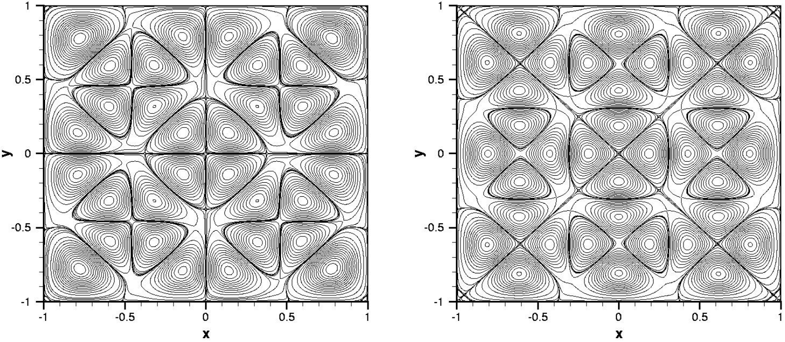


Figure: $\psi(x, y)$ contour plots, from $N = 96$ PrDi solver.

Moffatt eddies in the 2D Stokes eigenmodes (1)

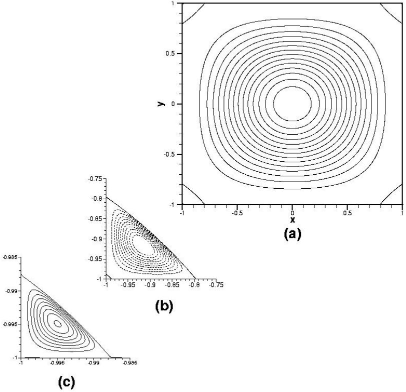


Figure: $\psi(x, y)$ contour plots, from $N = 96$ PrDi solver. Solid/dashed lines respectively correspond to positive (or zero)/negative levels.

Moffatt eddies in the 2D Stokes eigenmodes (2)

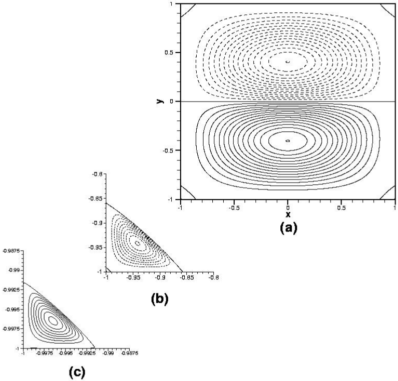


Figure: $\psi(x, y)$ contour plots, from $N = 96$ PrDi solver. Solid/dashed lines respectively correspond to positive (or zero)/negative levels.

Moffatt eddies in the 2D Stokes eigenmodes (3)

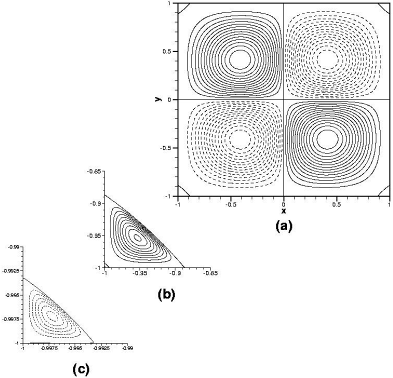
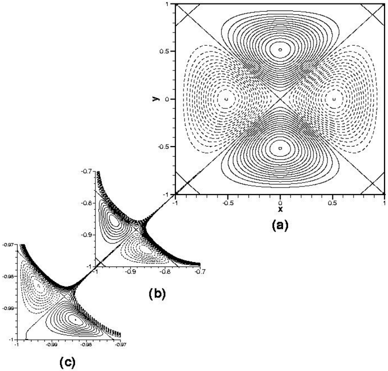


Figure: $\psi(x, y)$ contour plots, from $N = 96$ PrDi solver. Solid/dashed lines respectively correspond to positive (or zero)/negative levels.

Moffatt eddies in the 2D Stokes eigenmodes (4)



(a)

(b)

(c)

Figure: $\psi(x, y)$ contour plots, from $N = 96$ PrDi solver. Solid/dashed lines respectively correspond to positive (or zero)/negative levels.

Moffatt eddies in the 2D Stokes eigenmodes (5)

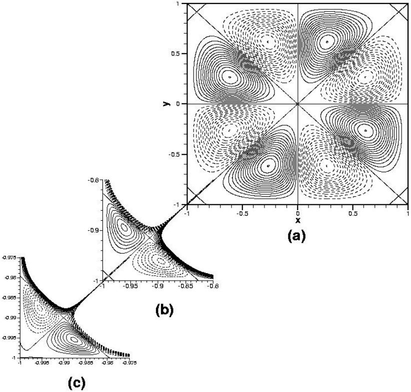


Figure: $\psi(x, y)$ contour plots, from $N = 96$ PrDi solver. Solid/dashed lines respectively correspond to positive (or zero)/negative levels.

a Stokes eigenmode in the cube (1)

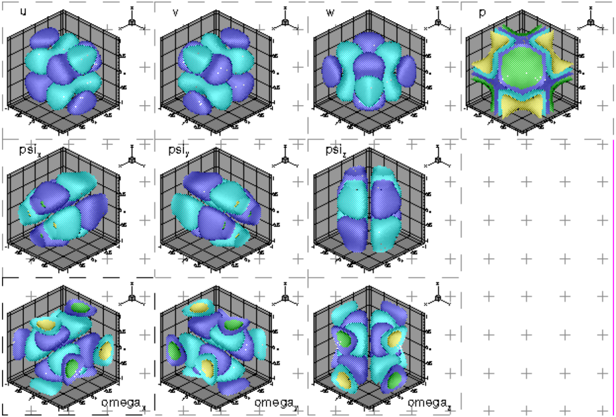
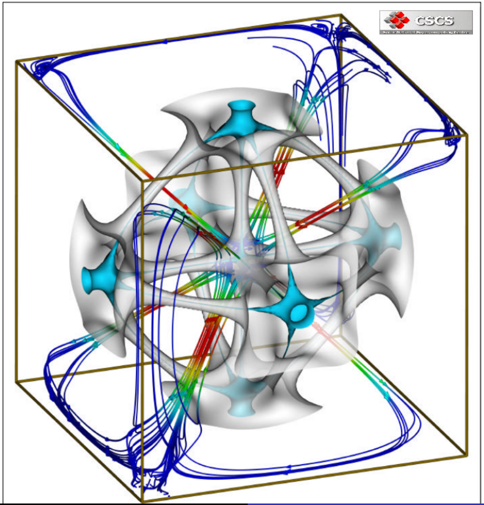


Figure: $\lambda = -45.366$, from the $N = 64$ PrDi solver.

Streamlines of a Stokes eigenmode in the cube



Corner streamlines of a Stokes eigenmode in the cube (1)

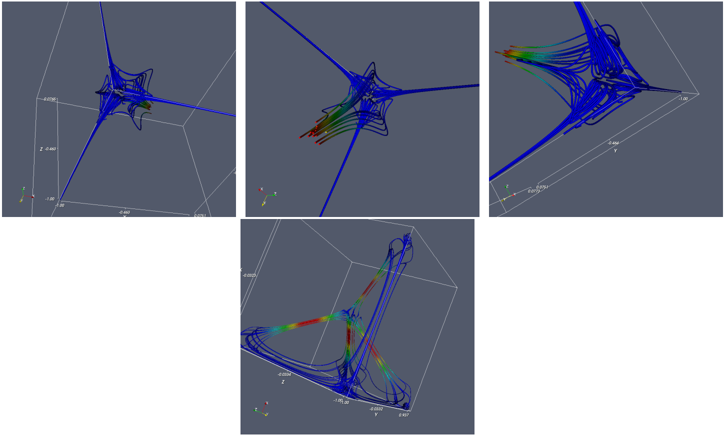


Figure: $\lambda = -45.366$, from the $N = 64$ PrDi solver.

Corner streamlines of a Stokes eigenmode in the cube (2)

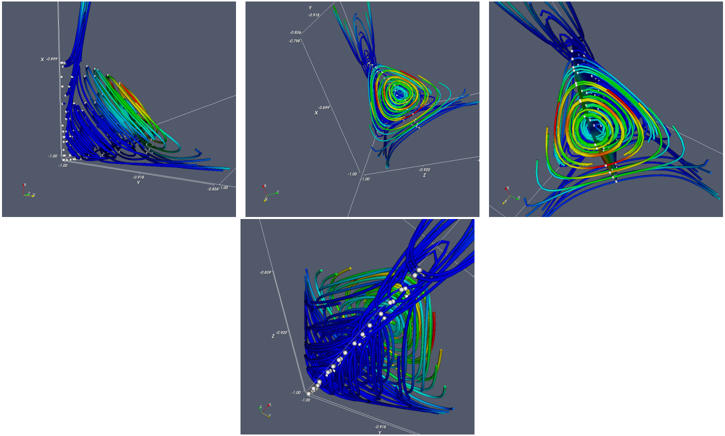


Figure: $\lambda = -36.680$, from the $N = 64$ PrDi solver.

Continuous Unsteady Stokes Problem

Let $(\vec{\mathbf{v}},\, p)$ be solutions of the non local relation

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} - \vec{\nabla}^2 \vec{\mathbf{v}} = -\vec{\nabla} p \ \text{ and } \ \vec{\nabla} \cdot \vec{\mathbf{v}} = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{\nabla} p = 0.$$

$$\begin{array}{ccc} \Downarrow & \Downarrow & \Downarrow \\ \Downarrow & \vec{\mathbf{v}} = \vec{\nabla} \wedge \vec{\mathbf{\Psi}} & \vec{\nabla} p = \vec{\nabla} \wedge \vec{\mathbf{\Pi}} \end{array}$$

$$\Rightarrow \qquad \frac{\partial \vec{\mathbf{\Psi}}}{\partial t} + \vec{\omega} + \vec{\mathbf{\Pi}} = 0 \ \ , \ \ \vec{\omega} = \vec{\nabla} \wedge \vec{\mathbf{v}}.$$

In the core part of the Stokes eigenmodes, in any 2D/3D domain,

$$\lambda\, \vec{\mathbf{\Psi}} + (\vec{\omega} - \vec{\omega}_0) \simeq 0,$$

where $\vec{\omega}_0$ is an offset vorticity, possibly zero.

(1) $\vec{\omega}$, $\vec{\boldsymbol{\Psi}}$ and $\vec{\boldsymbol{\Pi}}$ for a Stokes eigenmode in the square

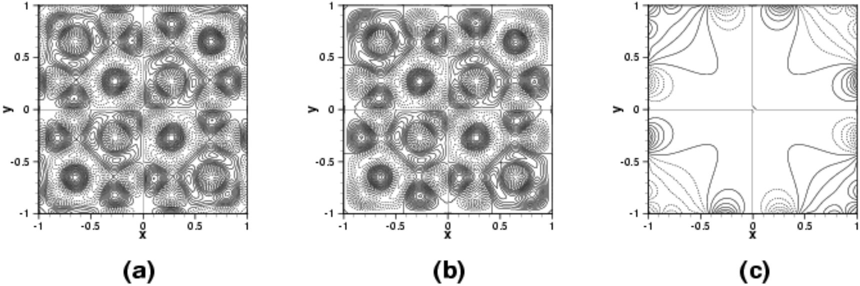


Figure: **(a)** $-\frac{\omega}{\lambda}$, **(b)** ψ and **(c)** $-\frac{\Pi}{\lambda}$; $\lambda = -331.966266$.

(2) $\vec{\omega}$-$\vec{\boldsymbol{\Psi}}$ correlation for a Stokes eigenmode in the square

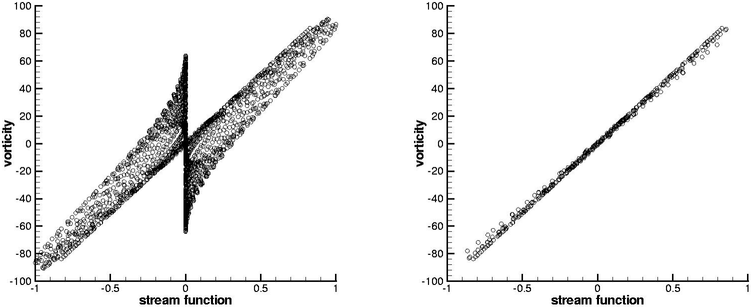
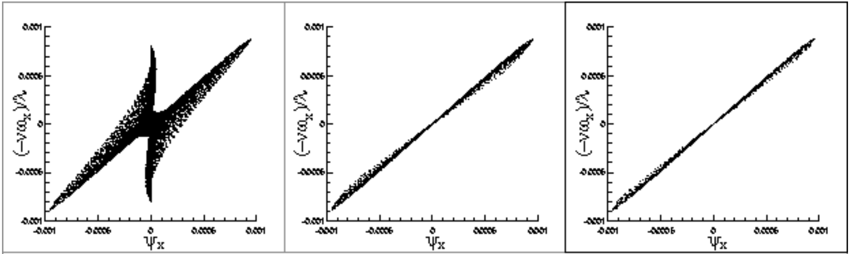


Figure: Scatter plot in the whole square, and in $[-0.6, 0.6]^2$.

(3) $\vec{\omega}$-$\vec{\mathbf{\Psi}}$ correlation for a Stokes eigenmode in the cube



(a) (b) (c)

Figure: Scatter plot $\left(\psi_x, -\frac{\omega_x}{\lambda}\right)$, for $\lambda = -45.366354$, respectively from $[-1,+1]^3$, $[-0.5967,+0.5967]^3$ and from $[-0.5556,+0.5556]^3$.

Continuous Unsteady (Navier-)Stokes Problem

Let us (1) write the dimensional Navier-Stokes equations,

$$\frac{\partial \vec{\mathbf{v}}}{\partial t} = \nu \, \vec{\nabla}^2 \vec{\mathbf{v}} - \frac{\vec{\nabla} p}{\rho} + \vec{\mathbf{s}} \;\;,\;\; \vec{\nabla} \cdot \vec{\mathbf{v}} = 0 \;\;,\;\; \rho \equiv \rho(p),$$

ν, ρ being respectively the momentum diffusivity and the density, $\vec{\mathbf{s}}$ standing for $-\left(\vec{\mathbf{v}} \cdot \vec{\nabla}\right) \vec{\mathbf{v}}$, possibly completed by any other source term, and (2) make the Helmholtz decompositions,

$$\vec{\mathbf{v}} = \vec{\nabla} \wedge \vec{\boldsymbol{\Psi}} - \vec{\nabla}\psi \;,\; \vec{\nabla} \cdot \vec{\boldsymbol{\Psi}} = 0 \;;\; \vec{\nabla}^2 \psi = 0,$$

$$\frac{\vec{\nabla} p}{\rho} = \vec{\nabla} \wedge \vec{\boldsymbol{\Pi}} - \vec{\nabla}\pi \;\;,\;\; \vec{\nabla} \cdot \vec{\boldsymbol{\Pi}} = \vec{\nabla}^2 \vec{\boldsymbol{\Pi}} = 0,$$

$$\vec{\mathbf{s}} = \vec{\nabla} \wedge \vec{\boldsymbol{\Sigma}} - \vec{\nabla}\sigma \;,\; \vec{\nabla} \cdot \vec{\boldsymbol{\Sigma}} = 0 \;;\; \vec{\boldsymbol{\Sigma}} \equiv \vec{\boldsymbol{\Sigma}}\left(\vec{\boldsymbol{\Psi}}, \psi\right) \;,\; \sigma \equiv \sigma\left(\vec{\boldsymbol{\Psi}}, \psi\right).$$

Its Potential Formulation

This leads to the following decomposition of the Navier-Stokes equations

$$\frac{\partial \psi}{\partial t} + \pi - \sigma\left(\vec{\mathbf{\Psi}}, \psi\right) = \theta(t),$$

$$\left(\frac{\partial}{\partial t} - \nu \vec{\nabla}^2\right)\vec{\mathbf{\Psi}} + \vec{\mathbf{\Pi}} - \vec{\mathbf{\Sigma}}\left(\vec{\mathbf{\Psi}}, \psi\right) = \vec{\nabla}\Theta \quad , \quad \vec{\nabla}^2\Theta = 0,$$

$\theta(t)$ is any function of time, and Θ any harmonic function of the space coordinates. No viscous control in the ψ balance equation: a pure advection dynamics.

¿ How to determine ψ ?

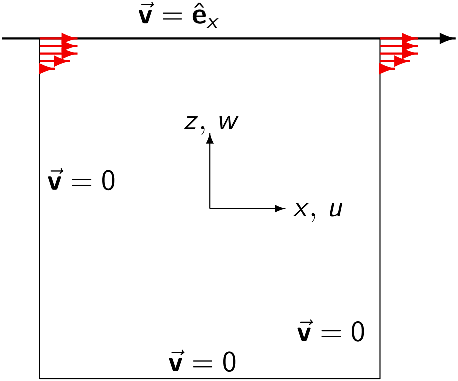
About ψ, the velocity scalar potential: Questions

It is fully determined by

$$\vec{\nabla}^2\psi = 0 \quad , \quad \text{and boundary conditions.}$$

¿ Where does its time dependency come from ?
¿ How to divide up the $\vec{\mathbf{v}}$ boundary conditions into $\vec{\boldsymbol{\Psi}}$ and ψ ?
¿ Is it just a matter of convenience to choose $\left.\frac{\partial\psi}{\partial n}\right|_{\partial\Omega} = \vec{\mathbf{v}}\cdot\vec{\mathbf{n}}|_{\partial\Omega}$,
and fix $\vec{\boldsymbol{\Psi}}$ from $\vec{\mathbf{v}}\cdot\vec{\mathbf{t}}|_{\partial\Omega}$ as proposed by
"G.J. Hirasaki and J.D. Hellums, Boundary conditions on the vector and scalar potentials in viscous three-dimensional hydrodynamics, in Quart. Applied Math., 1970." ?

Example : The driven cavity



Two (at least) possible models for the boundary conditions

They are :

1. $\psi = 0$ and $\vec{\mathbf{v}}|_{\partial\Omega} \Rightarrow \vec{\mathbf{\Psi}}\Big|_{\partial\Omega}$
2. $\frac{\partial\psi}{\partial n}\Big|_{\partial\Omega} = \vec{\mathbf{v}}\cdot\vec{\mathbf{n}}|_{\partial\Omega}$, and then fix $\vec{\mathbf{\Psi}}$ for completing $\vec{\mathbf{v}}|_{\partial\Omega}$

¿ WILL THE RESULTING FLOWS BE IDENTICAL ?