Une méthode de pénalisation par face pour l'approximation des équations de Navier-Stokes à nombre de Reynolds élevé

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Outline

State of the art: face penalty methods

The addition of a term penalizing the jump of the gradient over element edges

$$J(u_h, v_h) = \sum_K \int_{\partial K \setminus \partial \Omega} \gamma h_{\partial K}^{\alpha} [\nabla u_h \cdot n][\nabla v_h \cdot n] \, \mathrm{d}s$$

to the standard Galerkin formulation may be used to stabilize

- transport operators
- Stokes like systems
- symmetric Friedrichs systems

Error analysis for linear problems leads to (quasi) optimal apriori error estimates for continuous finite element spaces

$$V_h = \{v : v \in C^0(\Omega); v|_K \in P_k(K), \forall K \in \mathcal{T}_h\}.$$

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$$J(u_h, v_h) = \sum_K \int_{\partial K \setminus \partial \Omega} \gamma h_{\partial K}^{\alpha} [\nabla u_h \cdot n][\nabla v_h \cdot n] \, ds$$

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Theorem (Burman-Fernández-Hansbo (2004))

There exists an interpolation operator π_h^* on V_h^k such that

$$\|h^{\frac{1}{2}}(I - \pi_h^*)\nabla \mathbf{v}_h\|_{0,\Omega}^2 \leq \gamma \sum_{K \in \mathcal{T}_h} \int_{\partial K} h_K^2 [\![\nabla \mathbf{v}_h \cdot n]\!]^2,$$

with $[\![v]\!] \stackrel{\text{def}}{=} v^+ - v^-$ if $\partial K \subset \Omega$ and $[\![v]\!] \stackrel{\text{def}}{=} 0$ if $\partial K \subset \partial \Omega$.

Brief history

- Babuska & Zlamal, biharmonic operator, (1972).
- Douglas & Dupont, second order elliptic and parabolic problems, (1976).
- Burman & Hansbo, convection dominated limit, (2003).
- Burman & Ern, discrete maximum principle, (2004).
- Burman & Fernández & Hansbo, the Oseen's problem, (2004).
- Burman & Ern, *hp*-FEM for transport operators, (2005).
- Burman & Fernández, the incompressible Navier-Stokes equations, semidiscretization in space (2005).

Important related work:

- Guermond, subgrid viscosity, 1999.
- Codina, orthogonal subscale stabilization, 2000.
- Brezzi & Fortin, "A minimal stabilisation procedure", 2001.
- Becker & Braack, local projection stabilization, 2001.

Variational multiscale methods and projection based LES

- Knowing the exact solution $(\mathbf{u}, p)$ we could compute the *ideal projection* $(\pi_h \mathbf{u}, \pi_h p) \in [V_h]^d \times V_h$.
- Since the exact solution is unknown we have to do with a *working projection* given by a discrete scheme (typically Galerkin FEM).
- The working projection should be *stable* and *accurate* uniformly in the Reynolds number: *standard Galerkin has to be modified*.
- Assumption:
  - the Bernoulli hypothesis: *all fine to coarse interaction is dissipative*.
- We choose π_h, (the ideal projection) to be the L^2-projection.

Stabilized methods based on scale separation, the Euler equations

1. Let $W = H(\text{div}) \times L^2_0$, $U = (\mathbf{u}, p)$, $L(\mathbf{w})U = (\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla p$, $\pi^\perp = (I - \pi_h^*)$. Assume $\mathbf{f} \in [V_h]^d$. Find $U \in W$ such that

$$(\partial_t \mathbf{u} + L(\mathbf{u})U, \mathbf{v}) + (\nabla \cdot \mathbf{u}, q) = (\mathbf{f}, \mathbf{v}), \, \forall (\mathbf{v}, q) \in W.$$

Stabilized methods based on scale separation, the Euler equations

1. Find $U \in W$ such that

$$(\partial_t \mathbf{u} + L(\mathbf{u})U, \mathbf{v}) + (\nabla \cdot \mathbf{u}, q) = (\mathbf{f}, \mathbf{v}), \forall (\mathbf{v}, q) \in W.$$

2. Scale separation $U = U_h + \widetilde{U}$, $U_h = \pi_h U$
   - $\pi_h U$ is the L^2-projection of U onto $W_h = [V_h]^d \times V_h$
   - $\widetilde{U}$ orthogonal to the finite element space (c.f. Codina).

Stabilized methods based on scale separation, the Euler equations

1. Find $U \in W$ such that

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2. Scale separation $U = U_h + \widetilde{U}$, $U_h = \pi_h U$
3. Inserting $U_h + \widetilde{U}$ yields the formulation

$$\begin{aligned}(\partial_t \mathbf{u}_h + L(\mathbf{u}_h)U_h, \mathbf{v}_h) + (\nabla \cdot \mathbf{u}_h, q_h) &= (\mathbf{f}, \mathbf{v}_h) \\ &+ (T^{-1}(\pi^\perp L(\mathbf{u}_h)U_h, \pi^\perp \nabla \cdot \mathbf{u}_h), (\pi^\perp L(\mathbf{u}_h)V_h, \pi^\perp \nabla \cdot \mathbf{v}_h)) \\ &+ ((\tilde{\mathbf{u}} \cdot \nabla)\mathbf{u}, \mathbf{v}_h) \quad \forall V = (\mathbf{v}, q) \in W.\end{aligned}$$

4. T^{-1} is the solution operator for the fine scale equation

$$\begin{aligned}(\partial_t \tilde{\mathbf{u}} + (\mathbf{u} \cdot \nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}} \cdot \nabla)\mathbf{u}_h + \nabla \tilde{p}, \tilde{\mathbf{v}}) + (\nabla \cdot \tilde{\mathbf{u}}, \tilde{q}) \\ = (\pi^\perp L(\mathbf{u}_h)U_h, \tilde{\mathbf{v}}) + (\pi^\perp \nabla \cdot \mathbf{u}_h, \tilde{q}).\end{aligned}$$

Simplifications leading to edge oriented stabilization

1. We drop the fine to coarse interaction terms $((\tilde{\mathbf{u}} \cdot \nabla)\mathbf{u}, \mathbf{v}_h)$
2. Bernoulli hypothesis: approximate T^{-1} with a scaled diagonal matrix.
3. Stabilized FEM based on the projected residual: Find $U_h \in W_h$ such that

$$
\begin{aligned}
(\partial_t \mathbf{u}_h + L(\mathbf{u}_h)U_h, \mathbf{v}_h) + (\nabla \cdot \mathbf{u}_h, q_h) = (\mathbf{f}, \mathbf{v}_h) \\
-(\delta_u \pi^\perp L(\mathbf{u}_h)U_h, \pi^\perp L(\mathbf{u}_h)V_h)) - (\delta_{div} \pi^\perp \nabla \cdot \mathbf{u}_h, \pi^\perp \nabla \cdot \mathbf{v}_h)) \\
\forall V_h = (\mathbf{v}_h, q_h) \in W_h.
\end{aligned}
$$

4. Equivalent dissipation (recall $\pi^\perp = (I - \pi_h^*)$):

$$
\begin{aligned}
\|\pi^\perp L(\mathbf{u}_h)U_h\|_K^2 &\leq \sum_{e \in \mathcal{E}(K)} \int_e \gamma h_K [L(\mathbf{u}_h)U_h]^2 \, \mathrm{d}s \\
&\leq \sum_{e \in \mathcal{E}(K)} \int_e \gamma h_K \left\{ [(\mathbf{u}_h \cdot \nabla)\mathbf{u}_h]^2 + [\nabla p_h]^2 \right\} \, \mathrm{d}s
\end{aligned}
$$

Edge Stabilized FE, Navier-Stokes: Space Semi-Discretization

For all $t \in (0, T)$, find $(\mathbf{u}_h(t), p_h(t)) \in [V_h]^d \times V_h$ such that

$$\begin{cases} (\partial_t \mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) & = (\mathbf{f}, \mathbf{v}_h), \\ \qquad\qquad - b(q_h, \mathbf{u}_h) & = 0, \\ \qquad\qquad \mathbf{u}_h(0) & = \pi_h \mathbf{u}_0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in [V_h]^d \times V_h$, with

$$a(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) \stackrel{\text{def}}{=} (\mathbf{u}_h \cdot \nabla \mathbf{u}_h, \mathbf{v}_h) + (\nu \nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + \frac{1}{2}(\nabla \cdot \mathbf{u}_h, \mathbf{u}_h \cdot \mathbf{v}_h) + \text{bd terms}$$

$$b(p_h, \mathbf{v}_h) \stackrel{\text{def}}{=} -(p_h, \nabla \cdot \mathbf{v}_h) + \text{bd terms}$$

Edge Stabilized FE, Navier-Stokes: Space Semi-Discretization

For all $t \in (0, T)$, find $(\mathbf{u}_h(t), p_h(t)) \in [V_h]^d \times V_h$ such that

$$\begin{cases} (\partial_t \mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b(p_h, \mathbf{v}_h) + j_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \\ \qquad - b(q_h, \mathbf{u}_h) + j(p_h, q_h) = 0, \\ \qquad \mathbf{u}_h(0) = \pi_h \mathbf{u}_0, \end{cases}$$

for all $(\mathbf{v}_h, q_h) \in [V_h]^d \times V_h$, with

$$j_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{v}_h) \stackrel{\text{def}}{=} \sum_{K \in \mathcal{T}_h} \int_{\partial K} \gamma h_K^2 (1 + |\mathbf{u}_h \cdot n|^2) [\![\nabla \mathbf{u}_h]\!] : [\![\nabla \mathbf{v}_h]\!],$$

$$j(p_h, q_h) \stackrel{\text{def}}{=} \sum_{K \in \mathcal{T}_h} \int_{\partial K} \gamma h_K^2 [\![\nabla p_h]\!] \cdot [\![\nabla q_h]\!].$$

Convergence (selected results)

- $\mathbf{J}\big[\mathbf{u}_h;(\mathbf{u}_h,p_h),(\mathbf{v}_h,q_h)\big] = j_{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{v}_h)+j(p_h,q_h)$
- Triple-norm: $|||(\mathbf{v}_h,q_h)|||^2_{\mathbf{w}_h} \stackrel{\text{def}}{=} \|\nu^{\frac{1}{2}}\nabla\mathbf{v}_h\|^2_{0,\Omega}+\mathbf{J}[\mathbf{w}_h;(\mathbf{v}_h,q_h),(\mathbf{v}_h,q_h)]$.

Theorem (Velocity convergence, Burman & Fernández, 2005)

The following estimates hold (when $\nu < h$)

$$\|\pi_h\mathbf{u}-\mathbf{u}_h\|_{L^\infty((0,T);L^2(\Omega))} \leq h^{\frac{3}{2}}C(\mathbf{u},p)\mathrm{e}^{c(\mathbf{u})T},$$

$$\left(\int_0^T |||(\pi_h\mathbf{u}-\mathbf{u}_h,\pi_h p-p_h)|||^2_{\mathbf{u}_h}\,\mathrm{d}t\right)^{\frac{1}{2}} \leq h^{\frac{3}{2}}C(\mathbf{u},p,T)\mathrm{e}^{c(\mathbf{u})T},$$

$$\int_0^T \mathbf{J}[\mathbf{u}_h,(\mathbf{u}_h,p_h),(\mathbf{u}_h,p_h)]\,\mathrm{d}t \leq h^3C(\mathbf{u},p)\mathrm{e}^{c(\mathbf{u})T}$$

with $c(\mathbf{u})$ depending on $\|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}$ and $C(\mathbf{u},p)$ depending on $\|\mathbf{u}\|_{L^2(0,T;H^2(\Omega))}$, $\|p\|_{L^2(0,T;H^2(\Omega))}$, $\|\mathbf{u}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}$.

Energy consistency: monitoring artificial dissipation

For the Navier-Stokes equations there holds

$$\|\mathbf{u}(T)\|^2 + \int_0^T \|\nu^{\frac{1}{2}}\nabla \mathbf{u}\|^2 \mathrm{d}t = \|\mathbf{u}(0)\|^2 + (\mathbf{f}, \mathbf{u}).$$

Any reasonable numerical method will satisfy

$$\|\mathbf{u}_h(T)\|^2 + \int_0^T \left\{ \|\nu^{\frac{1}{2}}\nabla \mathbf{u}_h\|^2 + S(\mathbf{u}_h, p_h) \right\} \mathrm{d}t = \|\mathbf{u}_h(0)\|^2 + (\mathbf{f}, \mathbf{u}_h).$$

$S(\mathbf{u}_h, p_h)$ the artificial dissipation added for the method to remain stable.

- Define $D = \frac{\int_0^T S(\mathbf{u}_h, p_h)\,\mathrm{d}t}{\int_0^T \|\nu^{\frac{1}{2}}\nabla \mathbf{u}_h\|^2\,\mathrm{d}t}$:

$$\|\mathbf{u}_h(T)\|^2 + (1 + D)\int_0^T \|\nu^{\frac{1}{2}}\nabla \mathbf{u}_h\|^2 \mathrm{d}t = \|\mathbf{u}_h(0)\|^2 + (\mathbf{f}, \mathbf{u}_h).$$

Some remarks on face penalty stabilization

1. The interior penalty operator can be seen as a subgrid viscosity: starting from polynomial order 3 and onward the kernel is a C^1 space with approximation properties.
2. Scale separation by polynomial order instead of hierarchic meshes.
3. The dissipation ratio D measures the energy consistency and is (related to) an a posteriori error estimator.
4. For high Reynolds number flow theory predicts (P1 elements and sufficiently regular solution):

$$\text{computational error} \leq \text{numerical dissipation} = \text{stabilization} \leq Ch^3$$

Scale separation and the energy inequality

Let us now assume $\mathbf{f} = 0$ and consider the projection on mesh $\mathcal{T}_h$ of the exact solution

$$\|\pi_h \mathbf{u}(T)\|^2 + \|(I - \pi_h)\mathbf{u}(T)\|^2 + \int_0^T \|\nu^{\frac{1}{2}} \nabla \mathbf{u}\|^2 \mathrm{d}t = \|\mathbf{u}(0)\|^2,$$

but $(I - \pi_h)\mathbf{u}(T)$ represents the unresolved scales and hence

$$\|(I - \pi_h)\mathbf{u}(T)\|^2 \approx \int_{\xi_h}^{\infty} E(\xi) \, \mathrm{d}\xi$$

(where $E(\xi)$ denotes the energy distribution over the wave numbers)

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(where $E(\xi)$ denotes the energy distribution over the wave numbers) leading to

$$\frac{\|\mathbf{u}(0)\|^2 - \|\pi_h \mathbf{u}(T)\|^2}{\int_0^T \|\nu^{\frac{1}{2}} \nabla \mathbf{u}\|^2 \mathrm{d}t} \approx 1 + \frac{\int_{\xi_h}^{\infty} E(\xi)\, \mathrm{d}\xi}{\int_0^T \|\nu^{\frac{1}{2}} \nabla \mathbf{u}\|^2 \mathrm{d}t}$$

Scale separation and the energy inequality

- The continuous case:

$$\frac{\|\mathbf{u}(0)\|^2 - \|\pi_h \mathbf{u}(T)\|^2}{\int_0^T \|\nu^{\frac{1}{2}} \nabla \mathbf{u}\|^2 \mathrm{d}t} \approx 1 + \frac{\int_{\xi_h}^{\infty} E(\xi)\ \mathrm{d}\xi}{\int_0^T \|\nu^{\frac{1}{2}} \nabla \mathbf{u}\|^2 \mathrm{d}t}$$

- The discrete case:

$$\frac{\|\mathbf{u}_h(0)\|^2 - \|\mathbf{u}_h(T)\|^2}{\int_0^T \|\nu^{\frac{1}{2}} \nabla \mathbf{u}_h\|^2 \mathrm{d}t} = 1 + D$$

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- We conclude that if $\mathbf{u}_h \approx \pi_h \mathbf{u}$ is to hold then

$$D \approx \frac{\int_{\xi_h}^{\infty} E(\xi)\ \mathrm{d}\xi}{\int_0^T \|\nu^{\frac{1}{2}} \nabla \mathbf{u}\|^2 \mathrm{d}t}$$

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Remark: for standard Galerkin $D = 0$!

Scale separation and the energy inequality

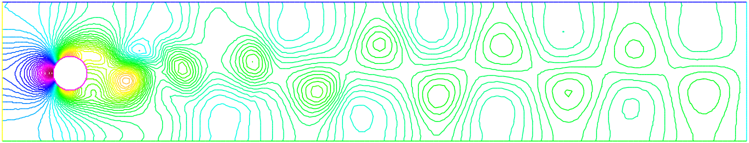
- Definition in 2D: $E(\xi) \sim \xi|\hat{\mathbf{u}}(\xi)|^2$
- In 2D there holds for isotropic decaying turbulence: $E(\xi) \sim \xi^{-3}$ (Kraichnan, 1967).
- If $\xi_h \approx h^{-1}$ is in the inertial range where $E(\xi) \approx \xi^{-3}$ then

$$D \approx C \frac{\int_{\xi_h}^{\xi_v} \xi^{-3} \,\mathrm{d}\xi}{\int_0^T \|\nu^{\frac{1}{2}} \nabla \mathbf{u}\|^2 \mathrm{d}t} \approx C \frac{\xi_h^{-2} - \xi_v^{-2}}{\int_0^T \|\nu^{\frac{1}{2}} \nabla \mathbf{u}\|^2 \mathrm{d}t}.$$

- Assuming ξ_v^{-2} negligeable we expect

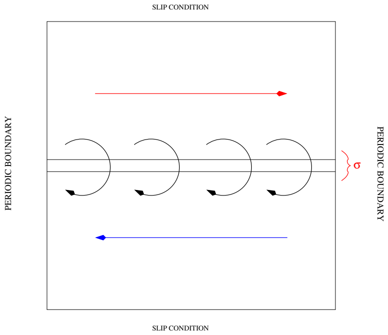
$$D \sim h^2$$

Numerical Results: Turek benchmark $Re = 100$ flow around a cylinder, P1/P1



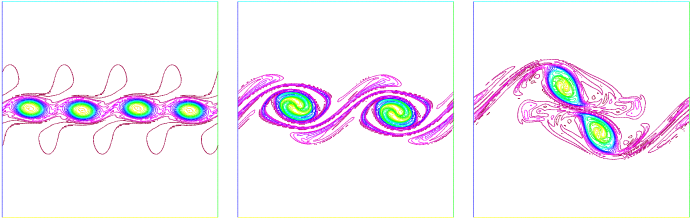
NoDOFs	dt	$C_{D_{max}}$	$C_{L_{max}}$	St	ΔP	D	$O(h^\alpha)$
8667	0.01	3.2518	1.0438	0.2994	2.4989	0.1031	-
33132	0.005	3.2390	1.0377	0.3016	2.4875	0.0230	2.16
131784	0.0025	3.2308	1.0262	0.3008	2.4697	0.0035	2.72
lower	-	3.22	0.99	0.2950	2.46	-	-
upper	-	3.24	1.01	0.3050	2.50	-	-

Numerical Results: $Re = 10000$ mixing layer



- Unit square, $\mathbf{u}_\infty = 1$, $\sigma = \frac{1}{28}$, $\nu = 3.571 \cdot 10^{-6} \rightarrow Re_\sigma = 10000$.
- Lesieur et al. proposed this problem as a model case for decaying 2D turbulence.
- They showed numerically that $E(\xi)$ decays between ξ^{-4} and ξ^{-3} for the streamwise velocity component (Fourier transform only in the x-variable).
- we expect: $c_1 h^3 < D < c_2 h^2$ to be consistent with Lesieur and $D \sim h^2$ to be consistent with Kraichnan.

Numerical Results: $Re = 10000$ mixing layer



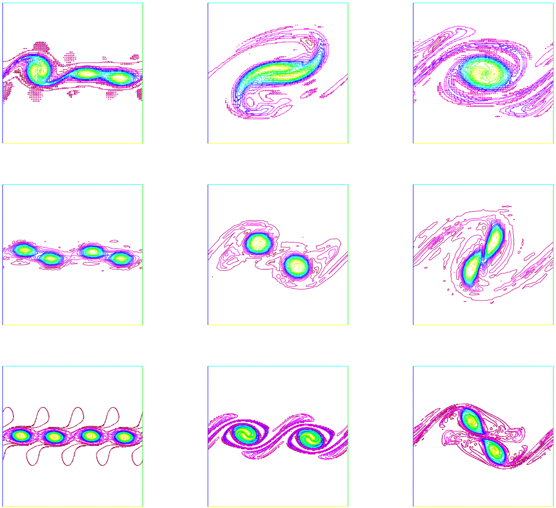
P1 el.	D	$O(h^{\alpha_D})$	$J(\mathbf{u}_h, p_h)$	P2 el.	D	$O(h^{\alpha_D})$	$J(\mathbf{u}_h, p_h)$
80	5.6	-	6E-4	40	0.38	-	5E-5
160	1.4	2.0	2E-4	80	0.1098	1.79	1.5E-5
320	0.3	2.22	4E-5	160	0.025	2.0	3.6E-6

The convergence of D implies $E(\xi) \sim \xi^{-}3$ coherent with the scaling law of Kraichnan and with the numerical results of Lesieur.

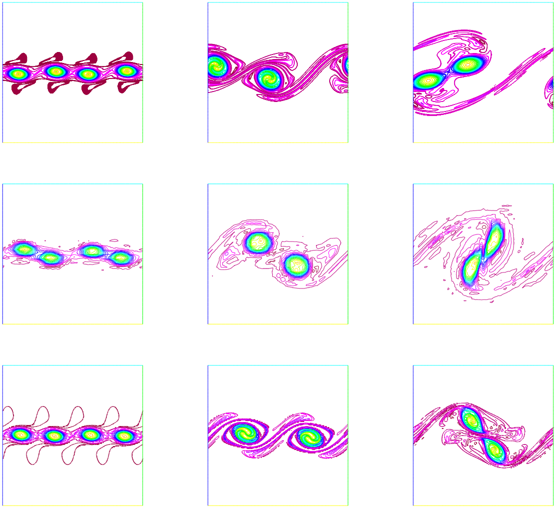
Conclusions and outlook

- Face oriented interior penalty methods work for incompressible flow at high Reynolds number.
- Interaction with turbulence?
- Future work focuses on complex flow problems such as:
  - Incompressible flow in 3D at high Reynolds number (turbulence)
  - Viscoelastic flow
  - Freesurface flow
  - Compressible flow

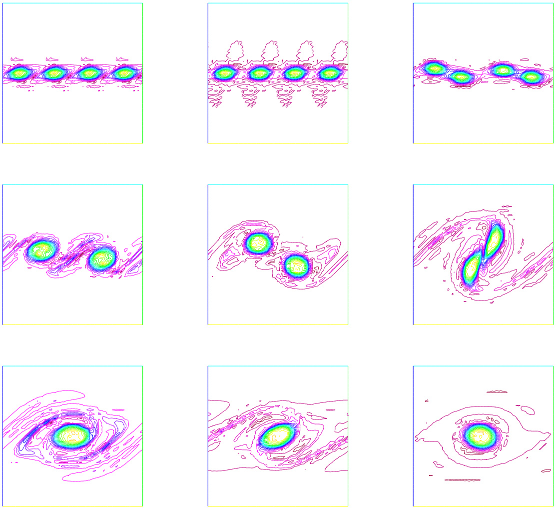
Mixing layer, Reynolds 10000, P1/80 $\times$ 80, P2/32 $\times$ 32, P2/160 $\times$ 160, t=50,80,100



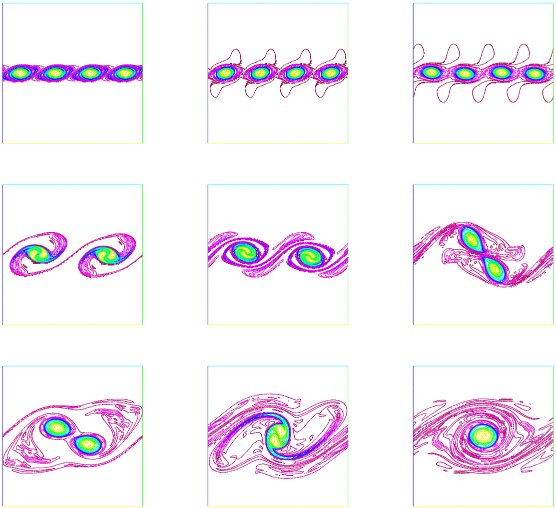
Mixing layer, Reynolds 10000, P1/320×320, P2/32×32, P2/160×160, t=50,80,100



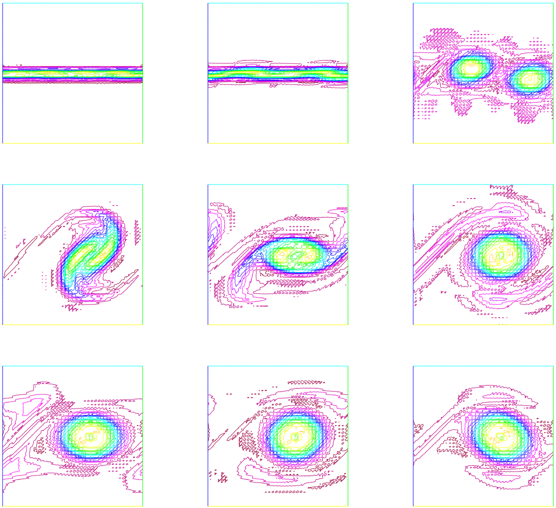
P2/P2, 32×32, t=20,30,50,70,80,100,120,140,200



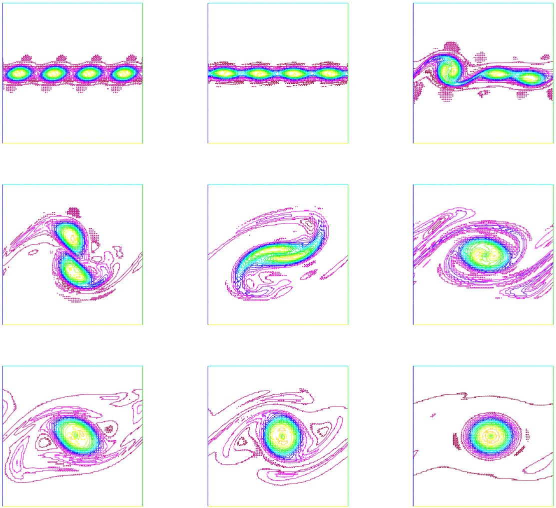
P2/P2, 160 × 160, t=20,30,50,70,80,100,120,140,200



P1/P1, 40×40, t=20,30,50,70,80,100,120,140,200



P1/P1, 80×80, t=20,30,50,70,80,100,120,140,200



P1/P1, 320×320, t=20,30,50,70,80,100,120,140,200