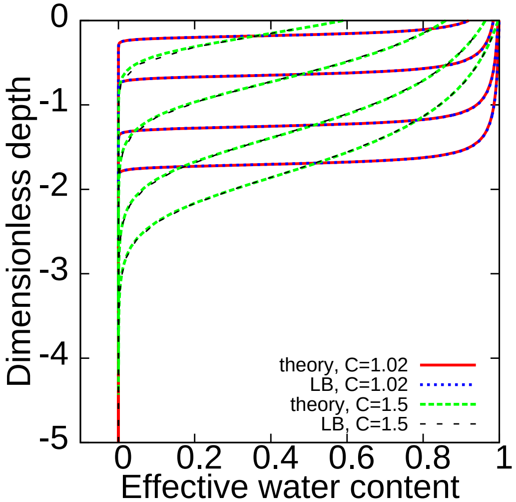
Infiltration into a semi-infinite column under constant flux

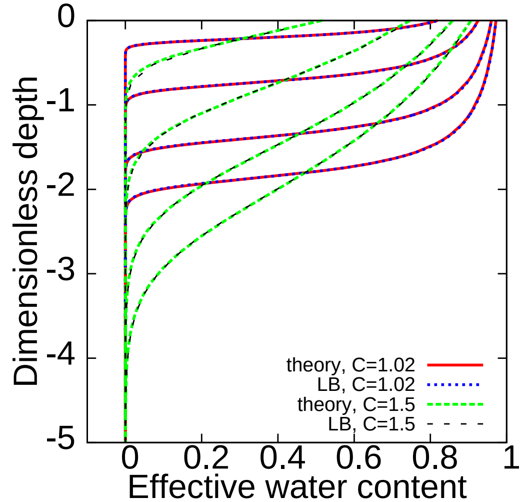
$K_r(\tilde{\theta}) = \frac{C-1}{C-\tilde{\theta}}(\beta_1\tilde{\theta} + \beta_2\tilde{\theta}^2)$, $K_r h'_{\tilde{\theta}}(\tilde{\theta}) = \frac{D_0 C^2}{(C-\tilde{\theta})^2}$, $\beta_1 + \beta_2 = 1$, $C > 1$

general form for Versatile Non-linear model of Broadbridge & White (1988), Watson et al.(1995)

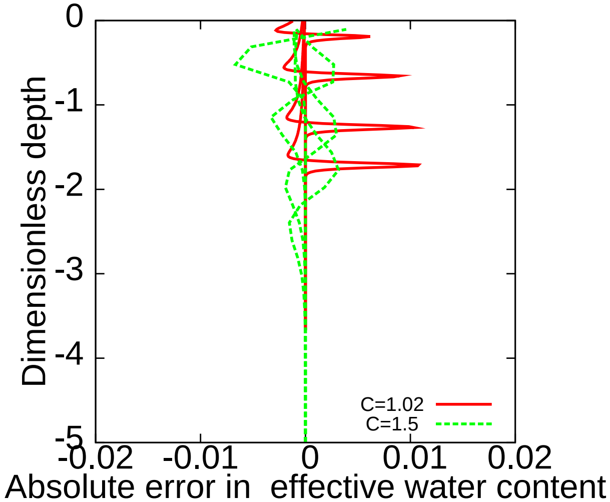
$\beta_1 = 0,\ \beta_2 = 1$:



$\beta_1 = \frac{C}{C-1},\ \beta_2 = -\frac{1}{C-1}$.



$\beta_1 = 0,\ \beta_2 = 1$:



- $C = 1.02$
- $C = 1.5$

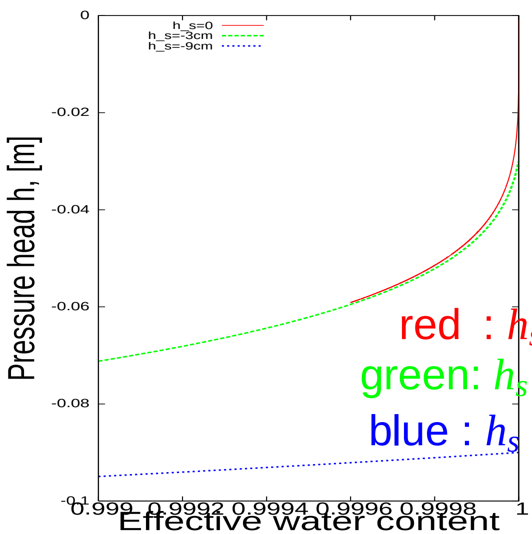
- Initial condition is $\tilde{\theta}(0,z) = 10^{-12}$, $U^{\text{lb}}/U \approx 1/30$, $L^{\text{lb}}/L = 200$
- Infiltration $\vec{u}_r = -q_{in}\vec{1}_z$ before ponding is modeled for $q_{in}/K_s = 1.2$

Steady state infiltration with VGM (van Genuchten 1980) and modified VGM (Vogel et al. 2001) models:

$\tilde{\theta}(h) = \beta_0(1+(-\alpha_0 h)^n)^{-m}$, $\beta_0 = (1+(-\alpha_0 h_s)^n)^m$,

$\alpha_0 > 0$, $n > 1$ and $m = 1 - 1/n$ as empirical parameters

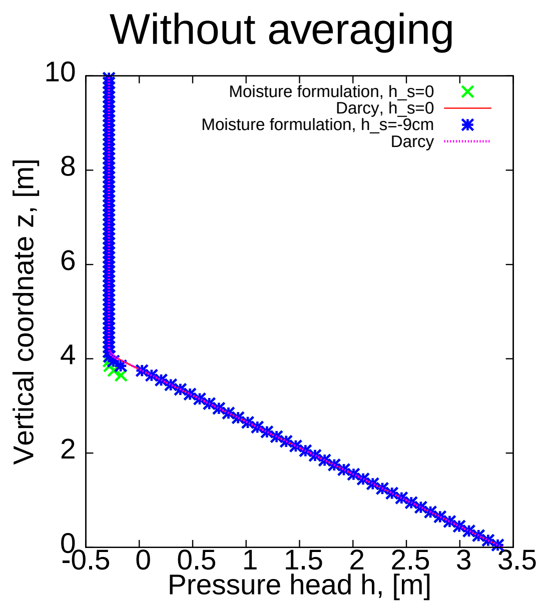
- Original model: $h'_{\tilde{\theta}}(h_s)$ is unbounded, $h_s = 0$
- Modified model: $h'_{\tilde{\theta}}(h_s)$ is bounded, $h_s < 0$
- Example: Sandy soil, $\alpha = 3.7m^{-1}$, $n = 5$

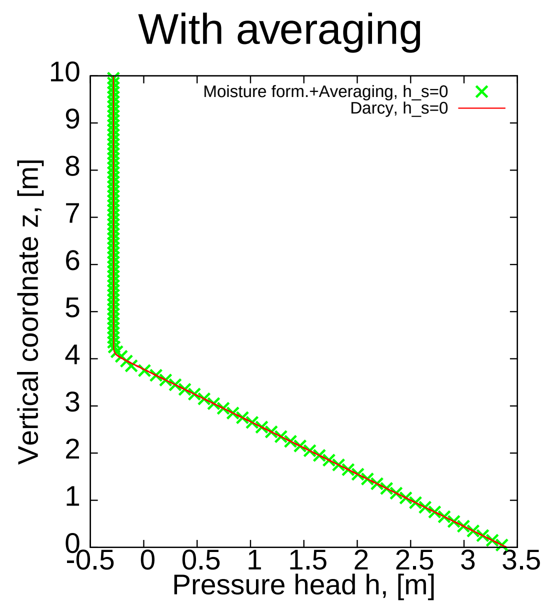


red : $h_s = 0$

green: $h_s = -3\ cm$

blue : $h_s = -9\ cm$





- Input flux is $q_{in} = 0.1K_s$, $K_s = 0.1m/h$, $h(z_L = 0) = 3.4\ m$. Box has 100 cells.

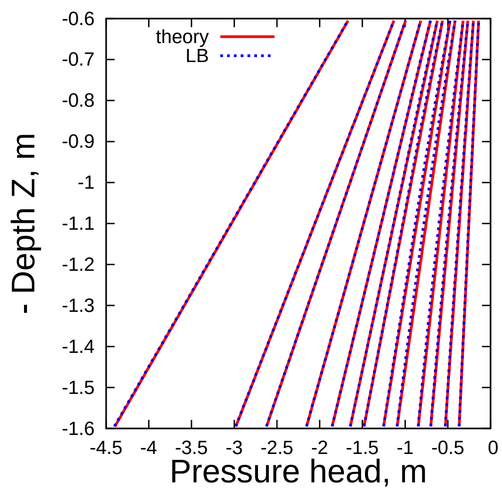
Exact variably saturated non-stationary solution

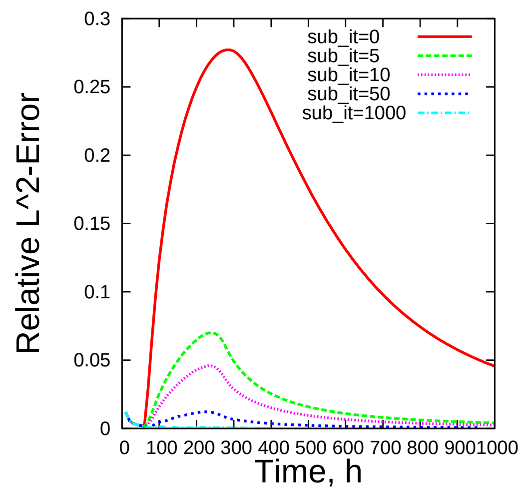
J.-P. Carlier (Cemagref, 2004) following "A class of exact solutions for Richards' equation", Barry et al.(1993)

a) Definition of retention curve:
$\tilde{\theta}(h) = \gamma \int_{-\infty}^{h} \frac{1}{h'+B} \partial_{h'} K_r dh'$
$\gamma^{-1} = \int_{-\infty}^{h_s} \frac{1}{h'+B} \partial_{h'} K_r dh'$

b) BCM hydraulic conductivity function: $K_r(h) = (h/h_s)^{-\beta}$,
$B = 0$ then $h(\tilde{\theta}) = h_s \tilde{\theta}^{-1/(\beta+1)}$

c) Linear solution within both zones: $h = \frac{Z}{A(t)}$, $h(t,0) = 0$,
$A(t) = 1 + W[-e^{(\frac{tK_S}{(\theta_s-\theta_r)\gamma}-1)}]$, with Lambert function $W(x)$,
$x = W(x) exp[W(x)]$.





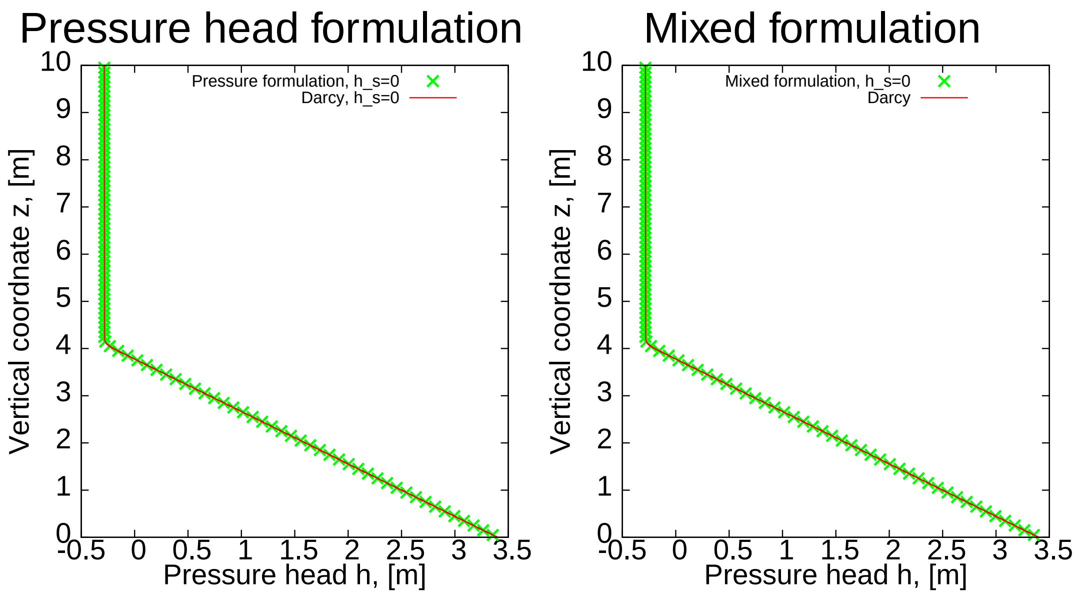
– Exact $h(t)$-solution is fixed at top $Z = 0.6m$ and the bottom $Z = 1.6\ m$

– Computations are started at $t = t_0 = 0.1h$, $h(t_0, Z = 1.6\ m) = -72m$

– Picture: $t = 20 \ldots 10^3\ h$

Steady state infiltration in Sandy soil using $h-$ and $\theta/h-$ formulations:

- Pressure head formulation does not use retention curves and has no approximation when $h'_{\tilde{\theta}}(h_s)$ is unbounded, $h_s = 0$.
- Mixed formulation does not use retention curves in unsaturated zone but needs $h'_{\tilde{\theta}}(1)$ to extrapolate them beyond air entry value h_s. Here, $P = h'_{\tilde{\theta}}(1 - 10^{-6}) = 3566.24\,m$.



- Pressure head h_L is fixed to $3.4\,m$ at the bottom $z_L = 0$.
- Input flux is $q_{in} = 0.1K_s$. Box has 100 cells.

Dispersion relation (D.d'Humières)

- Solution is looked in the form, $f = f_0 \exp(i[\vec{k}\cdot(\vec{r}-\vec{U}t)-\omega t])$, where $\omega(\vec{k}) = -i\nu(\vec{k})k^2$ $\nu(\vec{k}) = \sum_{\alpha,\beta} D_{\alpha\beta}\frac{k_\alpha k_\beta}{k^2}$ $\vec{k} = (k_x,k_y,k_z) = k(\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi)$.

- $z = \exp(-i\omega t)$ and f_0 are the eigenvalues and eigenvectors of the linear operator $\mathsf{S}^{-1}\cdot(\mathsf{I}+\mathsf{A}\cdot(\mathsf{I}-\mathsf{E}^{\text{eq.}}))$, $f^{\text{eq.}} = \mathsf{E}^{\text{eq.}}\cdot f$ $\mathsf{S} = \operatorname{diag}(\exp(\vec{k}\cdot\vec{C}_j))$

– Diffusion term:

$$\nu^{\text{lb}}(\vec{k}) = \nu(\vec{k})(1+\nu_1^{(\text{r})}(\vec{k})k^2+\cdots+\nu_n^{(\text{r})}(\vec{k})k^{2n}+\cdots)$$

– Optimal diffusion solution $\nu_1^{(\text{r})}(\vec{k}) = 0$ for any $\vec{k}$, c_s^2 and $a^{(e)}$ is

$$\Lambda^2(\lambda_D^{\text{opt}},\lambda_e^{\text{opt}}) = \frac{2}{9}\,,\ \lambda_D^{\text{opt}} = -3+\sqrt{3}\,,$$

– Advection term:

$$\vec{k}\cdot\vec{U}^{\text{lb}} = \vec{k}\cdot\vec{U}(1+u_1^{(\text{r})}(\vec{k})k^2+\cdots+u_n^{(\text{r})}(\vec{k})k^{2n}+\cdots)$$

– Optimal advection solution $u_1^{(\text{r})}(\vec{k}) = 0$ is BGK Model

$$\Lambda^2(\lambda_D^{\text{opt}},\lambda_e^{\text{opt}}) = \frac{1}{9}\,,\ \lambda_D^{\text{opt}} = \lambda_e^{\text{opt}} = -3+\sqrt{3}$$