Discontinuous Galerkin Methods for Anisotropic and Semi-Definite Diffusion with Advection

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Introduction

- We consider advection-diffusion-reaction problems with
  - discontinuous
  - anisotropic
  - semi-definite diffusivity
- The mathematical nature of the problem may not be uniform over the domain
- Indeed, because of anisotropy, the problem may be hyperbolic in one direction and elliptic in another
- The solution may be discontinuous across elliptic-hyperbolic interfaces

Model Problem

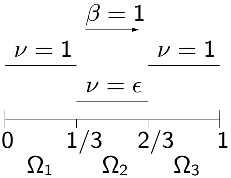
- $\Omega \subset \mathbb{R}^d$ bounded, open and connected Lipschitz domain
- $P_\Omega \stackrel{\text{def}}{=} \{\Omega_i\}_{i=1}^N$ partition of Ω into Lipschitz connected subdomains
- Consider the following problem:

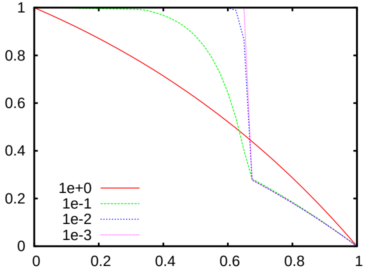
$$\nabla\cdot(-\nu\nabla u + \beta u) + \mu u = f$$

  - $\nu \in [L^\infty(\Omega)]^{d,d}$ symmetric piecewise constant on P_Ω is s.t. $\nu \geq 0$
  - $\beta \in [\mathfrak{C}^1(\overline{\Omega})]^d$
  - $\mu \in L^\infty(\Omega)$ is s.t. $\mu + \frac{1}{2}\nabla\cdot\beta \geq \mu_0$ with $\mu_0 > 0$

A One-Dimensional Example

$$\begin{cases} (-\nu u_\epsilon' + u_\epsilon)' = 0, & \text{in } (0,1), \\ u_\epsilon(0) = 1, \\ u_\epsilon(1) = 0. \end{cases}$$





$\lim_{\epsilon\to 0} u_\epsilon = \mathbb{I}_{\Omega_1 \cup \Omega_2}(x) + 3(x-1)\,\mathbb{I}_{\Omega_3}(x)$, discontinuous at $x = 2/3$

Goals

- At the continuous level, design suitable interface and BC's to define a well-posed problem
- At the discrete level, design a DG method that
  - does not require the *a priori* knowledge of the elliptic-hyperbolic interface
  - yields optimal error estimates in mesh-size that are robust w.r.t. anisotropy and semi-definiteness of diffusivity

Outline

Interface Conditions I

- Let
$$\Gamma \stackrel{\text{def}}{=} \{x \in \Omega;\ \exists \Omega_{i_1},\ \Omega_{i_2} \in P_\Omega,\ x \in \partial\Omega_{i_1} \cap \partial\Omega_{i_2}\},$$
where i_1 and i_2 are s.t. $\boxed{(n^t \nu n)|_{\Omega_{i_1}} \geq (n^t \nu n)|_{\Omega_{i_2}}}$
- We define the elliptic-hyperbolic interface as
$$I \stackrel{\text{def}}{=} \{x \in \Gamma;\ \ (n^t \nu n)(x)|_{\Omega_{i_1}} > 0,\ (n^t \nu n)(x)|_{\Omega_{i_2}} = 0\ \}$$
- Set, moreover,
$$I^+ \stackrel{\text{def}}{=} \{x \in I;\ \beta \cdot n_1 > 0\}, \quad I^- \stackrel{\text{def}}{=} \{x \in I;\ \beta \cdot n_1 < 0\}$$

Interface Conditions II

- For all scalar φ with a (possibly two-valued) trace on Γ, define

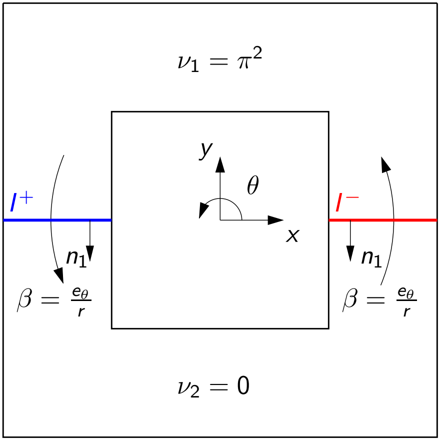
$$\{\varphi\} \stackrel{\text{def}}{=} \frac{1}{2}(\varphi|_{\Omega_{i_1}} + \varphi|_{\Omega_{i_2}}), \quad [\![\varphi]\!] \stackrel{\text{def}}{=} \varphi|_{\Omega_{i_1}} - \varphi|_{\Omega_{i_2}}$$

- We require that

$$\boxed{[\![u]\!] = 0, \text{ on } I^+} \qquad (E \longrightarrow H)$$

- Observe that continuity is not enforced on I^-
- When ν is isotropic the above conditions coincide with those derived in [Gastaldi and Quarteroni, 1989] in the one-dimensional case

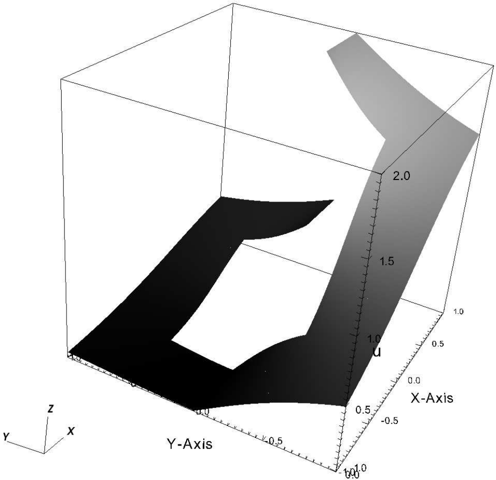
A Two-Dimensional Exact Solution I



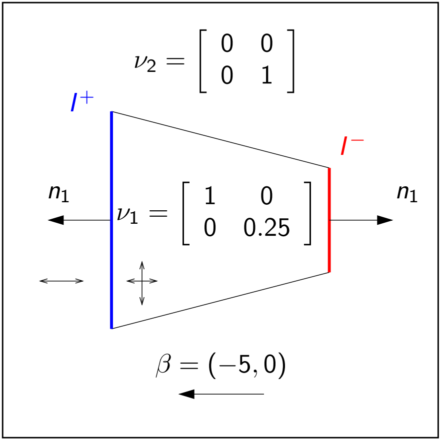
For a suitable rhs,

$$u = \begin{cases} (\theta - \pi)^2, & \text{if } 0 \le \theta \le \pi, \\ 3\pi(\theta - \pi), & \text{if } \pi < \theta < 2\pi. \end{cases}$$

A Two-Dimensional Exact Solution II



An Example with Strongly Anisotropic Diffusivity



Friedrichs-Like Mixed Formulation I

- We want to reformulate the problem so as to recover the symmetry and dissipativity (L-coercivity) properties of Friedrichs systems [Friedrichs, 1958]
- The problem in symmetric mixed formulation reads

$$\begin{cases} \sigma + \kappa \nabla u = 0, & \text{in } \Omega \setminus I, \\ \nabla\cdot(\kappa\sigma + \beta u) + \mu u = 0, & \text{in } \Omega, \end{cases} \qquad \text{(mixed)}$$

  where $\kappa \stackrel{\text{def}}{=} \nu^{1/2}$
- For $y = (y^\sigma, y^u)$, the advective-diffusive flux is defined as

$$\Phi(y) \stackrel{\text{def}}{=} \kappa y^\sigma + \beta y^u$$

Friedrichs-Like Mixed Formulation II

- The graph space is

$$W \stackrel{\text{def}}{=} \{y \in L;\quad \kappa\nabla y^u \in L_\sigma \text{ and } \nabla\cdot\Phi(y) \in L_u\ \}$$

with

$$L_\sigma \stackrel{\text{def}}{=} [L^2(\Omega \setminus I)]^d \quad L_u \stackrel{\text{def}}{=} L^2(\Omega) \quad L \stackrel{\text{def}}{=} L_\sigma \times L_u$$

- The space choice together with condition $(E \to H)$ yields

$$\boxed{\begin{aligned} &\{\Phi(z)\cdot n\} = 0, &&\text{on } \Gamma,\\ &[\![z^u]\!] = 0, &&\text{on } \Gamma \setminus I^-. \end{aligned}} \qquad (\text{cond. } \Gamma)$$

Friedrichs-Like Mixed Formulation III

- Define the zero- and first-order operators

$$\mathcal{L}(L;L) \ni K : z \mapsto (z^\sigma, \mu z^u)$$
$$\mathcal{L}(W;L) \ni A : z \mapsto (\kappa \nabla z^u, \nabla \cdot \Phi(z))$$

- The bilinear form

$$\boxed{a_0(z,y) \stackrel{\text{def}}{=} ((K+A)z, y)_L + \int_{I^+} (\beta \cdot n_1)[\![z^u]\!][\![y^u]\!]}$$

  is *L*-coercive whenever z and y are compactly supported
- a_0 will serve as a base for the construction of a weak problem with boundary and interface conditions weakly enforced

Boundary Conditions Weakly Enforced I

- Define the operators M and D s.t., for all $z,\ y \in W \times W$

$$\langle Dz, y\rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{D} z, \quad \langle Mz, y\rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{M} z,$$

where, for $\alpha \in \{-1, +1\}$,

$$\mathcal{D} = \left[\begin{array}{c:c} 0 & \kappa n \\ \hdashline (\kappa n)^t & \beta\cdot n \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{c:c} 0 & -\alpha\kappa n \\ \hdashline \alpha(\kappa n)^t & |\beta\cdot n| \end{array}\right]$$

- Observe that $M \geq 0$

Boundary Conditions Weakly Enforced II

$$a(z,y) \stackrel{\text{def}}{=} \underbrace{((K+A)z,y)_L + \int_{I^+} (\beta\cdot n_1)[\![z^u]\!][\![y^u]\!]}_{a_0(z,y)} + \tfrac{1}{2}\langle (M-D)(z), y\rangle_{W',W}$$

- a is L-coercive on W
- Let

$$\partial\Omega_E \stackrel{\text{def}}{=} \{x\in\partial\Omega;\ (n^t\nu n)(x)>0\},\quad \partial\Omega_H \stackrel{\text{def}}{=} \partial\Omega\setminus\partial\Omega_E.$$

Then

- $\boxed{\alpha=+1}$ Dirichlet on $\partial\Omega_E$/inflow on $\partial\Omega_H$ in $\mathrm{Ker}(M-D)$
- $\boxed{\alpha=-1}$ Neumann-Robin on $\partial\Omega_E$/inflow on $\partial\Omega_H$ in $\mathrm{Ker}(M-D)$

Main Result

Theorem

Let $f \in L_u$. Consider the problem

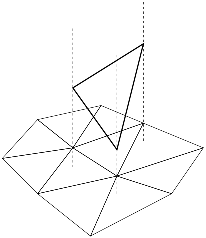
$$\begin{cases} \text{Find } z \in W \text{ such that, for all } y \in W, \\ a(z,y) = (f, y^u)_{L_u} \end{cases} \qquad \text{(weak)}$$

Then, (weak) is well-posed and its solution

- *solves (mixed) with BC's $(M - D)(z)|_{\partial\Omega} = 0$;*
- *satisfies interface conditions (cond. Γ)*

DG approximation I

- Discontinuous Galerkin methods rely on a piecewise fully discontinuous approximation



- To some extent, they can be seen as an extension of FV methods
- Their analysis can be performed exploiting many classical results valid for continuous Galerkin FE approximations

DG approximation II

- Pros
  - Discontinuous solutions are naturally handled so long as the discontinuities are aligned with the mesh
  - Convergence estimates only depend on local Sobolev regularity inside each element (high-order convergence even for poorly regular solutions)
  - There is great freedom in the choice of bases and of element shapes
  - *hp*-adaptivity can be easily implemented
  - Non-matching grids allowed
- Cons
  - High(er) computational cost

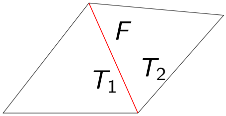
The discrete setting I

- Let $\{\mathcal{T}_h\}_{h>0}$ be a family of affine meshes of Ω compatible with P_Ω
- $\mathcal{F}_h^i$ will denote the set of interfaces, $\mathcal{F}_h^\partial$ the set of boundary faces and $\mathcal{F}_h \stackrel{\text{def}}{=} \mathcal{F}_h^i \cup \mathcal{F}_h^\partial$
- The discontinuous finite element space on $\mathcal{T}_h$ is defined as follows:

$$P_{h,p} \stackrel{\text{def}}{=} \{v_h \in L^2(\Omega);\ \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_p(T)\}$$

- We assume that mesh regularity and usual inverse and trace inequalities hold

The discrete setting II



- For all $\mathcal{F}_h^i \ni F = \partial T_1 \cap \partial T_2$ we define

$$\lambda_i \stackrel{\text{def}}{=} \sqrt{n^t \nu n|_{T_i}} \quad i \in \{1, 2\},$$

and, without loss of generality, we assume that $\boxed{\lambda_1 \geq \lambda_2}$

- Similarly, for $F \in \mathcal{F}_h^\partial$

$$\lambda \stackrel{\text{def}}{=} \sqrt{n^t \nu n}$$

- Observe that the discrete counterpart of $I^\pm$ do not need to be identified

Weighted Trace Operators

- For all $F \in \mathcal{F}_h^i$, let ω be a weight function s.t.

$$[L^2(F)]^2 \ni \omega = (\omega_1, \omega_2) \quad \omega_1 + \omega_2 = 1 \text{ for a.e. } x \in F$$

- For all $\mathcal{F}_h^i \ni F = \partial T_1 \cap \partial T_2$, for a.e. $x \in F$, set

$$\boxed{\{\varphi\}_\omega \stackrel{\text{def}}{=} \omega_1 \varphi|_{T_1} + \omega_2 \varphi|_{T_2} \quad [\![\varphi]\!]_\omega \stackrel{\text{def}}{=} 2\left(\omega_2 \varphi|_{T_1} - \omega_1 \varphi|_{T_2}\right)}$$

- When $\omega = (\frac{1}{2}, \frac{1}{2})$, the usual average and jump operators are recovered and subscripts are omitted

Generalities

The bilinear form a_h associated to a DG method for a linear PDE problem can be written as

$$a_h(u,v) = a_h^V(u,v) + a_h^i(u,v) + a_h^\partial(u,v)$$

where

- a_h^V corresponds to the standard Galerkin terms
- a_h^i contains interface terms intended
  - to penalize the non-conforming discrete components
  - to ensure the consistency of the method
- a_h^∂ collects boundary terms used to weakly enforce boundary conditions

Design Constraints

(C1) The bilinear form a_h is L-coercive and strongly consistent
(C2) The elliptic-hyperbolic interfaces are not identified a priori, but an automatic detection mechanism is devised instead
(C3) Suitable stabilizing terms are incorporated to control the fluxes

Design of the DG Bilinear Form I

Let S_F and M_F be two operators s.t.

$$\forall F \in \mathcal{F}_h^i, \quad S_F \geq 0,$$

$$\forall F \in \mathcal{F}_h^\partial, \quad M_F = \left[\begin{array}{c|c} 0 & -\alpha \kappa n_F \\ \hline \alpha(\kappa n_F)^t & M_F^{uu} \end{array}\right] \text{ and } M_F^{uu} \geq 0,$$

with associated seminorms $|\cdot|_M$ and $|\cdot|_J$ and consider

$$\begin{aligned} a_h(z,y) \stackrel{\text{def}}{=} &\sum_{T\in\mathcal{T}_h} \left[(Kz,y)_{L,T} + (Az,y)_{L,T}\right] \\ &- 2\sum_{F\in\mathcal{F}_h^i} (\{\Phi(z)\cdot n\},\{y^u\}_\omega)_{L_u,F} + ([\![z^u]\!], \tfrac{1}{4}[\![\Phi(y)\cdot n]\!]_\omega - \tfrac{\beta\cdot n_1}{2}\{y^u\})_{L_u,F} \\ &+ \sum_{F\in\mathcal{F}_h^i} (S_F([\![z^u]\!]),[\![y^u]\!])_{L,F} + \frac{1}{2}\sum_{F\in\mathcal{F}_h^\partial} ((M_F - \mathcal{D})z,y)_{L,F} \end{aligned}$$

Design of the DG Bilinear Form II

- We propose the following choices

$$\forall F \in \mathcal{F}_h^i, \quad \omega = \begin{cases} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}, \frac{\lambda_2}{\lambda_1+\lambda_2}\right), & \text{if } \lambda_1 > 0, \\ \left(\frac{1}{2}, \frac{1}{2}\right), & \text{otherwise} \end{cases}$$

$$M_F^{uu} \stackrel{\text{def}}{=} \frac{|\beta \cdot n|}{2} + \frac{\alpha+1}{2} \frac{\lambda^2}{h_F}, \qquad S_F \stackrel{\text{def}}{=} \frac{|\beta \cdot n|}{2} + \frac{{\lambda_2}^2}{h_F}$$

where by definition, $\lambda_2 = \min(\lambda_1, \lambda_2)$

- Then,

(i) a_h is *L*-coercive, i.e., for all y in $W(h)$, uniformly in h and κ,

$$a_h(y, y) \gtrsim \|y\|_L^2 + |y^u|_J^2 + |y^u|_M^2$$

(ii) a_h is strongly consistent

$$\forall y_h \in W_h, \quad a_h(z, y_h) = (f, y_h^u)_{L_u}$$

Basic Error Estimates

- The discrete problem is

$$\begin{cases} \text{Seek } z_h \in W_h \text{ such that} \\ a_h(z_h, y_h) = (f, y_h^u)_{L_u} \quad \forall y_h \in W_h \end{cases}$$

with $W_h = [P_{h,p_\sigma}]^d \times P_{h,p_u}$ and $p_u - 1 \leq p_\sigma$

- Define the natural energy norm

$$\|y\|_{h,\kappa}^2 \stackrel{\text{def}}{=} \|y\|_L^2 + |y^u|_J^2 + |y^u|_M^2 + \sum_{T \in \mathcal{T}_h} \|\kappa \nabla y^u\|_{L_\sigma, T}^2$$

- The main result, holding uniformly in κ, reads

$$\boxed{\|z - z_h\|_{h,\kappa} \lesssim h^{p_u} \|z\|_{[H^{p_\sigma+1}(\mathcal{T}_h)]^d \times H^{p_u+1}(\mathcal{T}_h)}}$$

Improved Convergence Estimates

- If the problem is uniformly elliptic,

$$\|z^u - z_h^u\|_{L_u} \lesssim h^{p_u+1} \|z\|_{[H^{p_\sigma+1}(\mathcal{T}_h)]^d \times H^{p_u+1}(\mathcal{T}_h)}$$

- If κ is isotropic,

$$\|z^u - z_h^u\|_{h,\beta} \stackrel{\text{def}}{=} \left(\sum_{T \in \mathcal{T}_h} h_T \|\beta \cdot \nabla (z^u - z_h^u)\|_{L_u,T}^2 \right)^{\frac{1}{2}} \lesssim h^{p_u} \left(h^{\frac{1}{2}} + \|\nu\|_{[L^\infty(\Omega)]^{d,d}} \right) \|z\|_{[H^{p_\sigma+1}(\mathcal{T}_h)]^d \times H^{p_u+1}(\mathcal{T}_h)}$$

Flux Formulation I

- Following engineering practice, the discrete problem can be equivalently formulated in terms of local problems
- For all $T \in \mathcal{T}_h$, for all $q^\sigma \in [\mathbb{P}_{p_\sigma}(T)]^d$,

$$(z_h^\sigma, q^\sigma)_{L_\sigma,T} - (z_h^u, \nabla\cdot(\kappa q^\sigma))_{L_u,T} + (\phi^\sigma(z_h^u), q^\sigma)_{L_\sigma,\partial T} = 0$$

- For all $T \in \mathcal{T}_h$, for all $q^u \in \mathbb{P}_{p_u}(T)$,

$$(\mu z_h^u, q^u)_{L_u,T} - (z_h^u, \beta\cdot\nabla q^u)_{L_u,T} - (z_h^\sigma, \kappa\nabla q^u))_{L_\sigma,T} + (\phi^u(z_h^\sigma, z_h^u), q^u)_{L_\sigma,\partial T} = (f, q^u)_{L_u,T}$$

Flux Formulation II

- For all $\mathcal{F}_h^i \ni F \subset \partial T$,

$$\begin{aligned}
\phi^u(z_h^\sigma, z_h^u) &= n_T^t\{\kappa z_h^\sigma\}_{\overline{\omega}} + (\beta\cdot n_T)\{z_h^u\} + (n_T\cdot n_F)S_F([\![z_h^u]\!]) \\
\phi^\sigma(z_h^u) &= (\kappa|_T n_T)\{z_h^u\}_{\overline{\omega}}
\end{aligned}$$

  with $\overline{\omega} \stackrel{\text{def}}{=} (1,1) - \omega$
- Similar expressions are obtained at boundary faces
- Note that ϕ^σ only depends on z_h^u, which allows the local elimination of z_h^σ

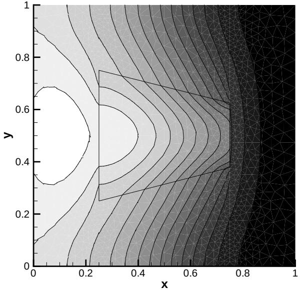
Increasing Computational Efficiency

- The σ-component of the unknown can be eliminated by solving reduced-size local problems.
- As a consequence, we end up with a discrete primal problem where the sole u-component of the unknown appears.
- The stencil of the local problems can be further reduced by devising variants of the method that take inspiration from [Baker, 1977, Arnold, 1982] and [Bassi et al., 1997].
- The primal formulation of the DG method was used in all the numerical test cases discussed below.
- Further details can be found in [Di Pietro et al., 2006].

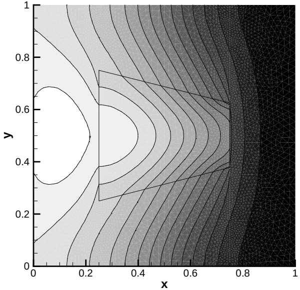
Convergence Results (Two-Dimensional Exact Solution)

h	$P_{h,1}$		$P_{h,2}$		$P_{h,3}$		$P_{h,4}$	
	err	ord	err	ord	err	ord	err	ord
$\|\|u - u_h\|\|_{h,\kappa}$								
1/2	3.15e+0		7.27e−1		1.74e−1		3.99e−2	
1/4	1.63e+0	0.95	2.05e−1	1.83	2.69e−2	2.70	3.51e−3	3.51
1/8	8.19e−1	0.99	5.32e−2	1.94	3.59e−3	2.91	2.51e−4	3.81
1/16	4.08e−1	1.00	1.34e−2	1.99	4.54e−4	2.98	1.63e−5	3.95
1/32	2.04e−1	1.00	3.36e−3	2.00				
$\|\|u - u_h\|\|_{L_u}$								
1/2	2.92e−1		3.30e−2		5.79e−3		1.17e−3	
1/4	7.49e−2	1.96	4.75e−3	2.80	4.62e−4	3.65	5.50e−5	4.41
1/8	1.91e−2	1.97	6.09e−4	2.96	3.26e−5	3.83	2.01e−6	4.77
1/16	4.86e−3	1.97	7.76e−5	2.97	2.10e−6	3.96	6.32e−8	4.99
1/32	1.23e−3	1.98	9.82e−6	2.98				

Example with Anisotropic Diffusivity I

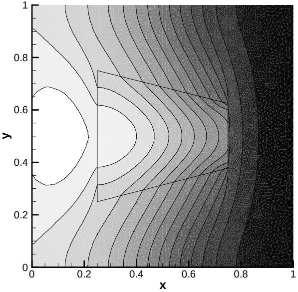


(a) $U_h = P_{h,1}$

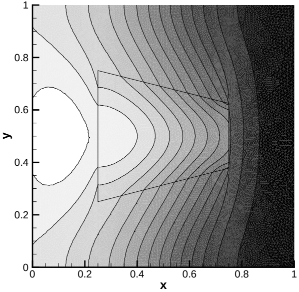


(b) $U_h = P_{h,2}$

Example with Anisotropic Diffusivity II



(c) $U_h = P_{h,3}$



(d) $U_h = P_{h,4}$

Conclusions

- A new DG method was designed, leading to optimal error estimates w.r.t. mesh-size
- The method is robust w.r.t. anisotropic and semi-definite diffusivity
- A key ingredient appears to be the use of diffusivity-dependent weighted averages

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