Quadrilateral et Hexahedral Pseudo-conform Finite Elements

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What is the problem?

Loss of convergence for the RT, BDM, BDFM when we use quadrilateral or hexahedral elements.

Standart model :

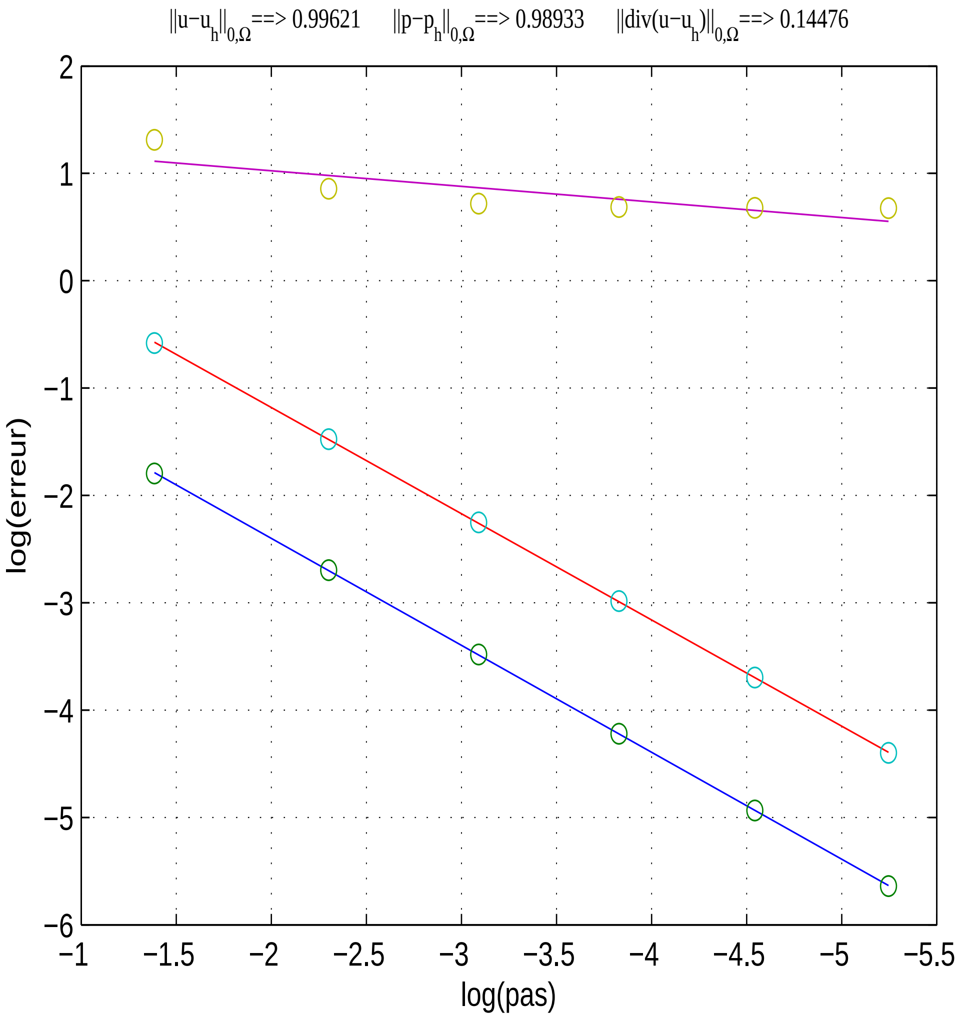
$$L_h = \left\{ \mathbf{v}_h \in H(div,\Omega); \mathbf{v}_h\big|_K \in RT_0(K) \right\}, M_h = \left\{ q_h \in L^2(\Omega); q_h\big|_K \in P_0(K) \right\}$$

find $(\mathbf{u}_h, p_h) \in L_h \times M_h$ solution of

$$\left| \begin{array}{l} \displaystyle\int_\Omega \mathrm{div}\mathbf{u}_h q_h dx + \int_\Omega \mathrm{f} q_h dx = 0 \quad \forall q_h \in M_h \\ \displaystyle\int_\Omega A^{-1}\mathbf{u}_h \mathbf{v}_h dx - \int_\Omega \nabla p_h \mathbf{v}_h dx = 0 \quad \forall \mathbf{v}_h \in L_h \end{array} \right.$$

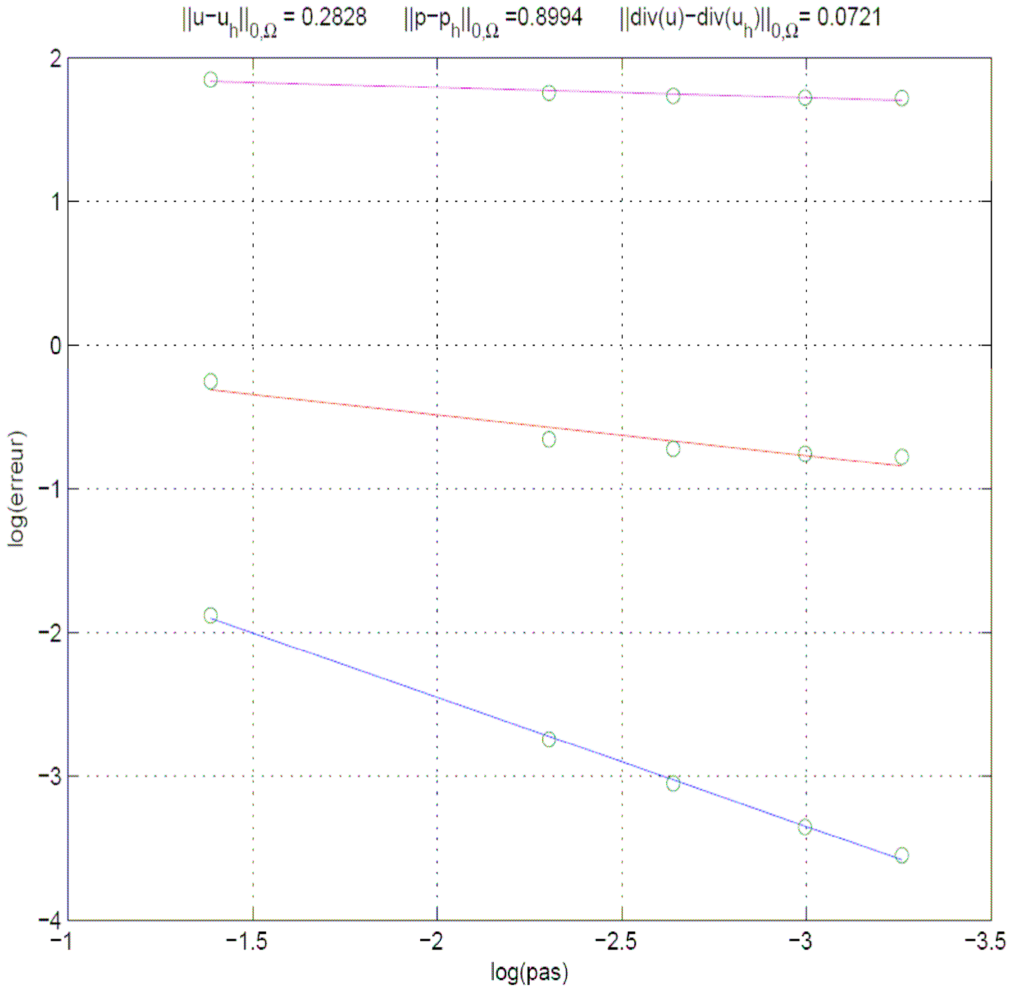
$\mathbf{u}_h$ does not converge in H(div) when the mesh is based on quadrilateral or hexahedral elements.

2D



Loss of convergence on div(**u**)

3D

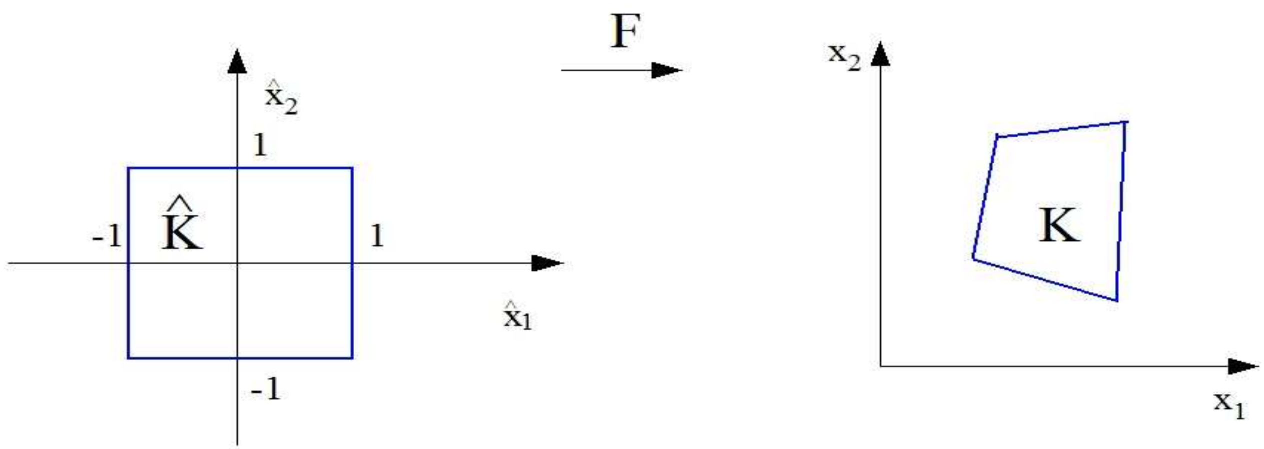


Loss of convergence on **u** and div(**u**)

Where does the problem come from?

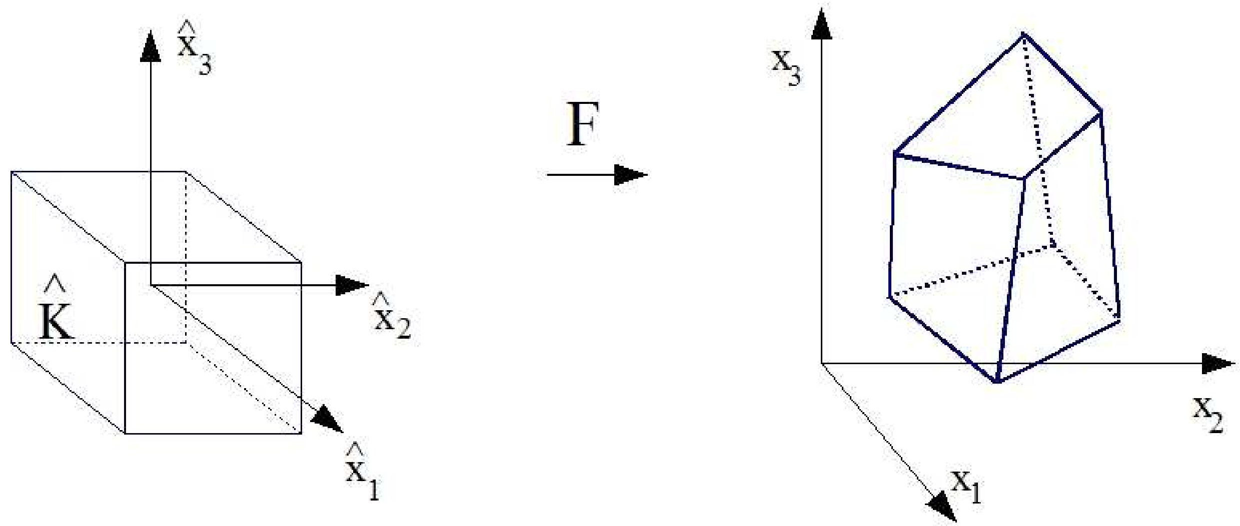
K : element of the triangulation

$\hat{K}$: reference element



F is bilinear

$$J_F = \det(DF) \in Q_1 \cap P_2$$



F is trilinear

$$J_F = \det(DF) \in Q_2 \cap P_4$$

Piola transform:

$$\mathbf{u}(\mathbf{x}) = \frac{1}{J_F(\hat{\mathbf{x}})} \mathbf{DF}(\hat{\mathbf{x}})\, \hat{\mathbf{u}}(\hat{\mathbf{x}}), \qquad p(\mathbf{x}) = \hat{p}(\hat{\mathbf{x}})$$

$$\int_{\partial K} \mathbf{u}\,.\,\mathbf{n}\; p ds = \int_{\partial \hat{K}} \hat{\mathbf{u}}\,.\,\hat{\mathbf{n}}\; \hat{p} d\hat{s}, \int_K div\, \mathbf{u}\; p dx = \int_{\hat{K}} div\, \hat{\mathbf{u}}\; \hat{p} dx$$

$$\hat{\mathbf{u}}(\hat{\mathbf{x}}) = \mathbf{M}(\hat{\mathbf{x}})\, \mathbf{u}(\mathbf{x})$$

Non linear relations

$$div\, \hat{\mathbf{u}}(\hat{\mathbf{x}}) = J_F(\hat{\mathbf{x}})\; div\, \mathbf{u}(\mathbf{x})$$

2D

$$\left\| \mathbf{u} - \boldsymbol{\pi}_h \mathbf{u} \right\|_{0,\Omega} \le Ch \left| u \right|_{1,\Omega}$$

$$\left\| div(\mathbf{u} - \boldsymbol{\pi}_h \mathbf{u}) \right\|_{0,\Omega} \le C \left\| div\, \mathbf{u} \right\|_{1,\Omega}$$

3D

$$\left\| \mathbf{u} - \pi_h\, \mathbf{u} \right\|_{0,\Omega} \le C \left| \mathbf{u} \right|_{1,\Omega}$$

$$\left\| div(\mathbf{u} - \boldsymbol{\pi}_h \mathbf{u}) \right\|_{0,\Omega} \le C \left\| div\, \mathbf{u} \right\|_{1,\Omega}$$

Solutions

- Increase the space of discretisation to control the non-linear part of the transformation F. (Arnold-Boffi-Falk, 2004 (2D))

2D

Parallelogram (RT_0)

Degrees of freedom: 4

Quadrilateral (ABF_0)

Degrees of freedom: 4+2=6

By using the same way, we obtain

3D

Parallelepiped (RT_0)

Degrees of freedom: 6

Hexahedron

Degrees of freedom: 36!!

Solutions

- « Remesh » the quadrilaterals or hexahedrals
Yu. Kuznetsov and S. Repin (J. Numer. Math 2005)

Obviously it is a possibility

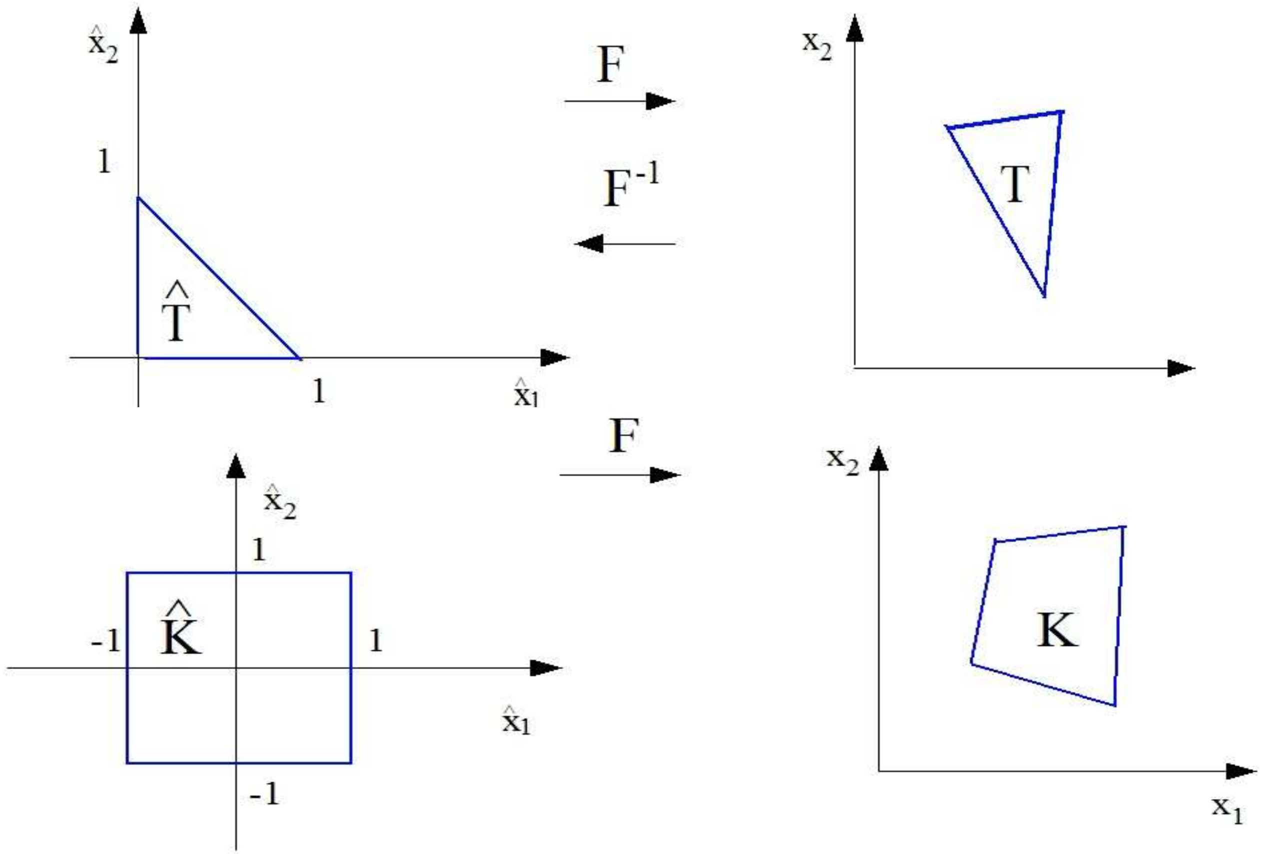
- Build pseudo-conform finite elements without adding degrees of freedom

We have chosen this last way

Cas 2D

Let us examine the Q_1 conform approximation on quadrilaterals. Why?

No loss of convergence with the Q1 finite element but the background is different of the P1 finite element on triangles



Work on $\hat{T}$ is equivalent to work on T

$$\hat{p}\big|_{\hat{T}} \in P_1 \Leftrightarrow p\big|_{T} \in P_1$$

$$\hat{p}\big|_{\hat{K}} \in Q_1 \Rightarrow p\big|_{K} \notin Q_1$$

$p\big|_{K}$ is not polynomial

Consequence: All the integrals must be calculated on $\hat{K}$

$$\nabla p_h = DF^{-1}\nabla \hat{p}_h$$

$$\int_K \nabla p_h . \nabla v_h dx = \int_{\hat{K}} DF^{-1}\nabla \hat{p}_h . DF^{-1}\nabla \hat{v}_h J_F d\hat{x}$$

Goal:

Construct a new finite element satisfying:

- p_h is polynomial on K
- Degrees of freedom are the same
- On parallelogram, recover the classical "Q1 finite element"

Price to pay:

- The degree of the polynomials must be increased.
- The approximation is pseudo-conform

Geometry and notations

$K \subset \mathrm{R}^2$: a convex quadrilateral

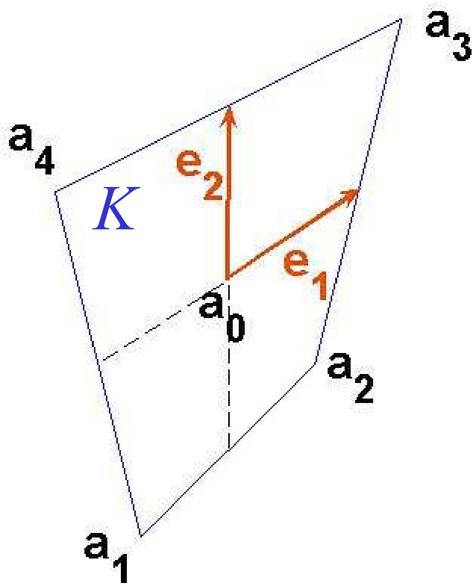
vertices : $\mathbf{a}_i$, edges : $\gamma_i = [\mathbf{a}_i, \mathbf{a}_{i+1}],\ i = 1,\ldots,4$

center of K : $\mathbf{a}_0 = \frac{1}{4}(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 + \mathbf{a}_4)$

$(\mathbf{e}_1, \mathbf{e}_2)$ the basis of $\mathbf{R}^2$ given by

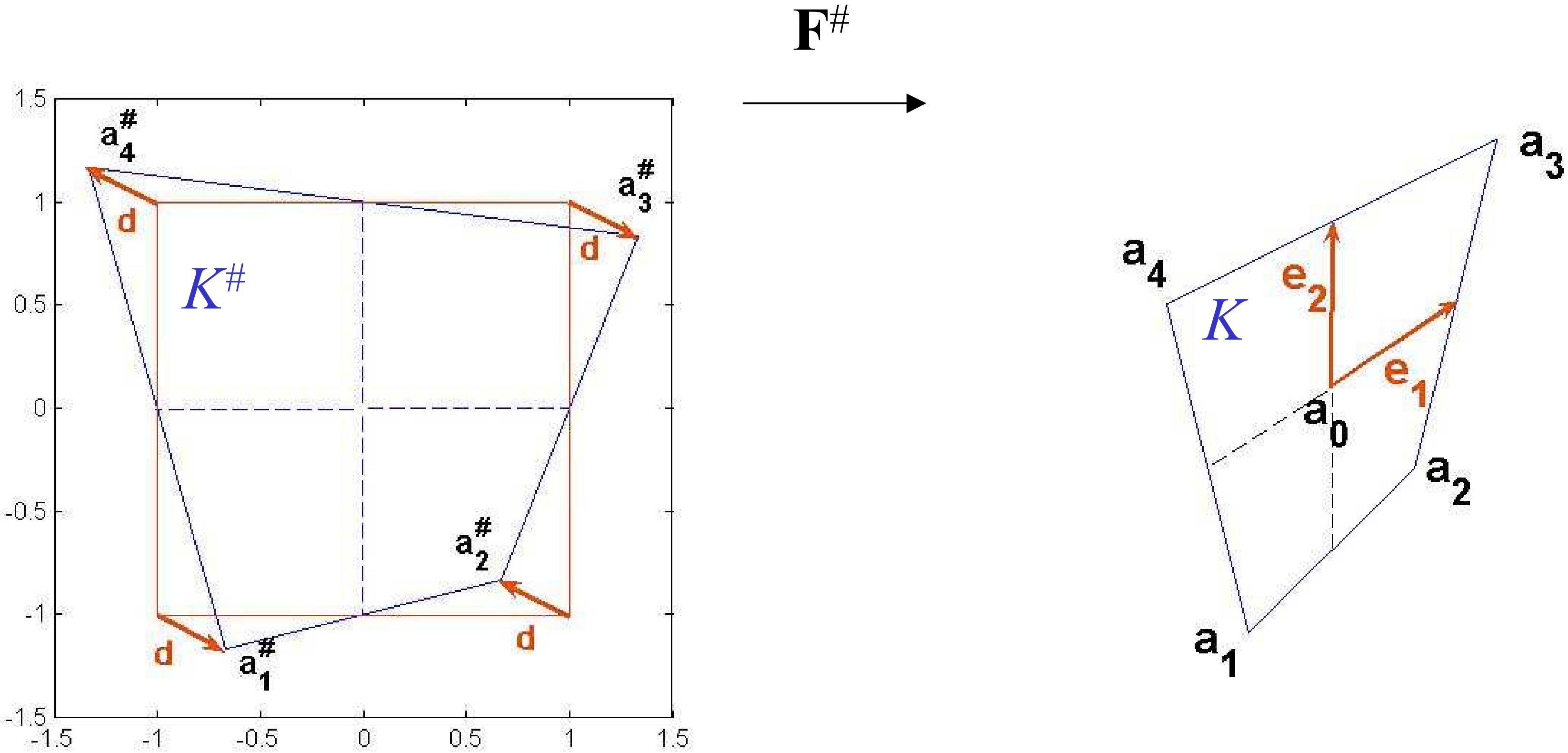
$$\mathbf{e}_1 = \frac{1}{4}(-a_1 + a_2 + a_3 - a_4)$$

$$\mathbf{e}_2 = \frac{1}{4}(-a_1 - a_2 + a_3 + a_4)$$



$\mathbf{F}^{\#}$: affine part of $\mathbf{F}$. By $\mathbf{F}^{\#}$, $K^{\#}$ is transformed into K

$K^{\#}$ is a deformation of the unit square $\hat{K}$



K convex quadrilateral $\Leftrightarrow \|\mathbf{d}\|_K = |d_1| + |d_2| < 1$

Construction of a new finite element $(K^{\#}, P_{K^{\#}}, \Sigma_{1,K^{\#}})$

We want $\Sigma_{1,K^{\#}} = \{ w \to w(\mathbf{a}_i^{\#}), \quad 1 \le i \le 4 \}$

Choice of $P_{K^{\#}}$?

$$\mapsto P_{K^{\#}} = Q_1$$

We obtain a non conforming approximation that does not satisfy the patch-test.
In the error estimation, we don't control the following term:

$$\sum_i \int_{\gamma_i} \frac{\partial \mathbf{u}}{\partial n} [\mathbf{u}_h - I_h \mathbf{u}] d\sigma$$

Solution : find $P_{K^{\#}}$ such that $\int_{\gamma_\mathrm{i}} [\mathbf{u}_h] d\sigma = 0$

We must have $P_1 \subset P_{K^\#}$

Proposition:

For any convex quadrilateral K there exist a polynomial $\omega_{K^\#}$ such that for the choice:

$$P_{K^\#} = Vect(P_1, \omega_{K^\#})$$

We have

1°) $(\mathrm{K}^\#, P_{K^\#}, \Sigma_{1,K^\#})$ is a finite element.

2°) $\forall w \in P_{K^\#} \quad \int_{\gamma_i^\#} w d\sigma = \frac{|\gamma_i|}{2}\{w(\mathbf{a}_i^\#) + w(\mathbf{a}_{i+1}^\#)\} \quad (i = 1\ldots,4)$

Remark : $\omega_{K^\#}$ is not unique

Moreover, it is of interest to obtain:

- $\omega_{K^\#}$ depend continously on the distorsion parameter $\mathbf{d}$
- at the limite $\mathbf{d} = 0$, the finite element $(\mathrm{K}^\#, P_{K^\#}, \Sigma_{1,K^\#}) = (\hat{\mathrm{K}}, Q_1, \hat{\Sigma})$

  the Lagrange reference finite element on the unit square

The simplest choice is to choose $\omega_{\mathrm{K}^\#}$ in $P_3 \cap Q_2$ satisfying :

$$\left| \begin{array}{l} \omega_{\mathrm{K}^\#}(\mathbf{a}_i) = (-1)^{i+1} \quad i = 1,\ldots,4 \\ \displaystyle\int_{\gamma_i^\#} \omega_{\mathrm{K}^\#} d\sigma = 0 \quad i = 1,\ldots,4 \end{array} \right.$$

$\omega_{K^\#}$ can be calculated explicitely according to $\mathbf{d}$ or numerically

Remark: We can computed the finite element basis in $P_3 \cap Q_2$ without calculating $\omega_{K^\#}$

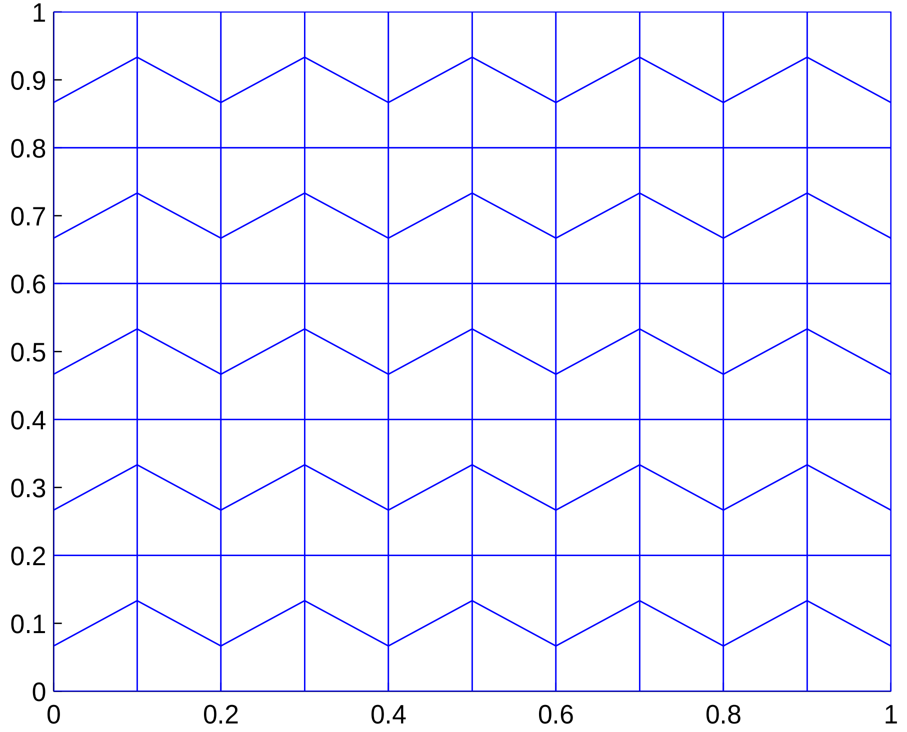
$$\omega_{K^\#}(x_1^\#, x_2^\#) = \frac{1}{den}(-3d_1 d_2(1-d_1^2)(1-d_2^2)(1-x_1^{\#^2}-x_2^{\#^2})$$

$$+(1-2d_1^2-2d_2^2+d_1^4+72d_1^2 2d_2^2+2d_2^4-32d_1^2 2d_2^4-32d_1^4 2d_2^2)x_1^\# x_2^\#$$

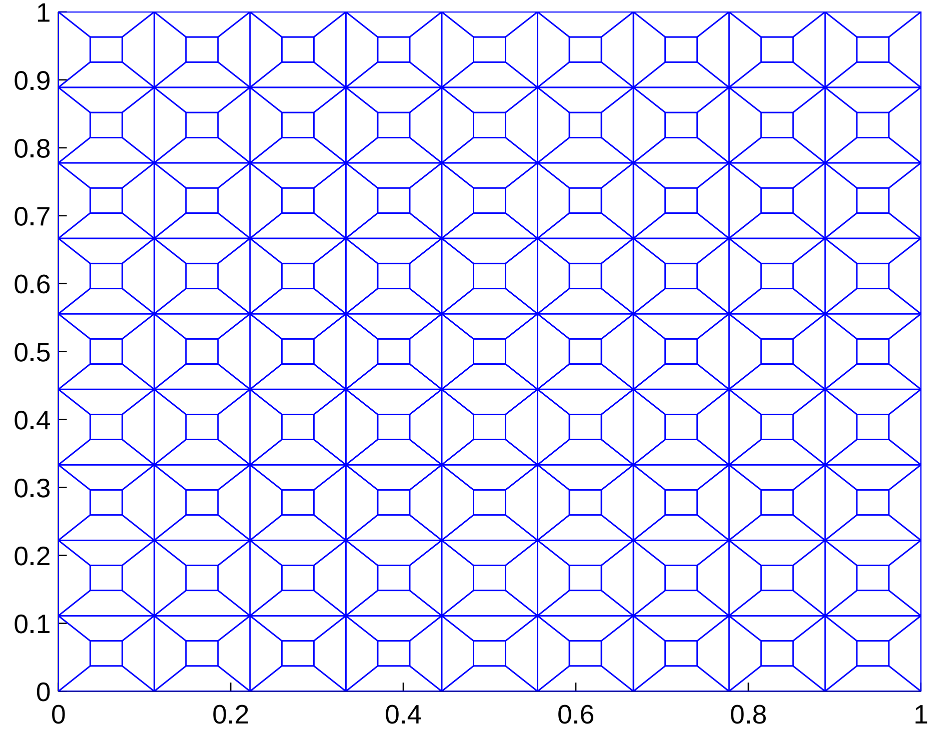
$$+(-1-d_1^2+2d_2^2+2d_1^4-d_1^2 d_2^2+d_2^4)d_2 x_1^{\#^2} x_2^\#$$

$$+(-1+2d_1^2-d_2^2-d_1^4-d_1^2 d_2^2+2d_2^4)d_1 x_1^\# x_2^{\#^2})$$

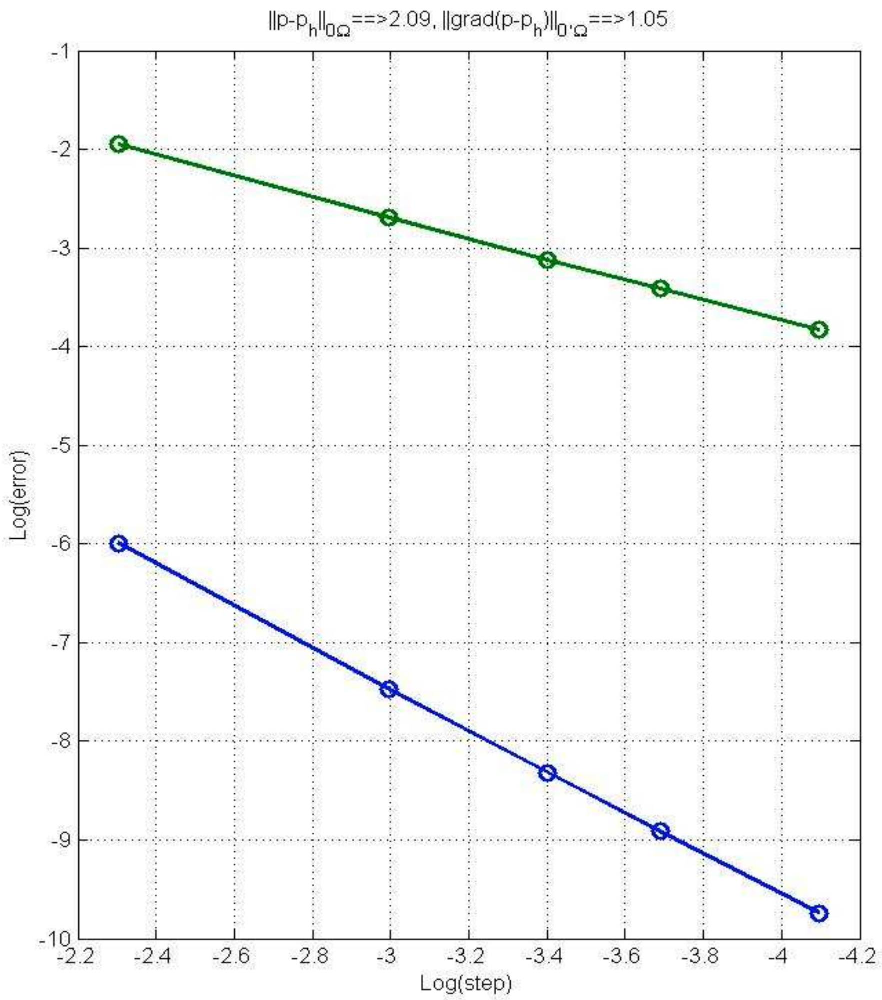
with $den = (1-d_1^2)(1-d_2^2)(1-(d_1+d_2)^2)(1-(d_1-d_2)^2)$



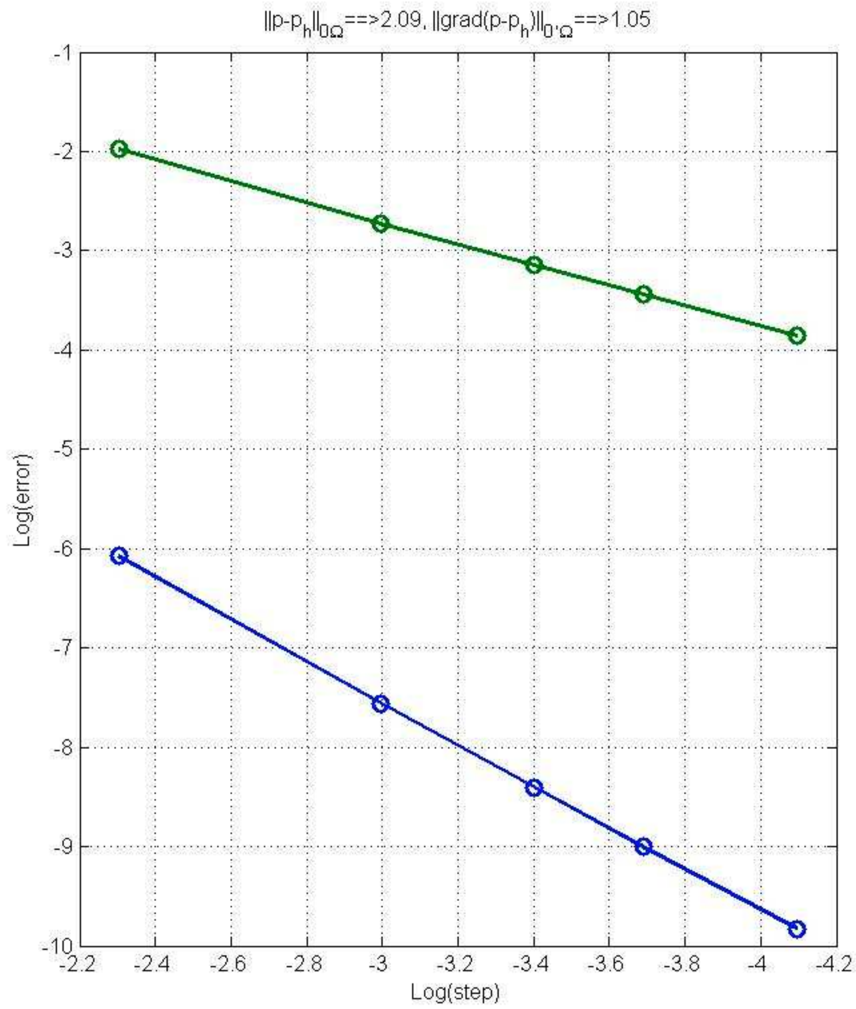
Mesh in “chevron”



Mesh in “alvéole”



Q_1 conform



Pseudo-conform method

Let us return to the initial problem

Construction of a new finite element $(\mathrm{K}^{\#}, P^{*}_{K^{\#}}, \Sigma^{*}_{1,K^{\#}})$

We want $\Sigma^{*}_{1,K^{\#}} = \left\{ \mathbf{w} \to \int_{\gamma_i^{\#}} \mathbf{w}.\mathbf{n}d\sigma, \quad 1 \le i \le 4 \right\}$

Proposition:

For any convex quadrilateral K there exist a polynomial vector $\boldsymbol{\psi}_{K^{\#}}$ such that for the choice:

$$P^{*}_{K^{\#}} = vect(P_0 \times P_0, \mathbf{x}^{\#}, \boldsymbol{\psi}_{K^{\#}})$$

We have

1°) $(\mathrm{K}^{\#}, P^{*}_{K^{\#}}, \Sigma^{*}_{1,K^{\#}})$ is a finite element.

2°) $\forall \mathbf{w} \in P^{*}_{K^{\#}}, \forall s \in P_1$

$$\int_{\gamma_i^{\#}} s\mathbf{w}.\mathbf{n}d\sigma = \left(\frac{1}{|\gamma_i|} \int_{\gamma_i} s d\sigma \right) \int_{\gamma_i} \mathbf{w}.\mathbf{n}d\sigma \quad (i = 1\ldots,4)$$

At the limite $\mathbf{d}=0$, the finite element $(K^{\#}, P^{*}_{K^{\#}}, \Sigma^{*}_{1,K^{\#}}) = (\hat{K}, P_{1,0}\times P_{0,1}, \hat{\Sigma}^{*}_{1})$ the RT reference finite element on the unit square

- The simplest choice is to choose $\boldsymbol{\psi}_{K^{\#}} = \mathbf{rot}(\omega_{K^{\#}})$

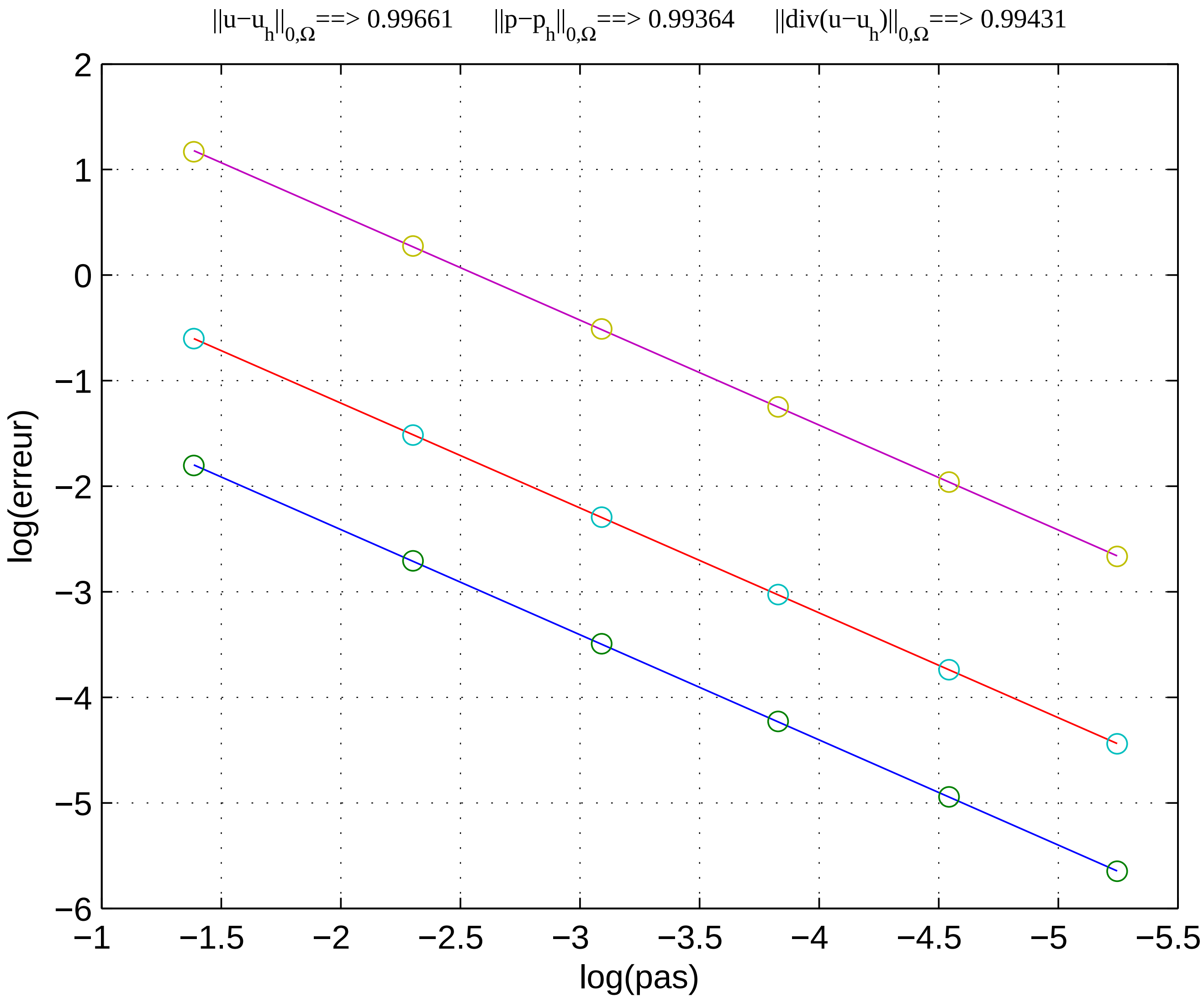
(De Rham diagram $H^{1} \xrightarrow{rot} H(div)$)

- Another choice is to choose $\boldsymbol{\psi}_{K^{\#}}$ in BDM_1 satisfying :

$$div(\boldsymbol{\psi}_{K^{\#}}) = 0, \int_{\gamma_i^{\#}} \boldsymbol{\psi}_{K^{\#}}.\mathbf{n}d\sigma = (-1)^{i}, \int_{\gamma_i^{\#}} s\boldsymbol{\psi}_{K^{\#}}.\mathbf{n}d\sigma = 0 \quad i=1,\dots,4$$

$$BDM_1 = vect(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}y\\0\end{pmatrix},\begin{pmatrix}0\\x\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}-x\\y\end{pmatrix},\begin{pmatrix}-x^{2}\\2xy\end{pmatrix},\begin{pmatrix}-2xy\\x_2^{2}\end{pmatrix})$$

Remark: We can computed the finite element basis in BDM_1 without calculated $\boldsymbol{\psi}_{K^{\#}}$



The convergence orders are good

Conclusion

- If the quadrilaterals K are parallelograms we have

$$(K^{\#}, P_{K^{\#}}, \Sigma_{1,K^{\#}}) = (\hat{K}, Q_1, \hat{\Sigma})$$
$$(K^{\#}, P^{*}_{K^{\#}}, \Sigma^{*}_{1,K^{\#}}) = (\hat{K}, P_{1,0} \times P_{0,1}, \hat{\Sigma}^{*}_{1})$$

and the approximations are conform.

- If the quadrilaterals K are not parallelograms the approximations are not conform.

  The use of $(K^{\#}, P_{K^{\#}}, \Sigma_{1,K^{\#}})$ allows us to have the continuity of p_h at the nodes and only the continuity of the mean values through each edge of the mesh.

  The use of $(\hat{K}, P_{1,0} \times P_{0,1}, \hat{\Sigma}^{*}_{1})$ allows us to have the continuity of the mean value of $\mathbf{u}_h.\mathbf{n}$ and the continuity of the first momentum of $\mathbf{u}_h.\mathbf{n}$ through each edge.

- The implementation is simple, but the basis functions depend on of the shape of K.

3D case

We apply the same approach with the 3D case.

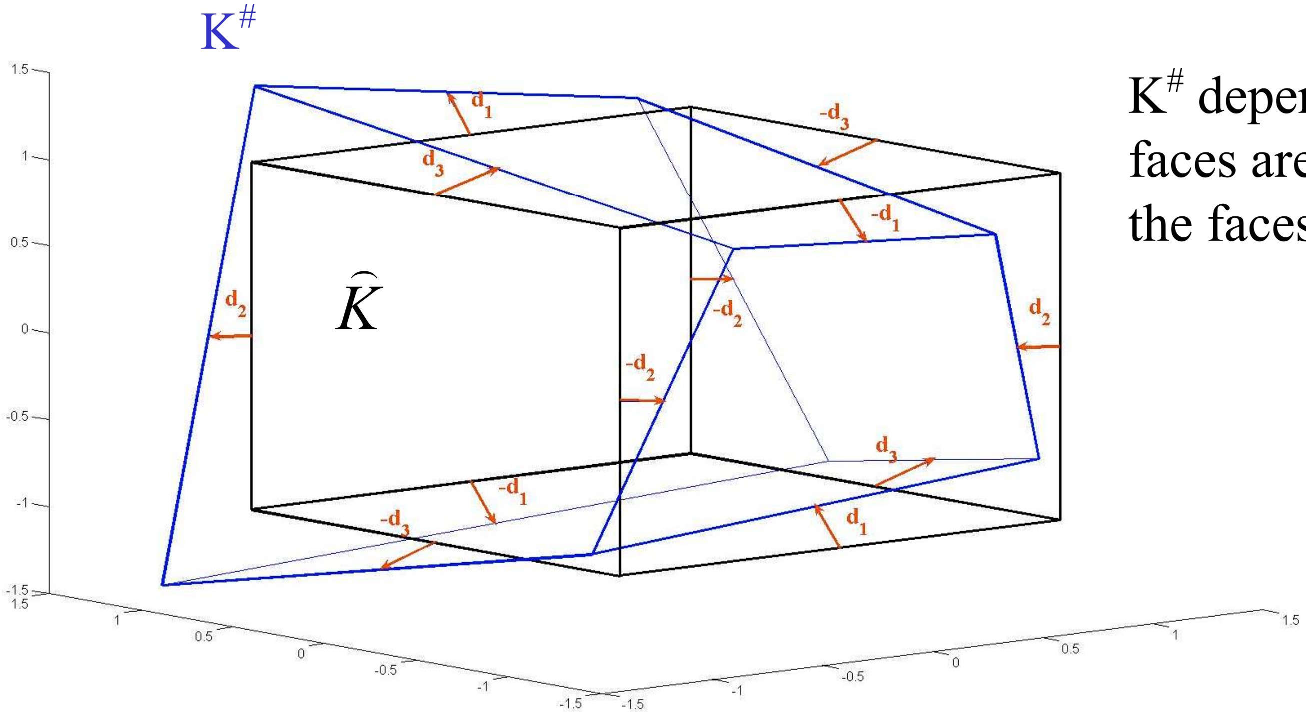
Problems with the hexahedrons

- Description of a hexahedron (deformation of the unit cube)

- Plane faces and non-plane faces? Is it a real problem?

- Parametrisation of the faces

3D case



$K^{\#}$ depends on 6 parameters if the faces are plane and 9 parameters if the faces are not plane.

$$\int_{\gamma_i^{\#}} v(\mathbf{x}^{\#})\,d\sigma^{\#} = \int_{\hat{\gamma}_i} v(\mathbf{x}^{\#}(\hat{\mathbf{x}}))\left\|\mathbf{M}(\hat{\mathbf{x}})\hat{\mathbf{n}}_i\right\|d\hat{\sigma}$$

$$\text{plane face}: \left\|\mathbf{M}(\hat{\mathbf{x}})\hat{\mathbf{n}}_i\right\| \in P_1$$

$$\text{non plane face}: \left\|\mathbf{M}(\hat{\mathbf{x}})\hat{\mathbf{n}}_i\right\| \text{ is not polynomial}$$

Construction of a new finite element $(K^{\#}, P_{K^{\#}}, \Sigma_{1,K^{\#}})$

$$\text{We want } \Sigma_{1,K^{\#}} = \left\{ w \to w(\mathbf{a}_i^{\#}), \quad 1 \le i \le 8 \right\}$$

How to built $P_{K^{\#}}$? (case of plane faces)

Let be $V_{K^{\#}}$ a polynomial space such that

- $dim(V_{K^{\#}}) = 14$
- $P_1 \subset V_{K^{\#}}$
- $\Sigma_{2,K^{\#}} = \left\{ w \to w(\mathbf{a}_i), i = 1...8; w \to \int_{\gamma_i^{\#}} v d\sigma^{\#}, i = 1...6) \right\}$

is unisolvant on $V_{K^{\#}}$ when $K^{\#} = \hat{K}$

Building of integration formulas

$$\forall v \in P_1(K^\#) \quad \int_{\gamma_i^\#} v d\sigma^\# = \sum_{j=1}^{4} \omega_{\gamma_i^\#, j} v(\mathbf{a}^\#_{l_i(j)}) \; i = 1,\ldots,6$$

where $(l_i(j), j = 1,\ldots,4)$ are serial numbers of the node of the face $\gamma_i^\#$,

$\omega_{\gamma_i^\#, j}$ depend only on the values of the cofacteur matrix of

the jacobian at points $(\mathbf{a}^\#_{l_i(j)}, \mathrm{j} = 1,\ldots,4)$

$$P^*_{K^\#} = \left\{ v \in V^*_{K^\#} ; \int_{\gamma_i^\#} s_j v d\sigma^\# = \sum_{j=1}^{4} \omega_{\gamma_i^\#, j} v(\mathbf{a}^\#_{l_i(j)}) \; i = 1,\ldots,6 \right\}$$

Possible choices of polynomial space $V_{K^\#}$

- $V_{K^\#} = \{1, x, y, z, xy, xz, yz, xyz, x^2, y^2, z^2, x^2y, y^2z, xz^2\} \subset P_3 \cap Q_2$

- $$V_{K^\#} = Q_1(K^\#) \oplus (1-x^2)(1-y^2)(1-z) \oplus (1-x^2)(1-y^2)(1+z)$$
$$\oplus (1-y^2)(1-z^2)(1-x) \oplus (1-y^2)(1-z^2)(1+x)$$
$$\oplus (1-z^2)(1-x^2)(1-y) \oplus (1-z^2)(1-x^2)(1+y)$$

Proposition:

For any quadrilateral $K^{\#}$ not too much deformed compared to unit cube we have

1°) $(\mathrm{K}^{\#}, P_{K^{\#}}, \Sigma_{1,K^{\#}})$ is a finite element.

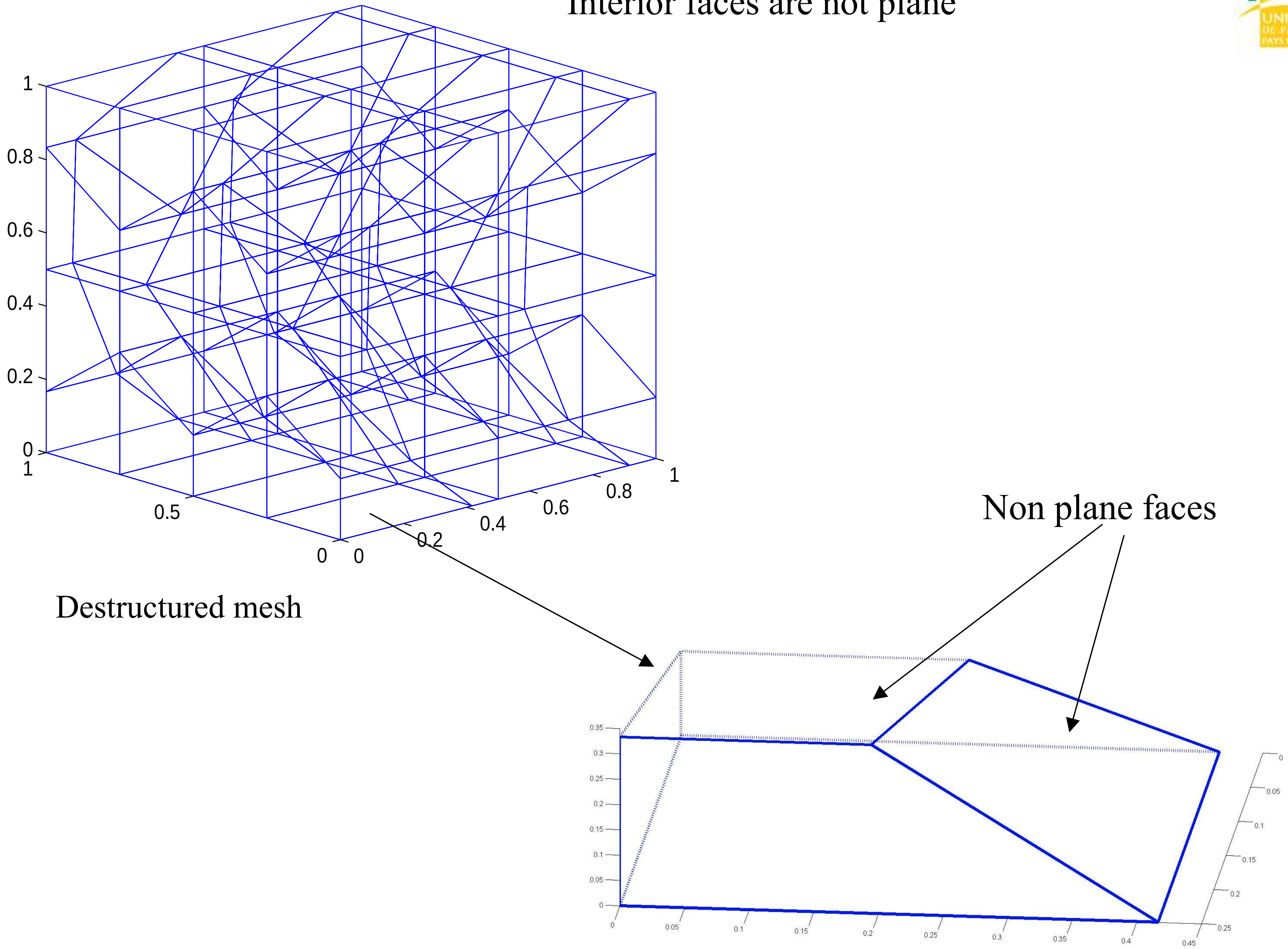
2°) $\forall v \in P_{K^{\#}} \int_{\gamma_i^{\#}} v d\sigma^{\#} = \sum_{j=1}^{4} \omega_{\gamma_i^{\#},j} v(\mathbf{a}^{\#}_{l_i(j)})$ $i = 1,\ldots,6$

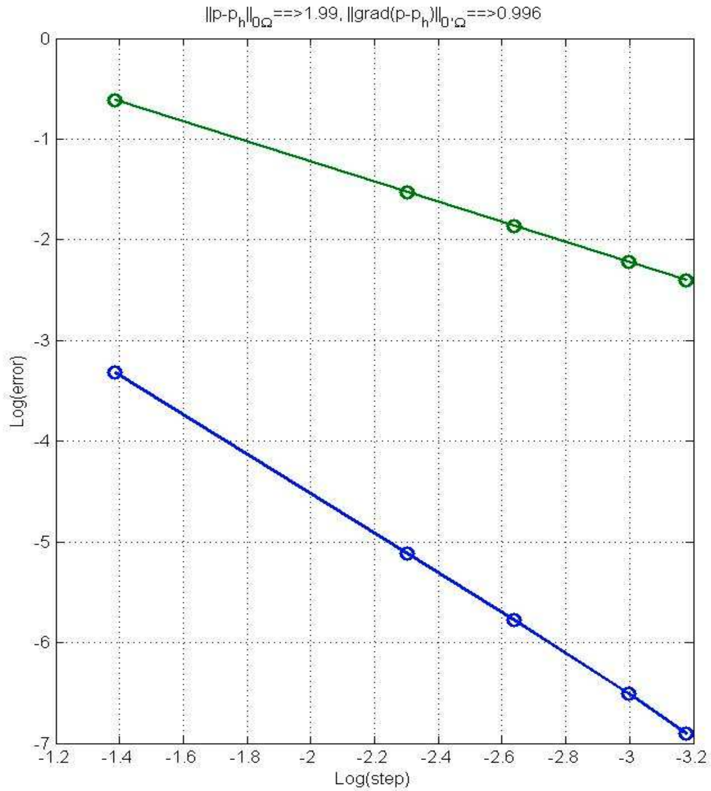
Remark: The main difference, when the faces are not plane, concerns the integration formulas:

$\omega_{\gamma_i^{\#},j}$ are explicitly known in the plane case

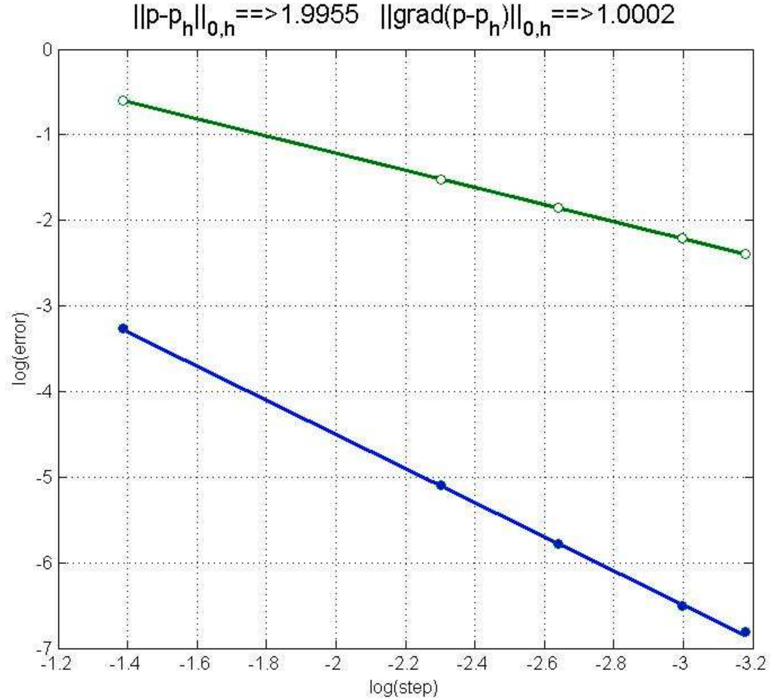
$\omega_{\gamma_i^{\#},j}$ must be numerically calculated in the non - plane case

Interior faces are not plane



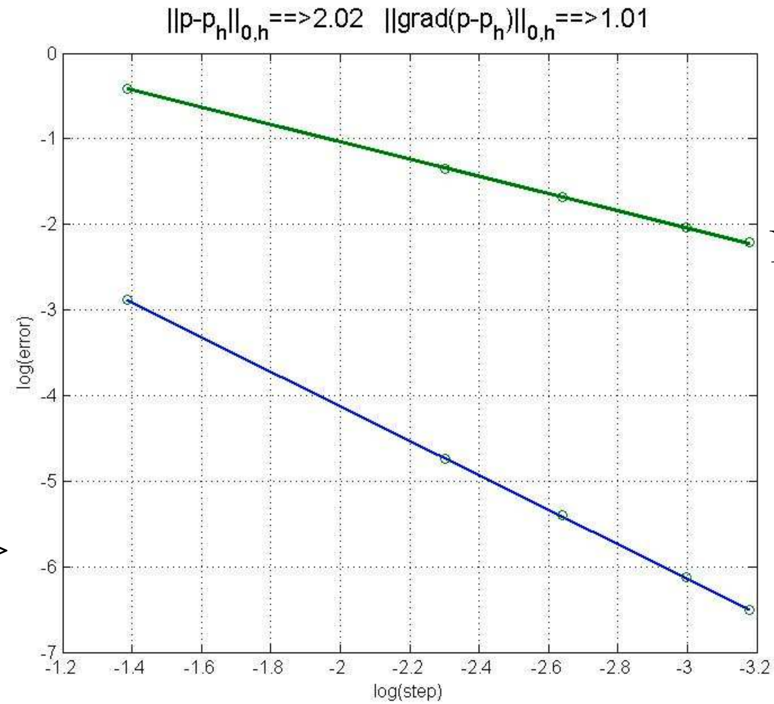


Q_1 conform



Plane faces

Pseudo-conform method==>



Non-plane faces

Construction of a new finite element $(\mathrm{K}^{\#}, P^{*}_{K^{\#}}, \Sigma^{*}_{1,K^{\#}})$

We want $\Sigma^{*}_{1,K^{\#}} = \left\{ \mathbf{w} \rightarrow \int_{\gamma_i^{\#}} \mathbf{w}.\mathbf{n}_i d\sigma, \quad 1 \leq i \leq 6 \right\}$

How to built $P^{*}_{K^{\#}}$?

Let be $V^{*}_{K^{\#}}$ a polynomial space such that

- $dim(V^{*}_{K^{\#}}) = 18$
- $P_{0,0,0} \subset V^{*}_{K^{\#}}$
- $\Sigma^{*}_{2,K^{\#}} = \left\{ \mathbf{u} \rightarrow \int_{\gamma_i^{\#}} \mathbf{u}.\mathbf{n} d\sigma^{\#}, i = 1\ldots6; w \rightarrow \int_{\gamma_i^{\#}} s_j \mathbf{u}.\mathbf{n} d\sigma^{\#}, i = 1\ldots6, j = 1,2) \right\}$

is unisolvant on $V_{K^{\#}}$ when $K^{\#} = \hat{K}$

Building of integration formulas

$$\forall \mathbf{v} \in P_{0,0,0}(K^{\#}) \quad \int_{\gamma_i^{\#}} s_j \mathbf{v}.\mathbf{n}_i d\sigma^{\#} = \omega_{\gamma_i^{\#},j} \int_{\gamma_i^{\#}} \mathbf{v}.\mathbf{n}_i d\sigma^{\#} \; j = 1,2; \, i = 1,\ldots,6$$

$\omega_{\gamma_i^{\#},j}$ depend only on values of the cofacteur matrix of the jacobian on $\gamma_i^{\#}$.

$$P^{*}_{K^{\#}} = \left\{ \mathbf{v} \in V^{*}_{K^{\#}} ; \int_{\gamma_i^{\#}} s_j \mathbf{v}.\mathbf{n}_i d\sigma^{\#} = \omega_{\gamma_i^{\#},j} \int_{\gamma_i^{\#}} \mathbf{v}.\mathbf{n}_i d\sigma^{\#} \; j = 1,2; \, i = 1,\ldots,6 \right\}$$

Possible choice of polynomial space $V^{*}_{K^{\#}} = BDM_1$

$$BDM_1 = vect\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}y\\0\\0\end{pmatrix}, \begin{pmatrix}z\\0\\0\end{pmatrix}, \begin{pmatrix}0\\x\\0\end{pmatrix}, \begin{pmatrix}0\\z\\0\end{pmatrix}, \begin{pmatrix}0\\0\\x\end{pmatrix}, \begin{pmatrix}0\\0\\y\end{pmatrix}, \begin{pmatrix}-x\\0\\z\end{pmatrix}, \begin{pmatrix}x\\-y\\0\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix}, \begin{pmatrix}x^2\\-2xy\\0\end{pmatrix}, \begin{pmatrix}-2xz\\0\\z^2\end{pmatrix}, \begin{pmatrix}-2xy\\y^2\\0\end{pmatrix}, \begin{pmatrix}0\\-xy\\xz\end{pmatrix}, \begin{pmatrix}xz\\-yz\\0\end{pmatrix}, \begin{pmatrix}xy\\0\\-yz\end{pmatrix} \right)$$

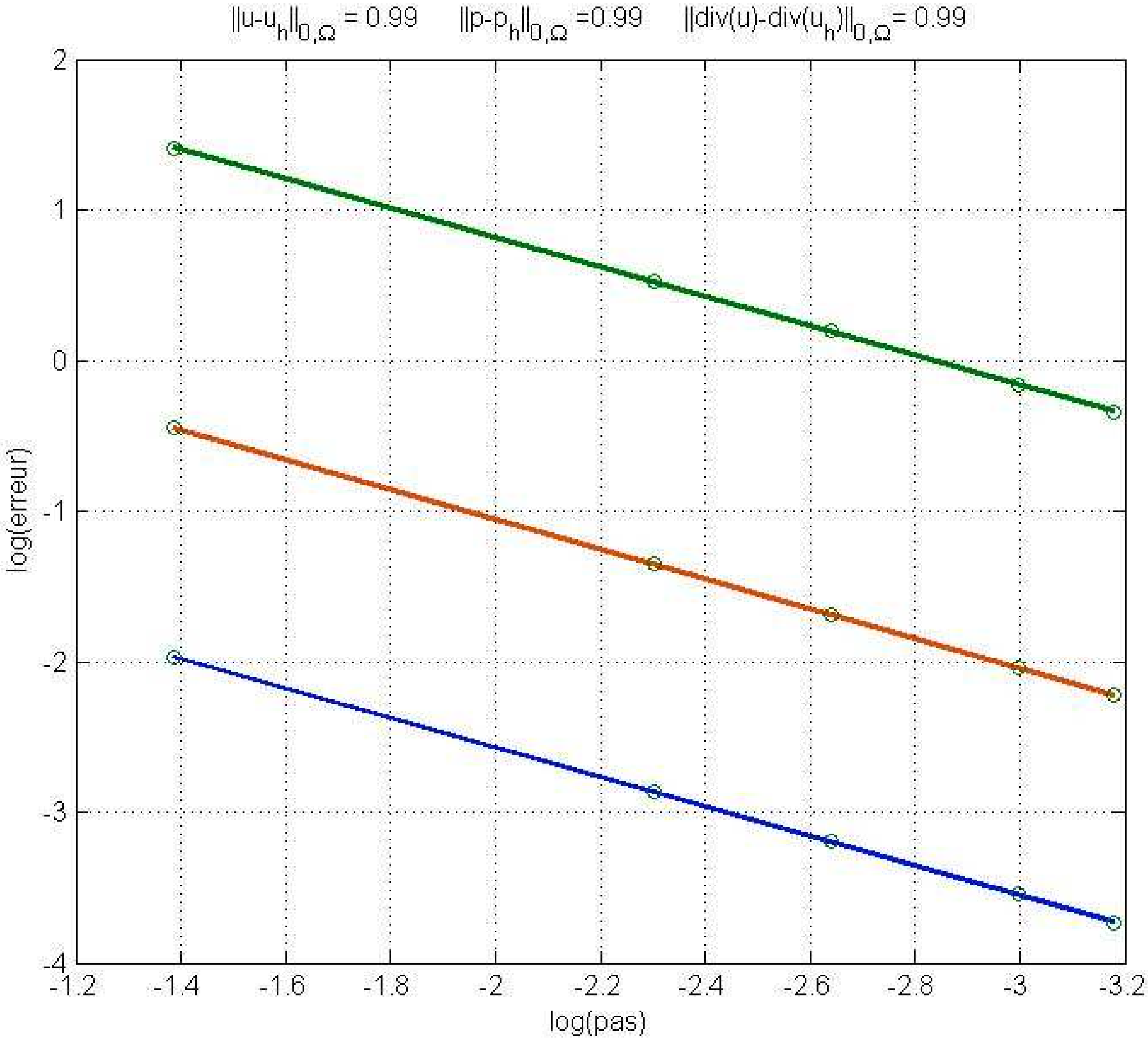
Proposition:

For any quadrilateral $K^{\#}$ not too much deformed compared to unit cube we have

1°) $(\mathrm{K}^{\#}, P^{*}_{K^{\#}}, \Sigma^{*}_{1,K^{\#}})$ is a finite element.

2°) $\forall \mathbf{v} \in P^{*}_{K^{\#}} \quad \int\limits_{\gamma_i^{\#}} s_j \mathbf{v}.\mathbf{n}_i d\sigma^{\#} = \omega_{\gamma_i^{\#},j} \int_{\gamma_i^{\#}} \mathbf{v}.\mathbf{n}_i d\sigma^{\#} \; j = 1,2; i = 1,\ldots,6$

Problem with the integration formula when the faces are not plane.



Pseudo-conform method (plane faces)

Conclusions

- 2D case:
  - The numerical results are agree with the theoretical results.
  - Possibility to generalise the results to FEM of higher degrees.
- 3D case:
  - Necessity to work on the geometry of the hexahedrons.
  - Clarify the problem of plane faces and non-plane faces.
  - Use the De Rham diagram to connect the H_1 case to the H(div) case

(De Rham diagram : $H^1 \xrightarrow{\nabla} H(curl) \xrightarrow{rot} H(div)$)