Numerical analysis of finite volume schemes

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Continuous spaces and discrete spaces

- Finite volume methods: discretization schemes for conservation laws: elliptic, parabolic, hyperbolic equations
- Different continuous spaces: H^1_0, $L^2(0, T, H^1)$, L^∞...
- Cell centred finite volume schemes:
  One discrete space: $H_{\mathcal{D}}(\Omega)$ = piecewise constant functions on the control volumes.
- Similar tools for the analysis of the continuous and discrete problems.

Principle of the finite volume method

- Conservation law $u_t + \mathrm{div}(F(u, \nabla u)) + s(u) = 0$ on $\Omega \subset \mathbb{R}^d$.
- Mesh $\mathcal{T}$ of Ω: $\Omega = \bigcup_{K \in \mathcal{T}} K$.

  control volumes K: polygonal subsets of Ω

  discretization $\mathcal{D}$= (mesh $\mathcal{T}$, edges $\mathcal{E}$,...)
- Balance equation:

$$\int_K u_t \, dx + \int_{\partial K} F(u, \nabla u) \cdot \mathbf{n} \, d\gamma(x) + \int_K s(u) \, dx = 0$$

$$\Longleftrightarrow \int_K u_t \, dx + \sum_{\sigma \in \mathcal{E}_K} \int_\sigma F(u, \nabla u) \cdot \mathbf{n} \, d\gamma(x) + \int_K s(u) \, dx = 0.$$

- Discrete unknowns u_K

$$u_\mathcal{D} \in H_\mathcal{D}(\Omega) : \; u_\mathcal{D} = \sum_{K \in \mathcal{T}} u_K 1_K$$

- Discretization of the fluxes $\int_\sigma F(u, \nabla u) \cdot \mathbf{n} \rightsquigarrow F_{K,\sigma}(u_\mathcal{D})$.

Example: linear convection equation

$$F(u, \nabla u) = \mathbf{v}u, v \in \mathbb{R}^d, \; s(u) = 0.$$

For $\sigma = \sigma_{KL}$, $v_{K,\sigma} = \displaystyle\int_\sigma \mathbf{v} \cdot \mathbf{n}_{K,\sigma}$,

$$F_{K,\sigma} = \begin{cases} v_{K,\sigma} \dfrac{u_K + u_L}{2} & \text{centred choice, unstable} \\ v_{K,\sigma}^+ u_K - v_{K,\sigma}^- u_L & \text{upwind choice, stable} \end{cases}$$

$$u_t + \mathrm{div}(\mathbf{v}u) = 0 \rightsquigarrow \frac{u_K^{n+1} - u_K^n}{\delta t} + \sum_{L \in \mathcal{N}_K} F_{K,\sigma} = 0$$

Example: linear convection diffusion reaction

$$F(u, \nabla u) = -\nabla u + \mathbf{v} u, \mathbf{v} \in \mathbb{R}^d, \; s(u) = bu, \; b \in \mathbb{R}.$$

$$\int_\sigma F(u) \cdot \mathbf{n}_{K,\sigma} = \int_\sigma -\nabla u \cdot \mathbf{n}_{K,\sigma} + \mathbf{v} \cdot \mathbf{n}_{K,\sigma} \; u$$

$$F_{K,\sigma} = -\frac{m_\sigma}{d_{KL}}(u_L - u_K) + v^+_{K,\sigma} u_K + v^-_{K,\sigma} u_L, \; \sigma = \sigma_{KL}$$

$$u_t - \Delta u + \mathrm{div}(\mathbf{v} u) + bu = 0 \rightsquigarrow$$

$$\frac{u_K^{n+1} - u_K^n}{\delta t} + \sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} + b m_K u_K^{n+1} = 0.$$

Consistency of the diffusive flux ?

Admissible meshes for diffusion operators

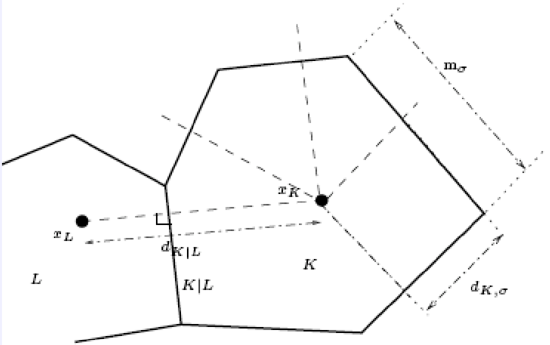

Figure 1: Notations for an admissible mesh

Examples: triangles, rectangles, Voronoï.

Weak formulation, convection diffusion, steady state

- Continuous problem:
  Find $u \in H_0^1(\Omega)$;
  $$\int_\Omega \nabla u(x) \cdot \nabla \phi(x) dx + \int_\Omega \mathrm{div}(\mathbf{v}u)(x)\phi(x)dx + \int_\Omega bu(x)\phi(x)dx = \int_\Omega f(x)\phi(x)dx, \ \forall \phi \in H_0^1(\Omega).$$
- $\mathcal{D} = (\mathcal{T}, \mathcal{E}, \mathcal{P})$: discretization of Ω
- FV scheme, weak form:
  Find $u_\mathcal{D} \in H_\mathcal{D}(\Omega)$;
  $$[u_\mathcal{D}, \phi]_\mathcal{D} + c_\mathcal{D}(u_\mathcal{D}, \phi) + \int_\Omega bu_\mathcal{D}(x)\phi(x)\, dx = \int_\Omega f(x)\phi(x)dx, \ \forall \phi \in H_\mathcal{D}(\Omega).$$
- $c_\mathcal{D}(u_\mathcal{D}, \phi) = \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_K} (v_{K,\sigma}^+ u_K - v_{K,\sigma}^- u_L)\phi_K.$

Weak formulation, convection diffusion

Inner product on $H_{\mathcal{D}}(\Omega)$ (P0 functions on the mesh)

$$[u,v]_{\mathcal{D}} = \sum_{\sigma\in\mathcal{E}_{\text{int}},\ \sigma=K|L} \frac{m_\sigma}{d_{KL}}(u_L-u_K)(v_L-v_K) + \sum_{\sigma\in\mathcal{E}_{\text{ext}},\ \sigma\in\mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}}\ u_K\ v_K$$

$$= \sum_{\sigma\in\mathcal{E}_{\text{int}},\ \sigma=K|L} m_\sigma d_{KL}\frac{u_L-u_K}{d_{KL}}\frac{v_L-v_K}{d_{KL}} + \sum_{\sigma\in\mathcal{E}_{\text{ext}},\ \sigma\in\mathcal{E}_K} m_\sigma d_{K,\sigma}\ \frac{u_K}{d_{K,\sigma}}\ \frac{v_K}{d_{K,\sigma}}$$

$m_\sigma d_{KL} = d\times$ area of "diamond cell"
(convex hull of σ, x_K, x_L)

Equivalence weak form / flux form

(WFV) $[u_{\mathcal{D}}, \phi]_{\mathcal{D}} + c_{\mathcal{D}}(u, \phi)$

$$+ \int_\Omega b u_{\mathcal{D}}(x)\phi(x)\, dx = \int_\Omega f(x)\phi(x) dx, \ \forall \phi \in H_{\mathcal{D}}(\Omega); ,$$

(FFV) $\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma} + b m_K u_K = m_K f_K.$

$(\Rightarrow)$ $\phi = 1_K$ in (WFV)

$(\Leftarrow)$ Multiply (FFV) by ϕ_K and sum over $K \in \mathcal{T}$.

Discrete norm

$u \in H_{\mathcal{D}}(\Omega)$ piecewise constant functions

$$||u||_{1,\mathcal{D}} = ([u,u]_{\mathcal{D}})^{1/2} = \Big(\sum_{\sigma \in \mathcal{E}_{\text{int}},\ \sigma = K|L} \frac{m_\sigma}{d_{KL}} (u_L - u_K)^2 + \sum_{\sigma \in \mathcal{E}_{\text{ext}},\ \sigma \in \mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}} u_K^2 \Big)^{1/2}$$

Poincaré inequality on $u \in H_{\mathcal{D}}$: $\|u\|_{L^2(\Omega)} \leq \mathrm{diam}(\Omega)\ ||u||_{1,\mathcal{D}}$

$\Rightarrow ||\cdot||_{1,\mathcal{D}}$ norm

Proof of convergence: main steps, coercive case

$$\frac{1}{2}divv + b \geq 0$$

1. A priori estimates on the approximate solution in the $H_{\mathcal{D}}$ norm and the L^2 norm :
   $\rightsquigarrow$ existence (and uniqueness) of $u_{\mathcal{D}}$ solution of the scheme.
   $\rightsquigarrow$ weak convergence in L^2, up to a subsequence, to some $\bar{u} \in L^2$.
2. Strong convergence ? $\bar{u} \in H_0^1$? :
   Consequence of the discrete Rellich theorem.
3. $\bar{u}$ weak solution of the continuous problem ?
   Passage to the limit in the scheme as $h_{\mathcal{D}} \to 0$ ($h_{\mathcal{D}}$ = mesh size).
4. $\bar{u}$ unique $\rightsquigarrow$ the whole sequence converges.
   By-product: existence of the solution to the continuous problem.

A priori estimate

- Take $\phi = u_{\mathcal{D}}$ in the scheme:

$$[u_{\mathcal{D}}, u_{\mathcal{D}}]_{\mathcal{D}} + c_{\mathcal{D}}(u_{\mathcal{D}}, u_{\mathcal{D}}) + \int_\Omega b u_{\mathcal{D}}^2(x)\, dx = \int_\Omega f(x) u_{\mathcal{D}}(x) dx.$$

- $c_{\mathcal{D}}(u_{\mathcal{D}}, u_{\mathcal{D}}) \geq 0$ (upstream choice)

$$\rightsquigarrow ||u_{\mathcal{D}}||_{1,\mathcal{D}}^2 \leq \|f\|_{L^2(\Omega)} \|u_{\mathcal{D}}\|_{L^2(\Omega)}$$

- By Poincaré,

$$||u_{\mathcal{D}}||_{1,\mathcal{D}} \leq diam(\Omega) \|f\|_{L^2(\Omega)}$$

existence of $u_{\mathcal{D}}$ and weak convergence in L^2.

Discrete Rellich theorem

- Control on the oscillations of the approximate solutions
- Estimate on the translates of $v \in H_{\mathcal{D}}(\Omega)$

$$\|v(\cdot + \eta) - v\|^2_{L^2(\Omega)} \leq |\eta| \ (|\eta| + 4h_{\mathcal{D}})) \, ||v||^2_{1,\mathcal{D}}.$$

Proof: mimics the continuous case

- Discrete Rellich : $(\mathcal{D}_n)_{n\in\mathbb{N}}$ Sequence of admissible discretizations ; $h_{\mathcal{D}_n} \to 0$. $u_n \in H_{\mathcal{D}_n}$; $\|u_n\|_{1,\mathcal{D}_n} \leq C$, there exists a subsequence $(u_n)_{n\in\mathbb{N}}$ and $\overline{u} \in H^1_0(\Omega)$ s.t. $u_n \to \overline{u}$ in $L^2(\Omega)$
  Proof: Consequence of the Kolmogorov theorem and the estimate on the translates.

Passage to the limit in the scheme

- Let $(\mathcal{D}_n)$ sequence of discretizations; $h_{\mathcal{D}_n} \to 0$
- Scheme ($b = 0$, $v = 0$):
  $$[u_{\mathcal{D}_n}, \phi]_{\mathcal{D}_n} = \int_\Omega f(x)\phi(x)\, dx, \ \forall \phi \in H_{\mathcal{D}_n}(\Omega).$$
- $\varphi \in C_c^\infty$, let $\phi = P_{\mathcal{D}_n}\varphi$:
  $P_{\mathcal{D}}$: interpolation operator. $P_{\mathcal{D}} \ : \ C(\Omega) \to H_{\mathcal{D}}(\Omega)$,
  $\varphi \mapsto P_{\mathcal{D}}\varphi$, $P_{\mathcal{D}}\varphi(x) = \varphi(x_K)$, p.p. $x \in K$, $\forall K \in \mathcal{T}$
- Lemma:
  $$[u_{\mathcal{D}_n}, P_{\mathcal{D}_n}\varphi]_{\mathcal{D}_n} \to \int_\Omega \nabla \bar{u}(x) \cdot \nabla \varphi(x) dx.$$
  (*Proof : conservativity and consistency of the flux*)
- $$\int_\Omega f(x) P_{\mathcal{D}_n}\varphi(x) dx \to \int_\Omega f(x)\varphi(x) dx.$$
- $\bar{u}$ is a weak solution, and by uniqueness: $\bar{u} = u$.

The non coercive case

$$\begin{cases} u : \Omega \to \mathbb{R} \\ -\Delta u + \mathrm{div}(\mathbf{v} u) + bu = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \qquad (P)$$

or:

$$\begin{cases} u : \Omega \to \mathbb{R} \\ -\Delta u + \mathbf{v} \cdot \nabla u + bu = f, \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \qquad (P^*)$$

Assumptions:
$f \in L^2(\Omega)$ or $H^{-1}(\Omega)$, $b \geq 0$,
$v \in C^1(\Omega)$ (no assumption on $\mathrm{div}\mathbf{v}$)
analysis also possible with non homogeneous Dirichlet, Robin, mixed conditions (open problem for Neumann conditions).

Non coercive case: technique of proof in the continuous case (Droniou, 2001)

A priori estimates on the truncates of u solution to (P) and on $\ln(1+|u|)$ in $H_0^1(\Omega)$.

Estimate on u solution to (P): $\|u\|_{H_0^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)})$

Proof of existence by topological degree for Problem (P)

A priori estimate on the solution w of (P^*), choosing as a test function a solution to (P) with r.h.s. $-\Delta w$.

Proof of existence by topological degree for Problem (P^*)

Existence and uniqueness to both (P) and (P^*) by a classical duality argument.

Non coercive case: convergence of the FV scheme

$f \in H^{-1}$ Droniou Gallouët 02, $f \in M(\Omega)$ Droniou Gallouët H. 03

$$\left\{ \begin{array}{l} u : \Omega \longrightarrow \mathbb{R} \\ -\Delta u + \mathrm{div}(\mathbf{v}u) + bu = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{array} \right. \qquad (P)$$

Upwind FV scheme $\leadsto AU = F$, with A such that $A_{i,j} \leq 0$, $A_{i,i} \geq 0$, and $A_{i,i} \geq -\sum_{k=1}^{N} A_{k,i}$.

$\leadsto A^t$ such that $A^t_{i,j} \leq 0$, $A^t_{i,i} \geq 0$, and $A^t_{i,i} \geq -\sum_{j=1}^{N} A^t_{i,j}$.
Hence $A^t v \geq 0 \Rightarrow v \geq 0$.

$\leadsto Av \geq 0 \Rightarrow v \geq 0 \leadsto$ existence and uniqueness.

Convergence: estimates with same test functions as the continuous case, weak convergence, passage to the limit in the scheme....

The parabolic case

$$\begin{cases} u : \Omega \times [0, T]; \\ u_t + \mathrm{div}(\mathbf{v}u) - \Delta u = 0 \text{ in } \Omega, \\ u(., t) = 0 \text{ on } \partial\Omega, \\ u(., 0) = u_0(x), \end{cases}$$

$u_0 \in L^2(\Omega), \mathbf{v} \in \mathbb{R}^d$

FV scheme, implicit Euler scheme in time:

$$\begin{cases} \dfrac{u_K^{n+1} - u_K^n}{\delta t} + \displaystyle\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = 0, \; n \geq 0, \\ u_K^0 = \dfrac{1}{m_K} \displaystyle\int_K u_0(x) dx. \end{cases}$$

$$F_{K,\sigma}^{n+1} = -\frac{m_\sigma}{d_{KL}} (u_L^{n+1} - u_K^{n+1}) + v_{K,\sigma}^+ u_K^{n+1} - v_{K,\sigma}^- u_L^{n+1}.$$

Estimates for the continuous parabolic problem

$$u \in L^2(0, T, H_0^1(\Omega)) \rightsquigarrow$$

$$\|u(\cdot + \eta, \cdot) - u(\cdot, \cdot)\|_{L^2(0,T,L^2(\Omega))} \leq C|\eta|$$

$$u \in L^2(0, T, H_0^1(\Omega)) \text{ and } u_t \in L^2(0, T, H^{-1}(\Omega)) \rightsquigarrow$$

$$\|u(\cdot, \cdot + \tau) - u(\cdot, \cdot)\|_{L^2(0,T,L^2(\Omega))} \leq C\tau^{\frac{1}{2}}.$$

Estimates for the discrete parabolic problem

Approximate solution $u_{\mathcal{D}} \in H_{\mathcal{D}}(\Omega \times (0, T))$ = set of piecewise constant functions on $K \times (t_n, t_{n+1})$

Estimates on $u_{\mathcal{D}}$: $\|u_{\mathcal{D}}\|_{L^\infty((0,T),L^2(\Omega))} \leq C$
and discrete version of $L^2(0, T, H_0^1(\Omega))$ estimate.

Estimate on the space translates:
$\|u_{\mathcal{D}}(\cdot + \eta, \cdot) - u_{\mathcal{D}}(\cdot, \cdot)\|_{L^2(0,T,L^2(\Omega)} \leq C\Big(|\eta|(|\eta| + h_{\mathcal{D}})\Big)^{\frac{1}{2}}$
Estimate on the time translates:
$\|u_{\mathcal{D}}(\cdot, \cdot + \tau) - u_{\mathcal{D}}(\cdot, \cdot)\|_{L^2(0,T,L^2(\Omega))} \leq C\tau^{\frac{1}{2}}$

$\rightsquigarrow$ Convergence of $u_{\mathcal{D}}$ to $\bar{u} \in L^2((0, T), H_0^1(\Omega))$ in $L^2((0, T), L^2(\Omega))$.

Passage to the limit in the scheme: $\bar{u} = u$, weak solution of the parabolic problem.

Incompressible Navier-Stokes equations

- Steady state Stokes equations: $u : \Omega \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$

$$\begin{cases} -\nu \Delta u + \nabla p = f \text{ in } \Omega, \\ \mathrm{div} u = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

- Weak formulation $E(\Omega) := \{v \in (H_0^1(\Omega))^d, \mathrm{div} v = 0\}$.

$$\begin{cases} u = (u^{(1)}, \ldots, u^{(d)})^t \in E(\Omega), \\ \nu \int_\Omega \nabla u : \nabla v \, dx = \int_\Omega f \cdot v \, dx, \quad \forall v \in E(\Omega). \end{cases}$$

with $\displaystyle\int_\Omega \nabla u : \nabla v \, dx = \sum_{i=1,d} \int_\Omega \nabla u^{(i)} \cdot \nabla v^{(i)} dx.$

Discrete divergence and gradient

- $H_{\mathcal{D}}(\Omega) \subset L^2(\Omega)$ piecewise constant functions on K cells.
- For $u \in (H^1)^d$, $\int_K \operatorname{div} u \, dx = \sum_{L \in \mathcal{N}_K} \int_{\sigma_{KL}} u \cdot \mathbf{n}_{K,\sigma_{KL}} d\gamma(x)$

  Centred discretization of $u \cdot \mathbf{n}$ on $\sigma_{KL} \rightsquigarrow$

  $$\text{For } u \in (H_{\mathcal{D}})^d, \ \operatorname{div}_{\mathcal{D}} u = \frac{1}{m_K} \sum_{L \in \mathcal{N}_K} m_{\sigma_{KL}} \mathbf{n}_{K,\sigma_{KL}} \cdot \frac{(u_K + u_L)}{2}.$$

- $$\int_\Omega \operatorname{div}_{\mathcal{D}} u \, p = - \int_\Omega u \cdot \nabla_{\mathcal{D}} p,$$

  $$\rightsquigarrow (\nabla_{\mathcal{D}} p)_K = \frac{1}{m_K} \sum_{L \in \mathcal{N}_K} m_{\sigma_{KL}} \mathbf{n}_{K,\sigma_{KL}} \frac{(p_L - p_K)}{2},$$

Properties of the discrete gradient

If $\|u^{(m)}\|_{\mathcal{D}_m} \leq C$ for all $m \in \mathbb{N}$. Then:

$\exists u^* \in H_0^1(\Omega)$ and a subsequence of $(\ u^{(m)})_{m\in\mathbb{N}}$ such that:

$u^{(m)} \to u^*$ as $m \to +\infty$ in $L^2(\Omega)$,

"Elliptic " lemma : $\forall \varphi \in C_c^\infty(\Omega)$,

$$\lim_{m\to+\infty}[u^{(m)}, P_{\mathcal{D}_m}\varphi]_{\mathcal{D}_m} = \int_\Omega \nabla u^* \cdot \nabla\varphi\, dx.$$

$\nabla_{\mathcal{D}_m} u^{(m)}$ weakly converges to ∇u^* in $L^2(\Omega)^d$ as $m \to +\infty$.

FV scheme

Colocated finite volume scheme: non coercive
$\rightsquigarrow$ penalized version

$$\begin{cases} (u,p) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega), \\ \nu[u,\phi]_{\mathcal{D}} - \displaystyle\int_\Omega p(x)\mathrm{div}_{\mathcal{D}}(\phi)(x)\,dx = \int_\Omega f(x)\cdot\phi(x)\,dx, \\ \qquad\qquad \forall \phi \in H_{\mathcal{D}}(\Omega)^d, \\ \displaystyle\int_\Omega \mathrm{div}_{\mathcal{D}}(u)(x)q(x)\,dx = -\langle p,q\rangle_{\mathcal{D},\lambda},\ \forall q \in H_{\mathcal{D}}(\Omega). \end{cases}$$

$$\langle \phi,\psi\rangle_{\mathcal{D},\lambda} = \frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}_K}\lambda_{K|L}\frac{\mathrm{m}_{K|L}}{d_{K|L}}(\phi_L-\phi_K)(\psi_L-\psi_K),$$ with

$\lambda : \mathcal{E} \to \mathbb{R}$.

Stabilization

- H^1 penalization

$$\langle \phi, \psi \rangle_{\mathcal{D},\lambda} = \frac{1}{2} \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}_K} \lambda_{K|L} \frac{\mathrm{m}_{K|L}}{d_{K|L}} (\phi_L - \phi_K)(\psi_L - \psi_K),$$

  with $\lambda : \mathcal{E} \to \mathbb{R}$.

- Stabilization *a la* Brezzi Pitkäranta:
  $\lambda_{K|L} = \beta {h_{\mathcal{D}}}^{\alpha}$, $\alpha \in (0, 2)$.
- Stabilization by clusters $\mathcal{C}_K$:

$$\lambda_{K|L} = \begin{cases} 0 & \mathcal{C}_K \neq \mathcal{C}_L, \\ \gamma, & \mathcal{C}_K = \mathcal{C}_L \end{cases}$$

  $\gamma \geq 0$.

- System equivalent to...

Penalized FV scheme

$$-\nu\left(\sum_{L\in\mathcal{N}_K}\tau_{KL}(u_L-u_K)+\sum_{\sigma\in\mathcal{E}_K\cap\mathcal{E}_{\text{ext}}}\tau_{K,\sigma}(-u_K)\right)+$$
$$\sum_{L\in\mathcal{N}_K}\frac{m_{\sigma_{KL}}\mathbf{n}_{K,\sigma_{KL}}}{2}(p_L-p_K)=\int_K f(x)\,dx,\ \forall K\in\mathcal{T},$$

$$\sum_{L\in\mathcal{N}_K}\frac{m_{\sigma_{KL}}\mathbf{n}_{K,\sigma_{KL}}}{2}\cdot(u_K+u_L)$$
$$+\sum_{L\in\mathcal{N}_K}\lambda_{KL}\tau_{KL}(p_L-p_K)=0,\ \forall K\in\mathcal{T}.$$

Estimates

- Estimate on velocity
  $\nu |u_{\mathcal{D}}|_{\mathcal{D}} \leq \mathrm{diam}(\Omega) \|f\|_{(L^2(\Omega))^d}$,
- Estimate on pressure (thanks to penalisation).
  $\nu \ |p_{\mathcal{D}}|^2_{\mathcal{D},\lambda} \leq \mathrm{diam}(\Omega) \|f\|_{(L^2(\Omega))^d}$.
- Hence existence and uniqueness of u and p.
- L^2 estimate on pressures $\|p_{\mathcal{D}}\|_{L^2(\Omega)} \leq C \|f\|_{(L^2(\Omega))^d}$
  Proof: use function $v \in H^1_0(\Omega)^d$ (Nečas) such that
  $\mathrm{div} v = p_{\mathcal{D}}$ and $\|v\|_{H^1_0(\Omega)^d} \leq C \|p_{\mathcal{D}}\|_{L^2(\Omega)}$

Passage to the limit in the scheme

- Estimates on the velocity translates
- Kolmogorov: convergence to some $\tilde{u} \in H_0^1$, up to a subsequence.
- Convergence of p to some $\tilde{p}$ weakly in L^2.
  $(\tilde{u}, \tilde{p})$ weak solution to the scheme ?
- YES ! Take $\varphi \in C_c^\infty(\Omega)$, and $v = P_{\mathcal{D}}(\varphi)$ in the scheme, and pass to the limit as $h_{\mathcal{D}}$ tends to 0, using weak consistency of the divergence (consistency of the normal fluxes) and weak convergence of the gradient.

Transient isothermal incompressible Navier Stokes

Continuous problem

$$\begin{cases} u_t - \nu \Delta u + u \cdot \nabla u + \dfrac{1}{\rho} \nabla p = f \\ \mathrm{div} u = 0, \end{cases}$$

+ B.C., + I.C. $\quad (u \cdot \nabla u)_k = \sum_{i=1,d} u^{(i)} \partial_i u^{(k)}.$

Time discretization by the Crank Nicolson scheme

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\delta t} + \nu \Delta u^{n+\frac{1}{2}} + u^{n+\frac{1}{2}} \cdot \nabla u^{n+\frac{1}{2}} + \dfrac{1}{\rho} \nabla p^{n+\frac{1}{2}} = f^{n+\frac{1}{2}} \\ \mathrm{div} u^{n+\frac{1}{2}} = 0 \end{cases}$$

with $u^{n+\frac{1}{2}} = \frac{1}{2}(u^n + u^{n+1})$, $p^{n+\frac{1}{2}} = \frac{1}{2}(p^n + p^{n+1})$

The weak continuous problem

Continuous problem

$E(\Omega) = \{v \in (H_0^1(\Omega))^d; \operatorname{div} v = 0\}$

$u \in L^2(0,T; E(\Omega)) \cap L^\infty(0,T; L^2(\Omega)^d)$,

$$\left\{ \begin{array}{l} -\int_0^T \int_\Omega u \cdot \partial_t \varphi \, dx \, dt - \int_\Omega u_0(x) \cdot \varphi(x,0) \, dx \\ +\nu \int_0^T \int_\Omega \nabla u : \nabla \varphi \, dx \, dt \; + \int_0^T \int_\Omega (u \cdot \nabla u)) \cdot \varphi) \, dx \, dt \\ \qquad = \int_0^T \int_\Omega f(x) \cdot \varphi \, dx \, dt \end{array} \right.$$

$\forall \varphi \in L^2(0,T; E(\Omega)) \cap C_c^\infty(\Omega \times (-\infty, T))^d$.

For $d = 3$, $u \in L^2(0,T; E(\Omega)) \cap L^\infty(0,T; L^2(\Omega)^d)$ yields $(u \cdot \nabla) u \in L^{4/3}(0,T,(E(\Omega))')$ so that :

$$u_t \in L^{4/3}(0,T,(E(\Omega))').$$

Finite volume scheme

Discretization of the nonlinear convection term:

$$\int_K (u \cdot \nabla) u(x,t) dx = \int_{\partial K} (u \cdot \mathbf{n}_K) u(x) d\gamma(x) =$$

$$\sum_{\sigma \in \mathcal{E}_K} \int_{\sigma_{KL}} (u \cdot \mathbf{n}_{K,\sigma}) u(x) d\gamma(x)$$

$$\rightsquigarrow \sum_{\sigma \in \mathcal{E}_K} \frac{m_{K|L} \mathbf{n}_{K,K|L}}{2} \cdot (u_K + u_L) \frac{u_K + u_L}{2}$$

(omitting the penalization terms)

$$b_{\mathcal{D}}(u,v,w) = \sum_{\sigma \in \mathcal{E}_K} \frac{m_{K|L} \mathbf{n}_{K,K|L}}{2} \cdot (u_K + u_L) \frac{v_K + v_L}{2} w_K$$

Finite volume scheme for incompressible NS

$$\begin{cases} (u_{\mathcal{D}}, p_{\mathcal{D}}) \in H_{\mathcal{D}}(\Omega)^d \times H_{\mathcal{D}}(\Omega), \\ \displaystyle\int_\Omega \frac{u_{\mathcal{D}}^{n+1} - u_{\mathcal{D}}^n}{\delta t} v \, dx + \nu [u_{\mathcal{D}}, v]_{\mathcal{D}} + b_{\mathcal{D}}(u_{\mathcal{D}}, u_{\mathcal{D}}, v) \\ \displaystyle - \int_\Omega p_{\mathcal{D}}(x) \mathrm{div}_{\mathcal{D}}(v)(x) \, dx = \int_\Omega f(x) \cdot v(x) \, dx, \forall v \in H_{\mathcal{D}}(\Omega)^d, \\ \displaystyle\int_\Omega \mathrm{div}_{\mathcal{D}}(u_{\mathcal{D}})(x) q(x) \, dx = -\langle p_{\mathcal{D}}, q \rangle_{\mathcal{D},\lambda}, \ \forall q \in H_{\mathcal{D}}(\Omega). \end{cases}$$

(All non specified $u_{\mathcal{D}}$ and $p_{\mathcal{D}}$ are $u_{\mathcal{D}}^{n+\frac{1}{2}}$ and $p_{\mathcal{D}}^{n+\frac{1}{2}}$).

$$u \in L^2(0, T, E(\Omega)) \rightsquigarrow$$

$$\|u(\cdot + \eta, \cdot) - u(\cdot, \cdot)\|_{L^2(0,T,(L^2(\Omega))^3)} \leq C\eta$$

$$E(\Omega) = \{v \in (H_0^1(\Omega))^3; \mathrm{div}\, v = 0\}$$

$$u \in L^2(0, T, E(\Omega)) \text{ and } u_t \in L^{\frac{4}{3}}(0, T, (E(\Omega))') \rightsquigarrow$$

$$\|u(\cdot, \cdot + \tau) - u(\cdot, \cdot)\|_{L^{\frac{4}{3}}(0,T,(L^2(\Omega))^3))} \leq C\tau^{\frac{1}{2}}.$$

A simple continuous estimate

$$u \in L^2(0,T,E(\Omega)) \text{ and } u_t \in L^1(0,T,(E(\Omega))') \rightsquigarrow$$

$$\|u(\cdot,\cdot+\tau) - u(\cdot,\cdot)\|_{L^1(0,T,(L^2(\Omega))^3))} \le C\tau^{\frac{1}{2}}.$$

BUT NO $L^2(0,T,(L^2(\Omega)^3)$ estimate.

Estimates for the discrete NS problem

Approximate solution $u_{\mathcal{D}} \in H_{\mathcal{D}}(\Omega \times (0,T))$ = set of piecewise constant functions on $K \times (t_n, t_{n+1})$

Estimate on $u_{\mathcal{D}}$: $\|u_{\mathcal{D}}\|_{L^\infty((0,T),(L^2(\Omega))^d)} \leq C$, $\|u_{\mathcal{D}}\|_{L^2((0,T),H_{\mathcal{D}}(\Omega))} \leq C$

Estimate on the space translates:

$\|u_{\mathcal{D}}(\cdot+\eta,\cdot) - u_{\mathcal{D}}(\cdot,\cdot)\|_{L^2((0,T,(L^2(\Omega)^d)} \leq C\big(|\eta|(|\eta|+h_{\mathcal{D}})\big)^{\frac{1}{2}}$

Estimate on the time translates:

$\|u_{\mathcal{D}}(\cdot,\cdot+\tau) - u_{\mathcal{D}}(\cdot,\cdot)\|_{L^1(0,T,(L^2(\Omega))^d)} \leq C\tau^{\frac{1}{2}}$

(Estimate in $L^{\frac{4}{3}}(0,T,(L^2(\Omega))^d)$ possible, but not $L^2(0,T,(L^2(\Omega))^d)$)

Convergence Navier-Stokes

$\rightsquigarrow$ Convergence of $u_{\mathcal{D}}$ to $\bar{u} \in L^2((0,T), E(\Omega))$ in $L^1((0,T), L^2(\Omega))$.

Passage to the limit in the scheme: $\bar{u} = u$, weak solution of the NS equations.

The semilinear parabolic case

$$\begin{cases} u : \Omega \times [0, T]; \\ u_t + \mathrm{div}(\mathbf{v} f(u)) - \varepsilon \Delta u = 0 \text{ in } \Omega, \\ u(.,t) = 0 \text{ on } \partial\Omega, \\ u(.,0) = u_0(x), \end{cases}$$

$u_0 \in L^\infty(\Omega), \mathbf{v} \in \mathbb{R}^d$, $f \in C^1(\mathbb{R}, \mathbb{R})$, $f' \geq 0$ (for simplicity)

FV scheme

$$\begin{cases} \dfrac{u_K^{n+1} - u_K^n}{\delta t} + \displaystyle\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^{n+1} = 0, \; n \geq 0, \\ u_K^0 = \dfrac{1}{m_K} \displaystyle\int_K u_0(x) dx. \end{cases}$$

$$F_{K,\sigma}^{n+1} = -\varepsilon \frac{m_\sigma}{d_{KL}} (u_L^{n+1} - u_K^{n+1}) + v_{K,\sigma}^+ f(u_K^{n+1}) + v_{K,\sigma}^- f(u_L^{n+1}).$$

Estimates for the discrete semilinear problem

Estimate on $u_{\mathcal{D}}$: $\|u_{\mathcal{D}}\|_{L^\infty((0,T),L^\infty(\Omega))} \rightsquigarrow$ weak $\star$ convergence (up to a subsequence) to $\bar{u}$

Estimate on the space translates:
$\varepsilon \|u_{\mathcal{D}}(\cdot + \eta, \cdot) - u_{\mathcal{D}}(\cdot, \cdot)\|^2_{L^2(\Omega\times(0,T)))} \leq C\eta(\eta + h_{\mathcal{D}})$

Estimate on the time translates:
$\varepsilon \|u_{\mathcal{D}}(\cdot, \cdot + \tau) - u_{\mathcal{D}}(\cdot, \cdot)\|^2_{L^2(\Omega\times(0,T)))} \leq C\tau$

$\rightsquigarrow$ Convergence of $u_{\mathcal{D}}$ to $\bar{u} \in L^2((0,T), H^1_0(\Omega))$ in $L^2((0,T), L^2(\Omega))$.

Passage to the limit in the scheme: $\bar{u} = u$, weak solution of the parabolic problem.

The hyperbolic case ($\varepsilon = 0$)

$$\begin{cases} u_t + \mathrm{div}(\mathbf{v} f(u)) = 0 \text{ in } \Omega, \\ u(\cdot, t) = 0, \text{ on } \{ x \in \partial\Omega, \mathbf{v} \cdot \mathbf{n}(x) \geq 0 \}, \\ u(\cdot, 0) = u_0, \end{cases}$$

$u_0 \in L^\infty(\Omega), \mathbf{v} \in \mathbb{R}^d$, $f \in C^1(\mathbb{R}, \mathbb{R})$, $f' \geq 0$ (for simplicity)

Estimate on $u_\mathcal{D}$: $\|u_\mathcal{D}\|_{L^\infty((0,T),L^\infty(\Omega))}$

$\rightsquigarrow$ weak $\star$ convergence (up to a subsequence) to $\bar{u}$ in $L^\infty((0,T), L^\infty(\Omega))$

not sufficient to pass to the limit in the scheme (even in the linear case).

Weak-BV inequality (Champier Gallouët 90):

$$\sum_{(K,L)\in\mathcal{E}} |v_{K,\sigma}|(f(u_K) - f(u_L))^2 \leq C.$$

Does not yield strong compactness.

Weak BV estimate

$$\sum_{(K,L)\in\mathcal{E}} |v_{K,\sigma}|(f(u_K) - f(u_L))^2 \le C.$$

Upwinding on u adds a diffusion term ($\mathbf{v} = (v_1, \ldots, v_d)^t$):

$$u_t + \mathrm{div}(\mathbf{v} f(u)) - h_{\mathcal{D}} \sum_{i=1}^{d} D_i(|v_i||f'(u)|D_i u) = 0$$

Weak BV inequality $\sim$ weak H^1 inequality:

$$\sum_{i=1}^{d} \|v_i f'(u) D_i u\|_{L^2} \le \frac{1}{\sqrt{h_{\mathcal{D}}}}.$$

Nonlinear weak $\star$ convergence, Young measures

$L^\infty(\Omega \times (0,T))$-estimate on $u_{\mathcal{D}} \rightsquigarrow \exists\, \bar{u} \in L^\infty(\Omega \times (0,T) \times (0,1))$ such that $u_{\mathcal{D}} \to \bar{u}$, as $h_{\mathcal{D}} \to 0$ (up to a subsequence) in the following sense:

$$\int_\Omega g(u_{\mathcal{D}}(x))\varphi(x)dx \to \int_0^1 \int_\Omega g(\bar{u}(x,\alpha))\varphi(x)dx\,d\alpha,$$

for all $\varphi \in L^1(\Omega \times (0,T))$ and all $g \in C(\mathbb{R},\mathbb{R})$,

that is:

$$g(u_{\mathcal{D}}) \to \int_0^1 g(\bar{u}(\cdot,\alpha))d\alpha,\ L^\infty(\Omega \times (0,T)) \text{ weak-}\star.$$

($\int_0^1 g(\bar{u}(x,\alpha))d\alpha = \int_{\mathbb{R}} g(s)d\nu_x(s)$, ν_x is a probability on $\mathbb{R}$)

Convergence to the entropy weak solution

1. $\bar{u}$ is an entropy weak process solution of

$$\begin{cases} u_t + \mathrm{div}(\mathbf{v} f(u)) = 0 \text{ in } \Omega, \\ u(\cdot, t) = 0, \text{ on } \{\, x \in \partial\Omega, \mathbf{v} \cdot \mathbf{n}(x) \geq 0\}, \\ u(\cdot, 0) = u_0, \end{cases}$$

2. If $\bar{u} \in L^\infty(\Omega \times (0, T) \times (0, 1))$, is an entropy weak process solution then:
   - $\bar{u}(x, \alpha)$ does not depends on α.
   - $\bar{u}$ is the unique entropy weak solution u.

   The proof uses the doubling variables method of Krushkov.
3. $u_{\mathcal{D}}$ converges to u in $(L^p(\Omega \times (0, T)))$ for all $p < \infty$.

Conclusions

- Cell centred finite volumes widely used in industrial codes:
  - addresses coupled problems (e.g. parabolic - hyperbolic - algebraic)
  - produces ”user friendly” codes, even for complex applications
- Analysis of the cell centred finite volume scheme presented here for
  - Elliptic problems with Dirichlet boundary conditions
  - Parabolic problems
  - Non linear hyperbolic equations
  - Transient incompressible Navier-Stokes

Conclusions

Analysis also exists for:

- Elliptic or parabolic problems with general boundary conditions (Bradji, Gallouët Herbin Vignal)
- Elliptic or parabolic problems with L^1 or measure data (Droniou Gallouët Herbin)
- Corner singularities (Djadel Nicaise)
- Nonlinear reaction diffusion equations (Eymard Gallouët, Hilhorst)
- Variational inequalities (Herbin, Marchand)
- Degenerate parabolic equations (Eymard, Gallouët , Herbin, Hilhorst ,Michel...)
- Hyperbolic equations with discontinuous fluxes (Bachmann, Gallouët, Vovelle)

Similar tools also used for

- A posteriori estimates and mesh adaptation (Kröner, Ohlberger)
- Numerical homogeneisation (Eymard, Gallouët)

Ongoing work and perspectives

- Anisotropic diffusion problems (Domelevo Omnes, Le Potier, Andreianov, Boyer, Hubert, , Droniou, Eymard Gallouët Herbin)
- Coupled systems: Ohmic losses (Bradji Herbin) semi-conductors (Chainais-Hillairet)
- Compressible Navier-Stokes equations (Eymard Herbin)
- Two phase flow in porous media, heteregeneous media (Enchéry, Cancès)
- Image processing (Mikula)
- Hyperbolic systems (Gallouët Hérard Seguin)
- . . .

Weak entropy solution and weak entropy process solution

- Weak entropy solution

  $u \in L^\infty(\Omega \times \mathbb{R}_+)$

  $$\int_{\mathbb{R}_+} \int_\Omega (\eta(u)\varphi_t + \Phi(u) \cdot \nabla\varphi)\, dx\, dt + \int_{\mathbb{R}_+} \eta(u_0(x))\varphi(x) dx + \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, \Phi(0)\, \varphi\, d\gamma(x) \geq 0,$$

  for all $\eta \in C^2(\mathbb{R})$, Φ s.t. $\Phi' = f'\eta'$, $\forall \varphi \in C_c^\infty([0, T[, \mathbb{R}_+)$ s.t. $\varphi = 0$ on $\{\mathbf{v} \cdot \mathbf{n} \geq 0\}$.

- Weak entropy process solution

  $u \in L^\infty(\Omega \times \mathbb{R}_+ \times (0, 1))$,

  $$\int_0^1 \int_{\mathbb{R}_+} \int_\Omega (\eta(u)\varphi_t + \Phi(u) \cdot \nabla\varphi\, dx\, dt d\alpha + \int_{\mathbb{R}_+} \eta(u_0(x))\, \varphi(x) dx + \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n}\, \Phi(0)\, \varphi\, d\gamma(x) \geq 0,$$

Proof of the lemma

- $$X_m = [u_m, P_{\mathcal{D}_m}\varphi]_{\mathcal{D}_m} = \sum_{\sigma\in\mathcal{E}_{\text{int}},\ \sigma=K|L} \frac{m_\sigma}{d_{KL}}(u_L - u_K)(\varphi_L - \varphi_K) + \sum_{\sigma\in\mathcal{E}_{\text{ext}},\ \sigma\in\mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}}\, u_K\, \varphi_K$$
- $\varphi \in C_c^\infty \Rightarrow m$ large enough so that $\varphi_K = 0$ if K is a neighbor to the edge

$$X_m = \sum_{\sigma\in\mathcal{E}_{\text{int}},\ \sigma=K|L} \frac{m_\sigma}{d_{KL}}(u_L - u_K)(\varphi_L - \varphi_K)$$

- Reordering:

$$X_m = \sum_{K\in\mathcal{T}} u_K \sum_{\sigma\in\mathcal{E}_K} F^*_{K,\sigma}(\varphi),\ \text{with } F^*_{K,\sigma}(\varphi) = \frac{m_\sigma}{d_{KL}}(\varphi_L - \varphi_K).$$

Proof of Lemma, sequel

$$F^*_{K,\sigma}(\varphi) = \int_{\partial K} -\nabla\varphi \cdot \mathbf{n}_{K,\sigma} d\gamma - m_\sigma R_{K,\sigma}(\varphi).$$

Consistency: If $\varphi \in C^2(\Omega)$, or $\varphi \in H^2(\Omega)$, $|R_{K,\sigma}| \leq C_\varphi h_{\mathcal{D}}$.

$$X_m = \sum_{K \in \mathcal{T}} u_K \sum_{\sigma \in \mathcal{E}_K} F^*_{K,\sigma}(\varphi) = Y_m + Z_m,$$

$$Y_m = -\sum_{K \in \mathcal{T}} u_K \int_{\partial K} \nabla\varphi \cdot \mathbf{n} d\gamma = -\int_\Omega u_{\mathcal{D}}(x) \Delta\varphi(x) dx$$

$$\to \int_\Omega u(x)\Delta\varphi(x)dx = \int_\Omega \nabla u(x) \cdot \nabla\varphi(x) dx \text{ as } h_{\mathcal{D}_m} \to 0.$$

$$Z_m = -\sum_{K \in \mathcal{T}} u_K \sum_{\sigma \in \mathcal{E}_K} m_\sigma R_{K,\sigma}(\varphi), \text{ and } |R_\sigma(\varphi)| \leq C_\varphi h_{\mathcal{D}}.$$

Proof of Lemma, end

$$Z_m = -\sum_{K\in\mathcal{T}} u_K \sum_{\sigma\in\mathcal{E}_K} m_\sigma R_{K,\sigma}(\varphi)$$

Conservativity: if $\sigma = \sigma_{KL}$, then $R_{K,\sigma}(\varphi) = -R_{L,\sigma}(\varphi)$.

$$|Z_m| \le \sum_{\sigma=K|L} m_\sigma |R_{K,\sigma}|\,|u_K - u_L|$$

$$|Z_m|^2 \le C_u h_{\mathcal{D}} \sum_{\sigma=K|L} d_{KL} m_\sigma \sum_{\sigma=K|L} \frac{m_\sigma}{d_{KL}} |u_K - u_L|^2$$

$$\le C_u h_{\mathcal{D}}\, d|\Omega|\, diam(\Omega) \|f\|_{L2(\Omega)}.$$

$Z_m \to 0$. ; Hence $\lim_{m\to\infty} [u_m, P_{\mathcal{D}_m}\varphi]_{\mathcal{D}_m}$.