Lattice Boltzmann schemes for the *Brinkman equation* in extremely heterogeneous porous media: bulk, boundary, interface

Gonçalo Silva & Irina Ginzburg

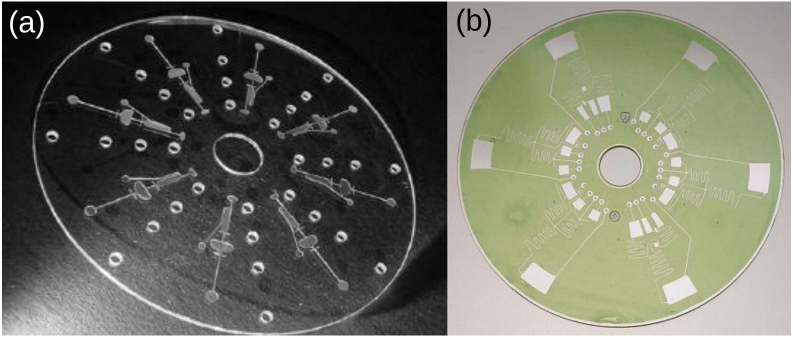
Groupe de travail “Schémas de Boltzmann sur réseau”

6th May 2015

1. Who am I?
   - PhD work
   - Experiments
   - Numerics
2. Context
   - Motivation
3. TRT-LBM
   - Properties
4. Channels
   - Bulk
   - Boundary/interface
   - Boundary/interface
   - Solutions
   - Conclusions
5. Non-channels
   - Benchmark description

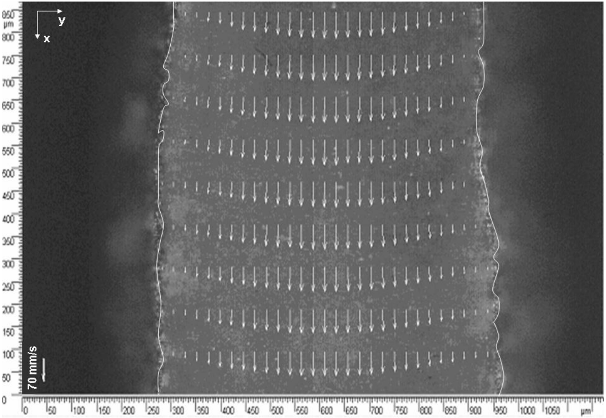
- Approximated solutions
- Stokes flow around solid cylinder
- Porous flow around solid cylinder
- Conclusions
- Biporous medium: Introduction
- Porous flow across porous cylinder $k_1 \ll k_2$
- Porous flow across porous cylinder $k_1 \gg k_2$
- Conclusions

Theoretical, numerical and experimental characterization of microfluidic channel flows in rotating platforms



Examples of lab-on-a-CD prototypes, (a) *image source*: Brenner *et al.* 2003; (b) *image source*: www.spaceref.com

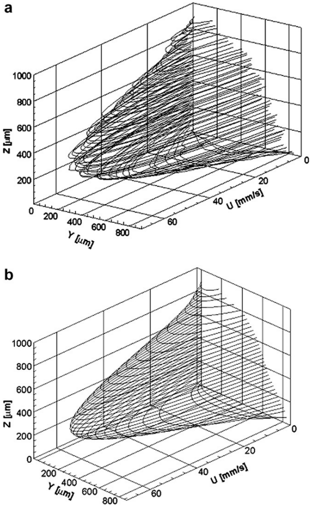
Example of micro-PIV measurement

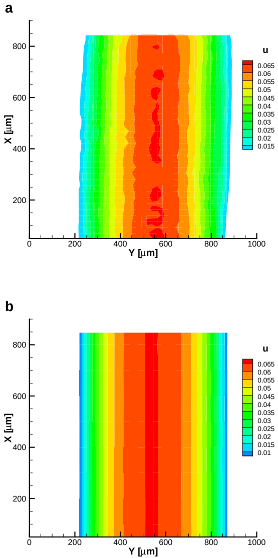


Experiments

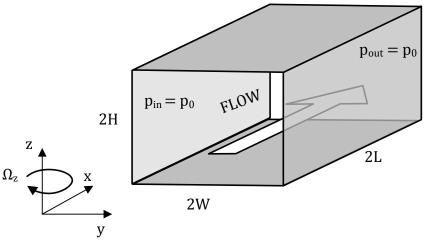
Micro-PIV

CFD



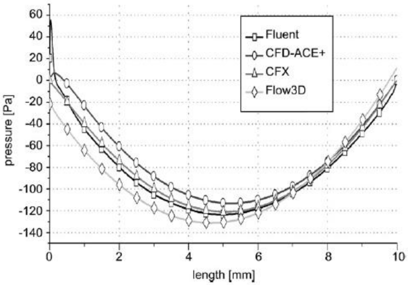


A simple numerical verification test



Centrifugal-driven rotating channel

flow setup



Streamwise pressure solution
predicted by several commercial CFD codes

– *image source*: Glatzela *et al.* 2008

LBM studies:

Physica A 390 (2011) 1085–1095

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

A study on the inclusion of body forces in the lattice Boltzmann BGK equation to recover steady-state hydrodynamics

Goncalo Silva, Viriato Semiao *

Mechanical Engineering Department, Instituto Superior Tecnico, Technical University of Lisbon, Av. Rovisco Pais, 1049-001 Lisbon, Portugal

J. Fluid Mech., page 1 of 22. © Cambridge University Press 2012 1
doi:10.1017/jfm.2012.83

First- and second-order forcing expansions in a lattice Boltzmann method reproducing isothermal hydrodynamics in artificial compressibility form

Goncalo Silva and Viriato Semiao†

TU Lisbon, Instituto Superior Tecnico, IDMEC, Department of Mechanical Engineering, P-1049001 Lisbon, Portugal

(Received 6 October 2011; revised 22 December 2011; accepted 7 February 2012)

Journal of Computational Physics 269 (2014) 259–279

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Truncation errors and the rotational invariance of three-dimensional lattice models in the lattice Boltzmann method

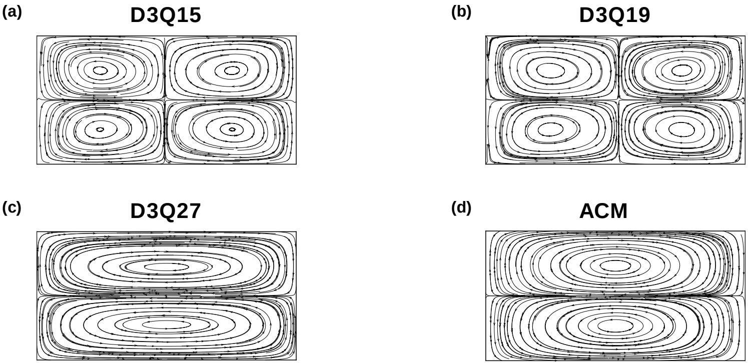
CrossMark

Goncalo Silva *, Viriato Semiao

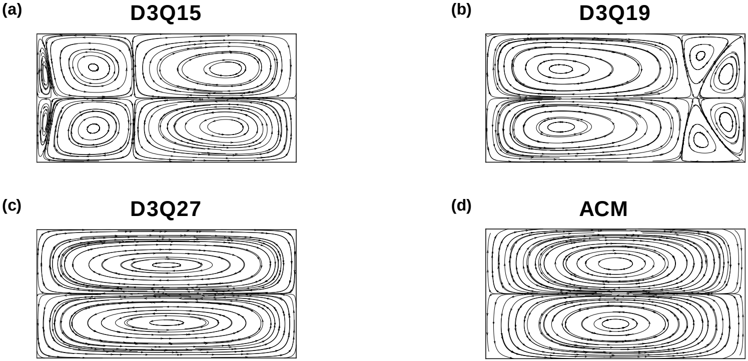
University of Lisbon, Instituto Superior Tecnico, IDMEC, Department of Mechanical Engineering, P-1049001 Lisbon, Portugal

Numerics

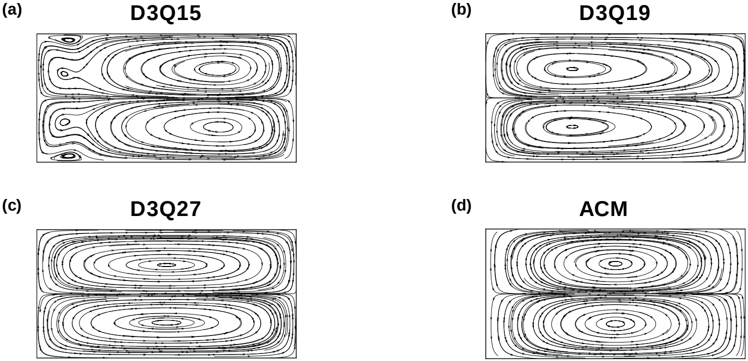
Results: $\mathrm{Ek}^{-1} \equiv \dfrac{\Omega_z H^2}{\nu} = 0.001$



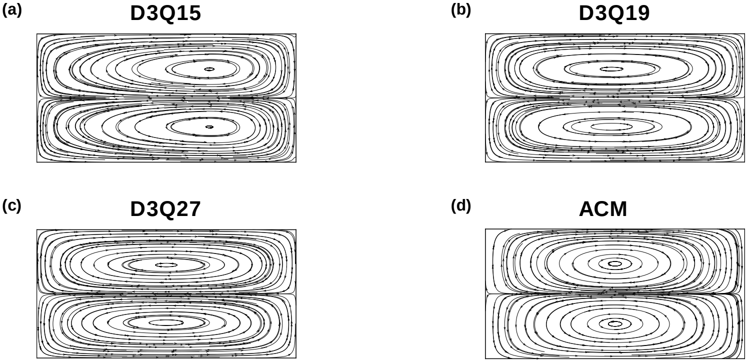
Results: $\mathrm{Ek}^{-1} \equiv \frac{\Omega_z H^2}{\nu} = 0.02$



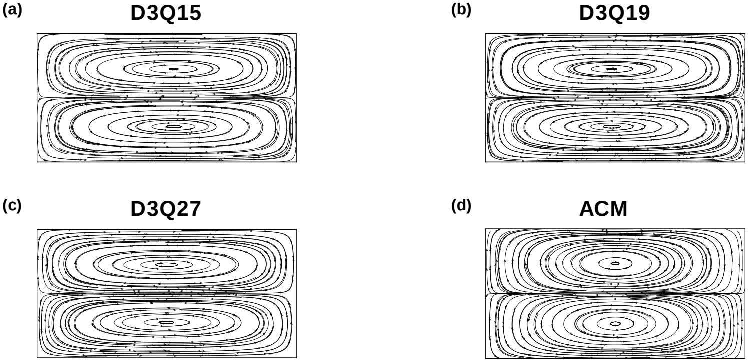
$$\textbf{Results:}\ \mathrm{Ek}^{-1} \equiv \frac{\Omega_z H^2}{\nu} = 0.05$$



Results: $\mathrm{Ek}^{-1} \equiv \frac{\Omega_z H^2}{\nu} = 0.1$



Results: $\mathrm{Ek}^{-1} \equiv \frac{\Omega_z H^2}{\nu} = 1$



Heterogeneous porous media systems

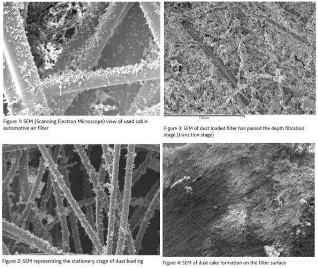


image source: www.ntu.edu.sg

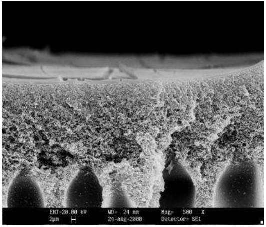


image source: www.memfil.com

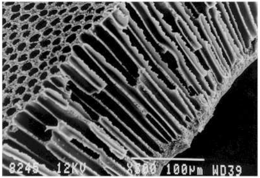
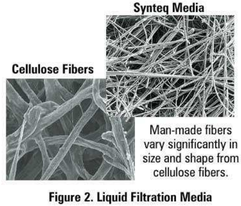


image source: www.ntu.edu.sg



Governing equations (nondimensional)

Stokes-Brinkman-Darcy equations

$$\vec{\nabla} \cdot \vec{u} = 0$$

$$f(\phi) \vec{\nabla} p = \Delta \vec{u} - \sigma^2 \vec{u}$$

subject to proper interface/solid BC.

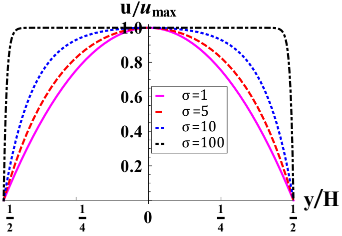
Solution is controlled by:

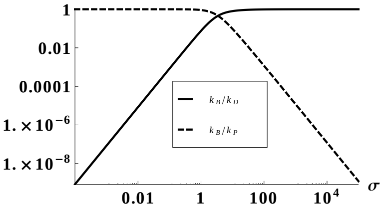
- $f(\phi)$ (herein = 1, for simplicity)
- $\sigma^2 = f(\phi) H^2 / k$

Given $\sigma \Rightarrow$ 3 dynamical regimes follow:

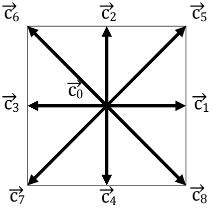
1. $\sigma < 1$: "Open" flow → Stokes
2. $\sigma \sim 1$: "Interface" flow → Brinkman
3. $\sigma > 1$: "Constant" flow → Darcy

Channel flow solutions:





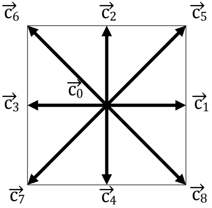
D2Q9



LB evolution equation:

1. Streaming: $f_q\left(\vec{r}+\vec{c}_q, t+1\right) = \tilde{f}_q\left(\vec{r}, t\right)$
2. TRT collision:
   $\tilde{f}_q\left(\vec{r}, t\right) = f_q\left(\vec{r}, t\right) + g_q^+ + g_q^-$
   where $g_q^\pm = -s^\pm\left(f_q^\pm - e_q^\pm\right)$ defined by:
   - Symmetric equilibrium $e_q^+ = t_q P(\rho)$
   - Anti-symmetric equilibrium
     $e_q^- = j_q + \Lambda^- F_q$
     with $j_q = t_q(\vec{j}\cdot\vec{c}_q)$ and $F_q = t_q(\vec{F}\cdot\vec{c}_q)$
   - Relaxation functions $\Lambda^\pm = \left(\frac{1}{s^\pm} - \frac{1}{2}\right) > 0$
     with $s^\pm \in]0, 2[$
   - Brinkman viscosity $\nu_B = \frac{\nu}{f(\phi)} = \frac{\Lambda^+}{3}$

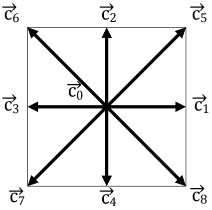
D2Q9



LB evolution equation:

3. Conservation:
   - Mass $\rho = \sum_{q=0}^{Q-1} f_q$
   - Momentum $\rho_0 \vec{u} = \vec{j} = \sum_{q=1}^{Q-1} \vec{c}_q f_q - \frac{1}{2}\vec{F}$
4. Invariance:
   - Steady-state solutions are fixed by $\Lambda = \Lambda^+\Lambda^-$ (viscosity-independence)
5. BF scheme:
   - Brinkman force term: $\vec{F} = \vec{F}^p - B_f \vec{j}$ with $B_f = \frac{\nu}{k}$
6. IBF scheme:
   - Introduce redefine symmetric relaxation function, e.g. $\Lambda_*^+ = \frac{3(4+B)\Lambda^+}{4(3+2B\Lambda)}$ where $B = \frac{\sigma^2}{H^2}$ or $B = \frac{B_f}{\nu_B}$

Steady-state recurrence equations:



1. One pair of linear combinations per link:

$$g_q^+ = \bar{\Delta}_q e_q^- - \Lambda^- \bar{\Delta}_q^2 e_q^+ + \left(\Lambda - \tfrac{1}{4}\right) \bar{\Delta}_q^2 g_q^+$$
$$g_q^- = \bar{\Delta}_q e_q^+ - \Lambda^+ \bar{\Delta}_q^2 e_q^- + \left(\Lambda - \tfrac{1}{4}\right) \bar{\Delta}_q^2 g_q^-$$

2. And another pair per link:

$$\bar{\Delta}_q g_q^- = \bar{\Delta}_q^2 e_q^+ - \Lambda^+ \bar{\Delta}_q^2 g_q^+$$
$$\bar{\Delta}_q g_q^+ = \bar{\Delta}_q^2 e_q^- - \Lambda^- \bar{\Delta}_q^2 g_q^-$$

where first and second central linkwise finite differences are:

- $\bar{\Delta}_q \psi = \frac{1}{2}\left(\psi(\vec{r} + \vec{c}_q) - \psi(\vec{r} - \vec{c}_q)\right)$
- $\bar{\Delta}_q^2 \psi = \psi(\vec{r} + \vec{c}_q) - 2\psi(\vec{r}) + \psi(\vec{r} - \vec{c}_q)$

General solution:

- Exact steady macroscopic equations:

$$\sum_{q=0}^{Q-1} g_q^+ = 0, \quad \sum_{q=1}^{Q-1} g_q^- \vec{c}_q = \vec{F}$$

- Taking the moments of recurrence equations $g_q^\pm$:

1. $\overline{\nabla} \cdot \vec{j} = \Lambda^- \bar{\Delta}^2 P - \left(\Lambda - \frac{1}{4}\right) \sum_{q=1}^{Q-1} \bar{\Delta}_q^2 g_q^+$
2. $\overline{\nabla} P - \vec{F} = \frac{\Lambda^+}{3} \bar{\Delta}^2 \vec{j} + \Lambda \sum_{q=1}^{Q-1} \bar{\Delta}_q^2 F_q^- \vec{c}_q - \left(\Lambda - \frac{1}{4}\right) \sum_{q=1}^{Q-1} \bar{\Delta}_q^2 g_q^- \vec{c}_q$

- $\overline{\nabla} \cdot \vec{j} = \sum_{q=1}^{Q-1} \bar{\Delta}_q j_q$
- $\bar{\Delta}^2 P = \sum_{q=1}^{Q-1} \bar{\Delta}_q^2 P_q$
- $\overline{\nabla} P = \sum_{q=1}^{Q-1} \bar{\Delta}_q P_q \vec{c}_q$
- $\bar{\Delta}^2 \vec{j} = 3 \sum_{q=1}^{Q-1} \bar{\Delta}_q^2 j_q \vec{c}_q$

- Isotropic correction (Brinkman eq.):

  $\Lambda \sum_{q=1}^{Q-1} \bar{\Delta}_q^2 F_q^- \vec{c}_q = -\frac{\Lambda B_f}{3} \bar{\Delta}^2 \vec{j}$

- Anisotropic correction (Brinkman eq.):

  $\left(\Lambda - \frac{1}{4}\right) \sum_{q=1}^{Q-1} \bar{\Delta}_q^2 g_q^- \vec{c}_q = ?$

$\boxed{\text{Channel solution:}}$

Straight/diagonal channel in rotated coordinate system (x', y'):

1. Momentum: $\sum_{q=1}^{Q-1} g_q^- c_{qx'} = F_{x'}$
2. Periodic condition: $g_q^- = 0$ if $c_{qy'} = 0$

$\Longrightarrow \boxed{g_q^- = 3t_q F_{x'}(y') c_{qx'} c_{qy'}^2}$

Anisotropic correction (Brinkman eq.):

$$\left(\Lambda - \frac{1}{4}\right) \sum_{q=1}^{Q-1} \bar{\Delta}_q^2 g_q^- \vec{c}_q = -3\left(\Lambda - \frac{1}{4}\right) B_f \bar{\Delta}_{y'}^2 j_{x'} \sum_{q=1}^{Q-1} t_q c_{qx'}^2 c_{qy'}^2 = -\left(\Lambda - \frac{1}{4}\right) B_f \bar{\Delta}_{y'}^2 j_{x'}$$

Total correction (Isotropic+Anisotropic):

$$-\frac{\Lambda B_f}{3} \bar{\Delta}_{y'}^2 j_{x'} + \left(\Lambda - \frac{1}{4}\right) B_f \bar{\Delta}_{y'}^2 j_{x'} = B_f \left(\frac{8\Lambda - 3}{12}\right) \bar{\Delta}_{y'}^2 j_{x'}$$

Channel solution:

Brinkman forcing modifies effective viscosity of Laplacian term

$$\boxed{\overline{\nabla}_{x'} P - F_{x'} = \frac{\Lambda^+}{3} \rho_0 \left(1 + \delta\right) \bar{\Delta}^2_{y'} u_{x'}}$$

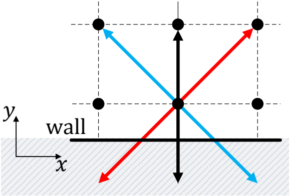
where $\delta = B \left(\frac{8\Lambda - 3}{12} \right)$ with $B = \frac{3B_f}{\Lambda^+}$

Observations:

- Consistency requires viscosity independent solutions. Impossible for BGK since $\Lambda = \left(\tau - \frac{1}{2}\right)^2$ makes $\delta(B, \nu)$!
- No numerical oscillations require $\delta > -1$, straight channel: $B < \frac{12}{3 - 8\Lambda}$.
- Idea of IBF, giving $\Lambda^+ = 3\nu_B$, $B_f = \nu / k$ and Λ, redefine: $\Lambda^+ \to \Lambda^+_\star \left(\nu_B, B_f, \Lambda\right)$. Example, $\Lambda^+_\star = \frac{9(4+B)\nu_B}{4(3+2B\Lambda)}$ makes $\delta = 0$.

Solid wall:

- Bounce-back: $f_q\left(\vec{r}_b, t+1\right) = \tilde{f}_{\bar{q}}\left(\vec{r}_b, t\right)$
  with $\vec{c}_{\bar{q}} = -\vec{c}_q$ and $\vec{r}_b + \vec{c}_q \in$ solid



For a straight channel of width H, the bounce-back closure relation reads:

$$u_x \pm \frac{1}{2}\alpha^+ \bar{\Delta}_y u_x + \frac{1}{8}\alpha^- \bar{\Delta}_y^2 u_x \Big|_{y_{\text{wall}} = \pm\frac{H}{2} \mp \frac{1}{2}} = 0$$

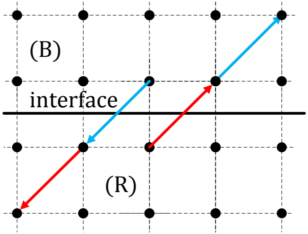
with $\alpha^+ = (1+\delta)$ and $\alpha^- = (1+\delta)\frac{16}{3}\Lambda$
(different coefficient values for IBF)

$\boxed{\text{Interface:}}$

- Implicit interface

$$\begin{cases} f_q(\vec{r}^{(B)}, t+1) = \tilde{f}_q(\vec{r}^{(R)}, t) \\ \tilde{f}_{\bar{q}}(\vec{r}^{(B)}, t) = f_{\bar{q}}(\vec{r}^{(R)}, t+1) \end{cases}$$

with $\vec{r}^{(B)} = \vec{r}^{(R)} + \vec{c}_q$



For stratified channels the steady-state closure relation for the (implicit) interface reads:

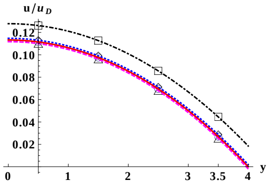
$$\|\nu_i(\bar{\Delta}_y u_x^{(i)} \pm \tfrac{1}{2}\bar{\Delta}_y^2 u_x^{(i)})\|_{Y_i \mp \frac{H}{2}} = 0, \quad \nu_i = \nu_B^{(i)}(1+\delta_i),$$

$$\|u_x^{(i)} \pm \tfrac{1}{2}\alpha^+ \bar{\Delta}_y u_x^{(i)} + \tfrac{1}{8}\alpha^- \bar{\Delta}_y^2 u_x^{(i)}\|_{Y_i \mp \frac{H}{2}} = 0$$

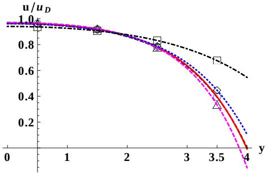
with $\alpha^+ = (1+\delta)$ and $\alpha^- = (1+\delta)\frac{16}{3}\Lambda$
(different coefficient values for IBF)

Bounded channel

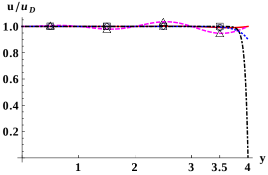
Solutions width $H = 8$ with $\Lambda = \{\frac{1}{512}, \frac{3}{16}, \frac{3}{8}, 2\}$



Stokes-Brinkman regime ($\sigma = 1$)



Brinkman regime ($\sigma = 8$)

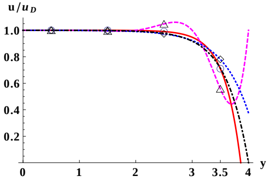


Darcy-Brinkman regime ($\sigma = 160$)

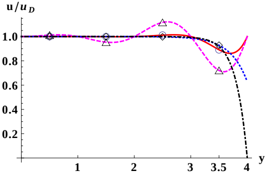
Bounded channel

Solutions width $H = 8$ with $\Lambda = \{\frac{1}{512}, \frac{3}{16}, \frac{3}{8}\}$

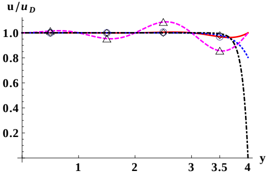
BF:



Brinkman regime ($\sigma = 20$)

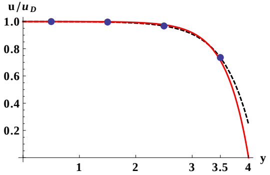


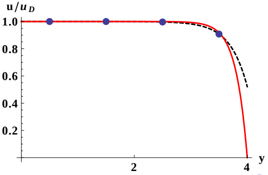
Brinkman regime ($\sigma = 40$)

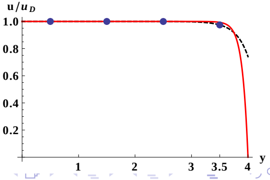


Brinkman regime ($\sigma = 80$)

IDF:

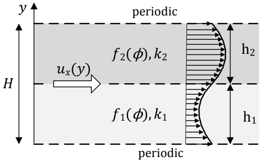




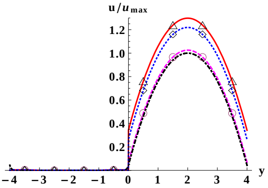


Unbounded channel

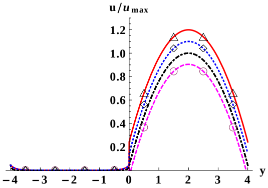
Solutions width $H = 8$ with $\Lambda = \{\frac{1}{512}, \frac{1}{8}, \frac{3}{16}\}$



Periodic unbounded channel



$\sigma \approx 0.26,\ k_2/k_1 = 10^6$ (BF)



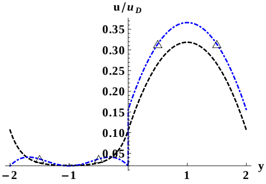
$\sigma \approx 0.26,\ k_2/k_1 = 10^6$ (IBF)

$\sigma \approx 2.8,\ k_2/k_1 = 10^2$:

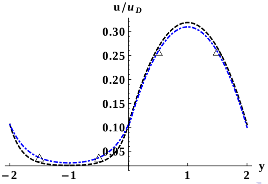
$H = 4$ and $\Lambda = \frac{1}{8}$

$H = 8$ and $\Lambda = \frac{1}{8}$

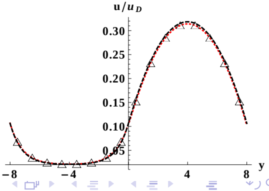
BF



IBF

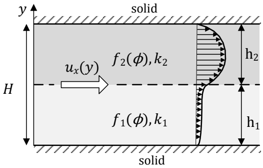


IBF(≈BF)

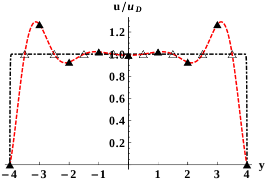


Bounded layered channel: FEM vs TRT (BF)

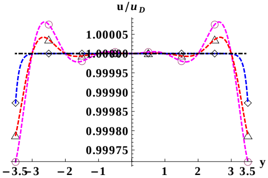
Solutions width $H = 8$ with $\Lambda = \{\frac{1}{512}, \frac{3}{16}, \frac{3}{8}\}$



Bounded stratified channel

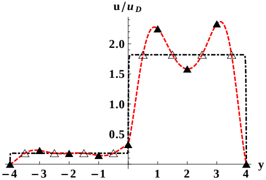


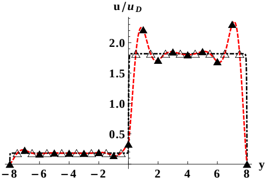
$\sigma = 10^3,\ k_2 = k_1$ (FEM/BF)

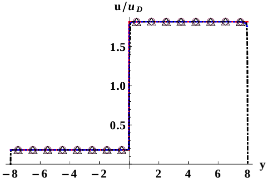


Zoom in: $\sigma = 10^3,\ k_2 = k_1$ (BF)

$H = \{8, 16\}$ and $\Lambda = \frac{1}{8}$ $\qquad$ $\sigma = 10^3,\ k_2/k_1 = 10$ $\qquad$ $H = 16$ and $\Lambda = \{\frac{1}{8}, \frac{3}{16}, \frac{3}{8}\}$







Brinkman flow in bounded channels: boundary condition

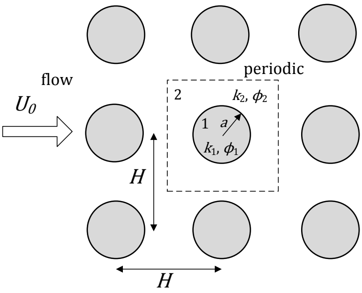
- Physical solution experiences three regimes: Stokes, Brinkman, Darcy.
- Discrete solution is controlled by bulk and boundary approximations.
- Bulk error in BF is vanished for $\Lambda = 3/8$. For $\Lambda < 3/8$ may lead to wiggles in bulk solution. Bulk error is absent by construction in IBF.
- Bounce-back boundary error is always present (already at the 1st order) for $B \neq 0$ (when $B = 0$, it is exact for $\Lambda < 3/16$).
- Bounce-back prescribes boundary implicitly. It allows smoother bulk accommodation on wall. Enforcement on wall nodes triggers oscillations, e.g. FEM.

Brinkman flow in layered channels: interface condition

- Discrete solution is controlled by bulk and interface approximations.
- Implicit interface prescription may lead to jumps in continuity conditions.
- Interface continuity is only exact for $B = 0$ (Stokes). Interface jumps increase for $B \gg 1$ (Darcy). They decrease for $\Lambda \ll 1$, but produce wiggles. Solution: IBF.

Benchmark description

Idealized geometry:

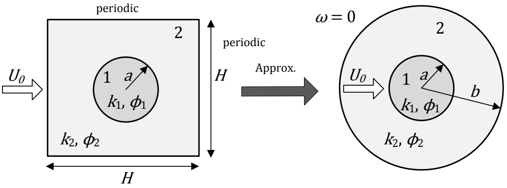


Characteristic parameters:

- Obstacles configuration:
  Square array arrangement
- Obstacles concentration:
  $\boxed{c = \frac{\pi a^2}{H^2}}$

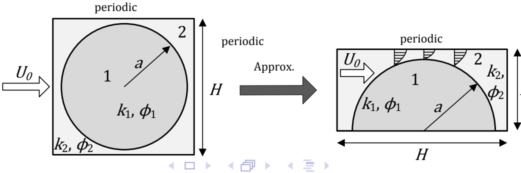
Cell model (dilute limit):

$c < 1$

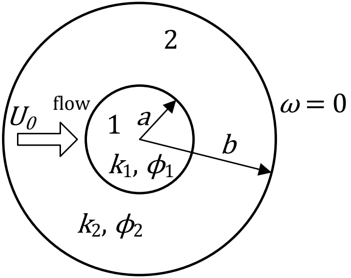


Lubrication theory (high concentration limit):

$c \sim 1$



Cell model:



Main features:

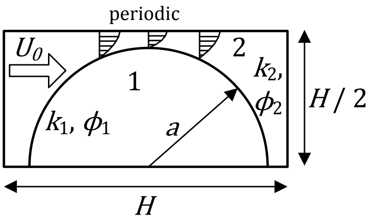
- Ratio of particle to cell volume c remains invariant
- Independent of packing arrangement (in principle requires $c \ll 1$)
- Approximation on outer BC: $\omega = 0$ (Kuwabara model)
- Polar coordinates: $(x, y) \mapsto (r, \theta)$
- Streamfunction formulation: $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ and $u_\theta = -\frac{\partial \psi}{\partial r}$

General solution (Brinkman):

$$\psi^{(i)}(r,\theta) = \left(\frac{C_1^{(i)}}{r} + C_2^{(i)} r + C_3^{(i)} I_1(\sigma_1 r) + C_4^{(i)} K_1(\sigma_1 r)\right)\sin(\theta)$$

subject to proper interface/solid BC

Lubrication theory:



Main features:

- Ratio of particle to cell volume c remains invariant
- Depends on packing arrangement
- Approximation in bulk: flow asymptotically unidirectional (Lubrication model)
- Caution: singular behaviour when $H/2 - a \to 0$

General solution (Brinkman):

$$u_x^{(i)}(y) = -\frac{1}{\sigma_i^2}\frac{\mathrm{d}p}{\mathrm{d}x} + C_1^{(i)}\exp(y\sigma_i) + C_2^{(i)}\exp(-y\sigma_i)$$

subject to proper interface/solid BC

Permeability computations:

Cell model

$$F = \int_0^{2\pi} \left(\left(-p^{(2)} + \tau_{rr}^{(2)}\right) \cos(\theta) - \tau_{r\theta}^{(2)} \sin(\theta) \right)_{r=b} b \, \mathrm{d}\theta$$

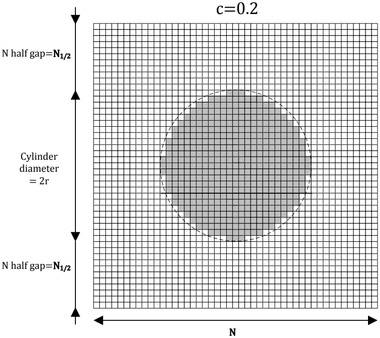
$$k_{eff} = \frac{U_0}{F/V} = \frac{U_0 \, \pi b^2}{F}$$

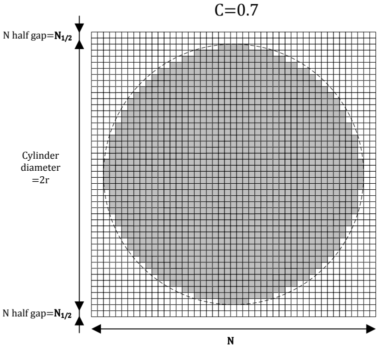
Lubrication theory

$$Q = \int_0^{a(x)} u_x^{(1)} \, \mathrm{d}y + \int_{a(x)}^{H/2} u_x^{(2)} \, \mathrm{d}y = \frac{-f(x)}{\mu} \frac{\mathrm{d}p}{\mathrm{d}x}$$

$$k_{eff} = \frac{4a}{H} \left(\int_0^a \frac{1}{f(x)} \, \mathrm{d}x \right)^{-1}$$

Domain discretization

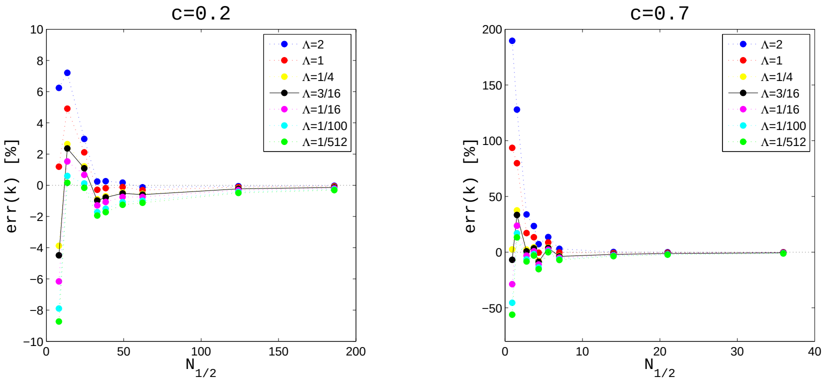




Solid cylinder is discretized in a regular grid by *bounce-back* (i.e. as staircase shape), introducing a discretization error of $O(N^{-1})$

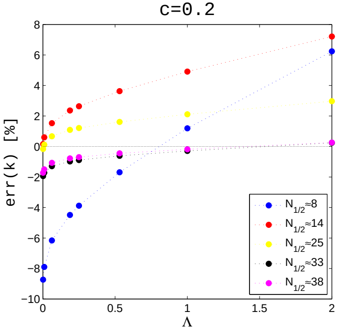
Note: Numerical solutions are evaluated for one half gap size – $\underline{N_{1/2}}$

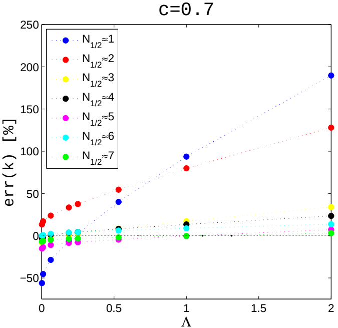
Effect of lattice resolution on accuracy



is non-monotonic and with large dependency on Λ at coarse resolutions

Effect of Λ on accuracy



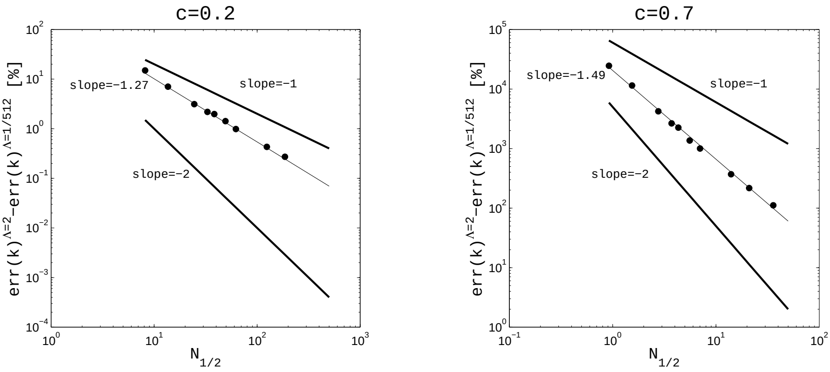


For fixed lattice resolution Λ controls Stokes solution accuracy through the location of *bounce-back* solid boundary

Remarks:

1. Staircase representation of cylinder reduces its true hydraulic diameter $\Rightarrow$ makes permeability solution converge to a smaller value
2. Large Λ reduces hydraulic diameter of cylinder $\Rightarrow$ under-estimates permeability at fixed grid resolution, i.e. $\frac{\partial \mathrm{err(k)}}{\partial \Lambda}|_N > 0$

Effect of lattice resolution on accuracy

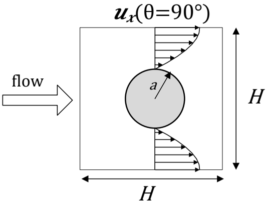


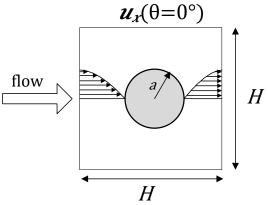
becomes monotonic taking difference between any two Λ solutions

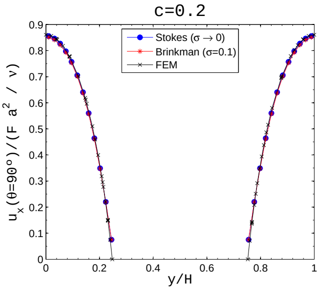
- Spatial convergence rate is between 1^{st}– and 2^{nd}–order

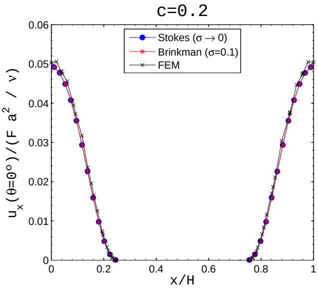
Porous flow around solid cylinder

$$\boxed{\text{Stokes regime}} \iff \vec{\nabla} p = \Delta \vec{u} - \underbrace{\cancel{\sigma^2 \vec{u}}}_{\sigma \to 0}$$



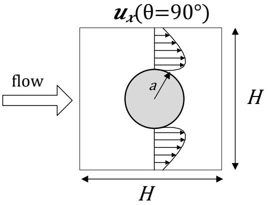


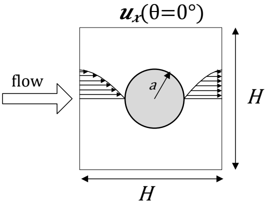


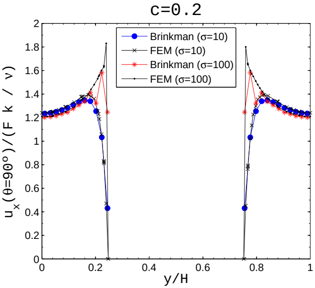


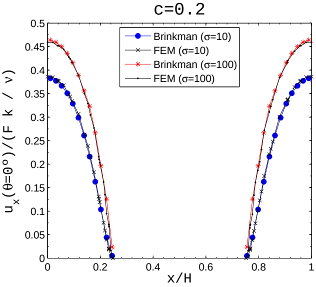
Porous flow around solid cylinder

$$\boxed{\text{Brinkman regime}} \iff \vec{\nabla} p = \Delta \vec{u} - \underbrace{\sigma^2 \vec{u}}_{\sigma \sim O(1)}$$



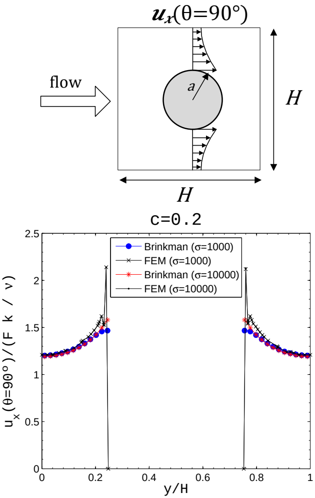


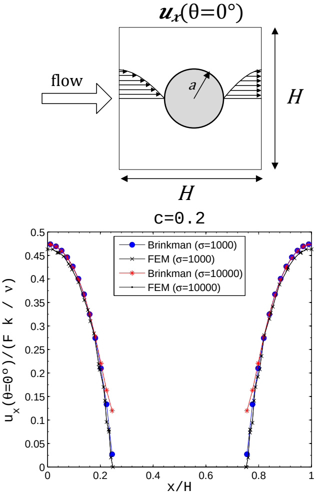




Porous flow around solid cylinder

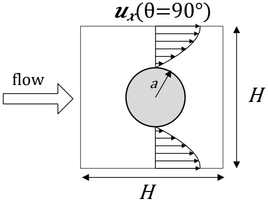
$\boxed{\text{Darcy regime}}$ $\Longleftrightarrow$ $\vec{\nabla} p = \cancel{\Delta \vec{u}} - \underbrace{\sigma^2 \vec{u}}_{\sigma \to \infty}$

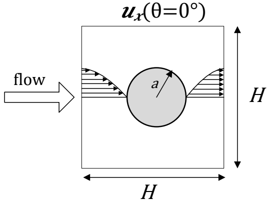


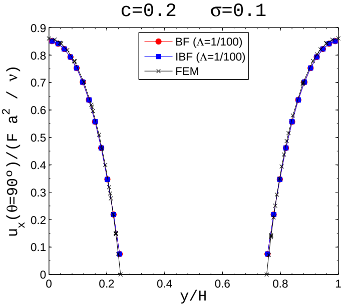


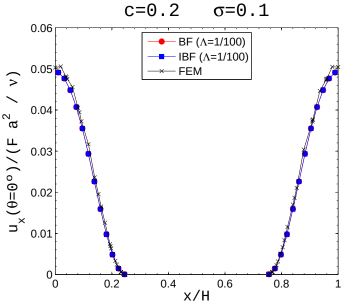
Porous flow around solid cylinder

Stokes regime (with IBF)

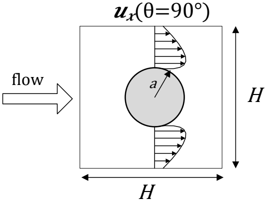


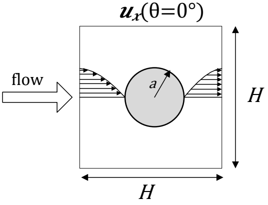


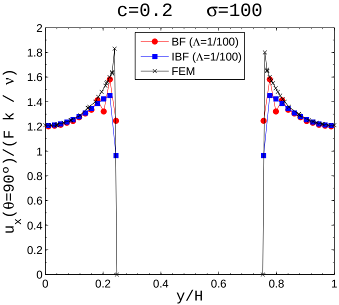


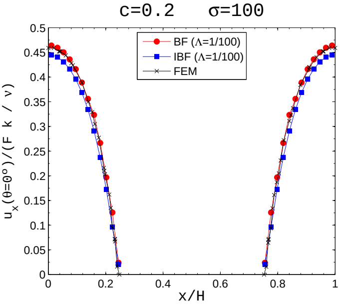


Brinkman regime (with IBF)

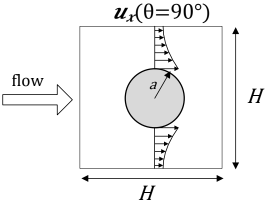


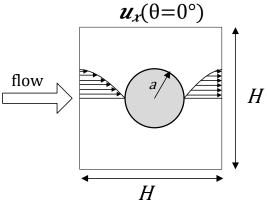


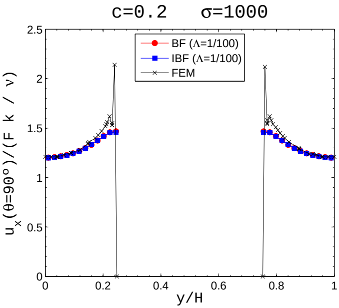


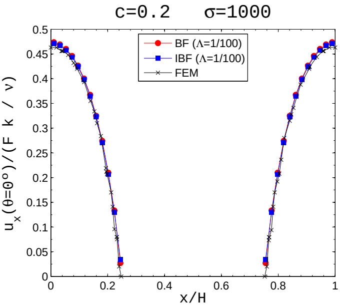


Darcy regime (with IBF)



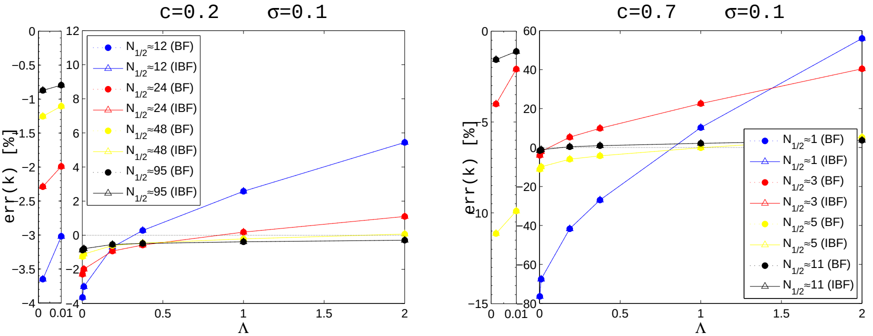






Effect of Λ on accuracy

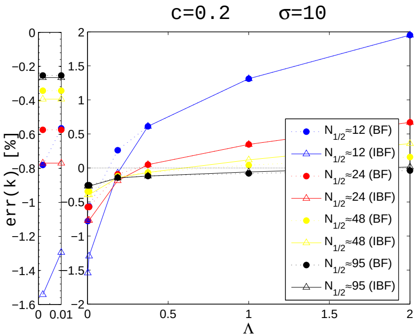
Stokes regime

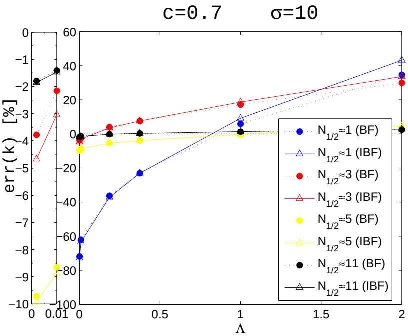


Essentially similar to TRT Stokes solver $\Rightarrow$ Λ controls the accuracy through the location of *bounce-back* solid boundary

Effect of Λ on accuracy

Brinkman regime

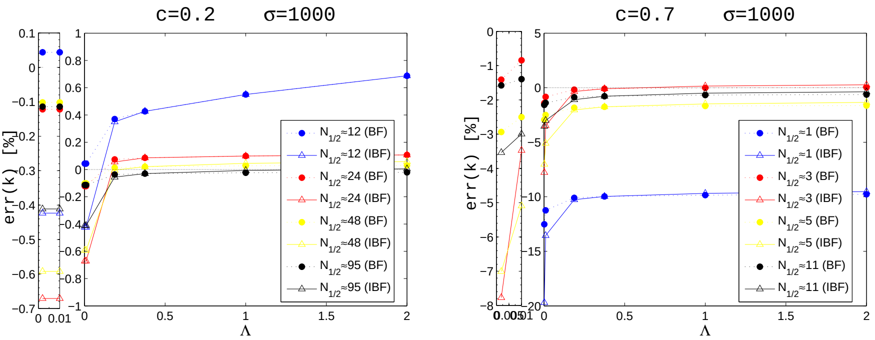




Brinkman correction (in bulk and boundary) now plays a role
$\Rightarrow$ smaller influence of Λ over accuracy (compared to Stokes)

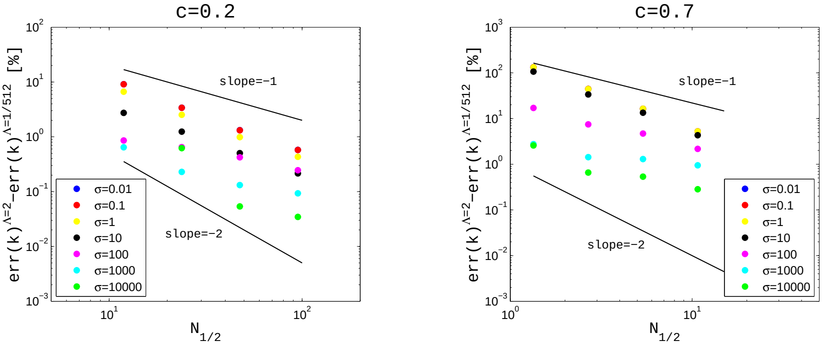
Effect of Λ on accuracy

Darcy regime



Accuracy becomes independent of Λ
except at small Λ values

Effect of lattice resolution on accuracy



$\forall\,\sigma$ spatial convergence rate between 1^{st}- and 2^{nd}-order

(Data shown for BF model)

Stokes flow around solid cylinder

- Role of Λ controls the location of the bounce-back solid boundary (by varying the size of cylinder)
- Dependency on Λ is larger at coarser lattice resolutions
- Dependency on lattice resolution indicates spatial convergence rate between 1^{st}– and 2^{nd}–order

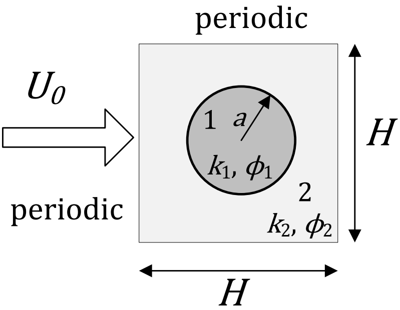
Porous flow around solid cylinder

- 3 physical regimes: Stokes $\sigma \ll 1$, Brinkman $\sigma \sim O(1)$, Darcy $\sigma \gg 1$
- Stokes regime: role of Λ controls the location of the bounce-back solid boundary $\Rightarrow$ accuracy is worst and larger dependency on Λ
- Brinkman regime: role of Λ controls both bulk and boundaries errors (Brinkman correction). Better accuracy for k. Yet, velocity solutions may experience wiggles $\Leftarrow$ IBF corrects them ☺
- Darcy regime: role of Λ tends to cease, except at small Λ values
- Effect of lattice resolution indicates spatial convergence rate between 1^{st}– and 2^{nd}–order $\forall$ σ regimes

Two physical problems to distinguish from:

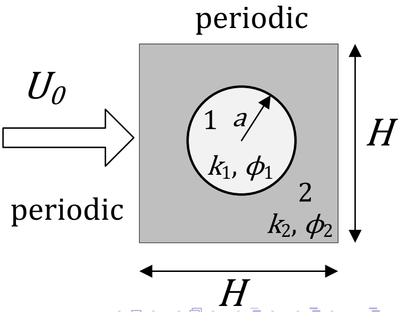
1) Porous cylinder is less permeable than outside porous medium

$$\frac{k_1}{k_2} < 1$$



2) Porous cylinder is more permeable than outside porous medium

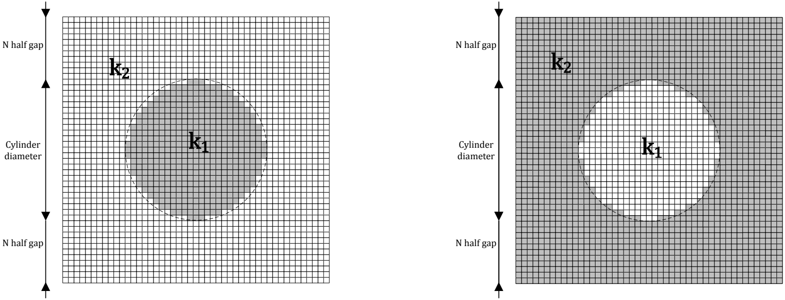
$$\frac{k_1}{k_2} > 1$$



Biporous medium: Introduction

In both cases, **interface** continuity conditions are set *implicitly* and approximate a *staircase* cylinder

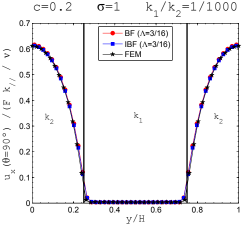
$$\boxed{\frac{k_1}{k_2} < 1} \qquad \boxed{\frac{k_1}{k_2} > 1}$$



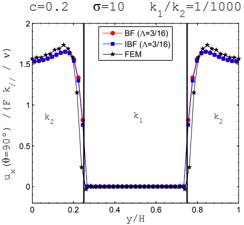
Porous flow across porous cylinder $k_1 \ll k_2$

$u_x(y)$ at vertical ($\theta = 90°$) midplane for $\frac{k_1}{k_2} = \frac{1}{1000}$ and $c = 0.2$

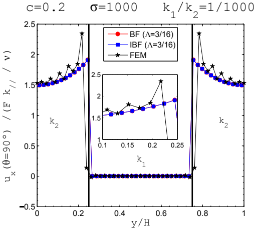
"Stokes-type flow"



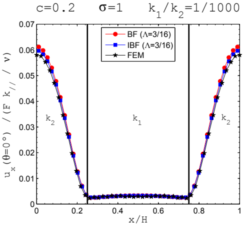
"Brinkman-type flow"

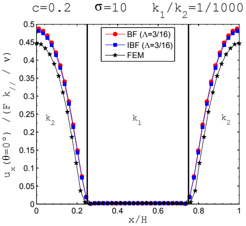


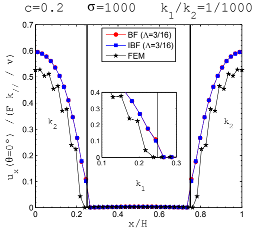
"Darcy-type flow"



$u_x(x)$ at horizontal ($\theta = 0°$) midplane for $\frac{k_1}{k_2} = \frac{1}{1000}$ and $c = 0.2$

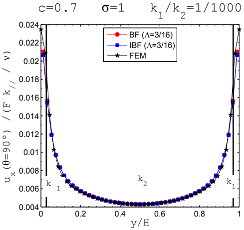




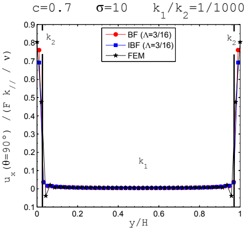


$u_x(y)$ at vertical ($\theta = 90°$) midplane for $\frac{k_1}{k_2} = \frac{1}{1000}$ and $c = 0.7$

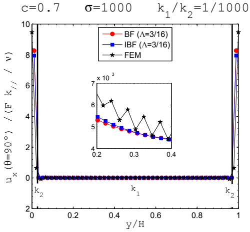
"Stokes-type flow"



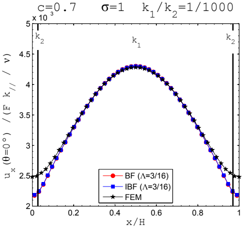
"Brinkman-type flow"

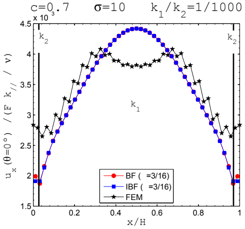


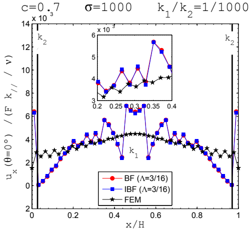
"Darcy-type flow"



$u_x(x)$ at horizontal ($\theta = 0°$) midplane for $\frac{k_1}{k_2} = \frac{1}{1000}$ and $c = 0.7$



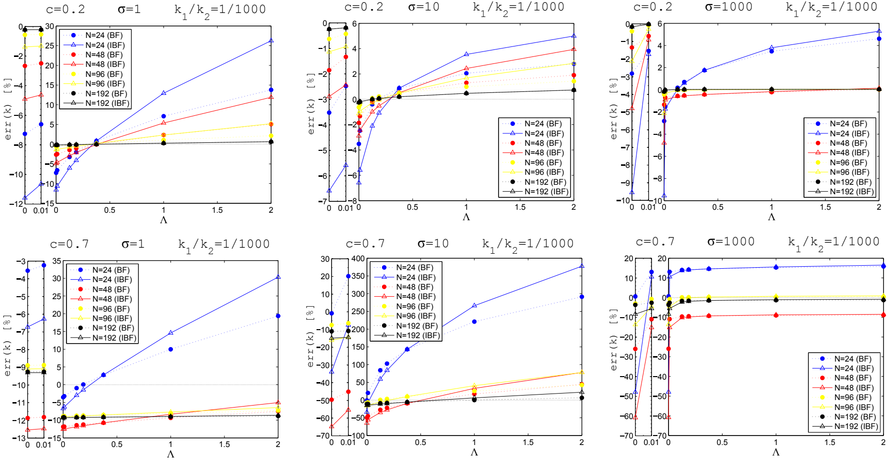




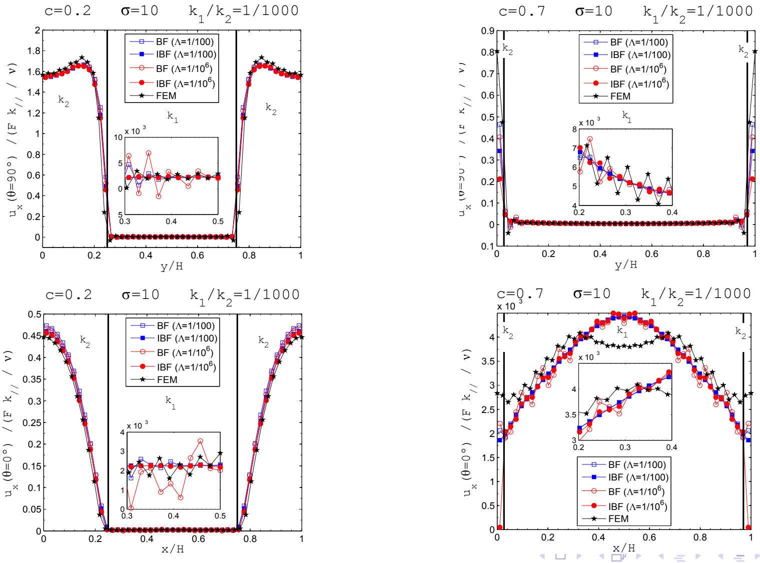
Porous flow across porous cylinder $k_1 \ll k_2$

Effect of Λ on accuracy for $\frac{k_1}{k_2} = \frac{1}{1000}$

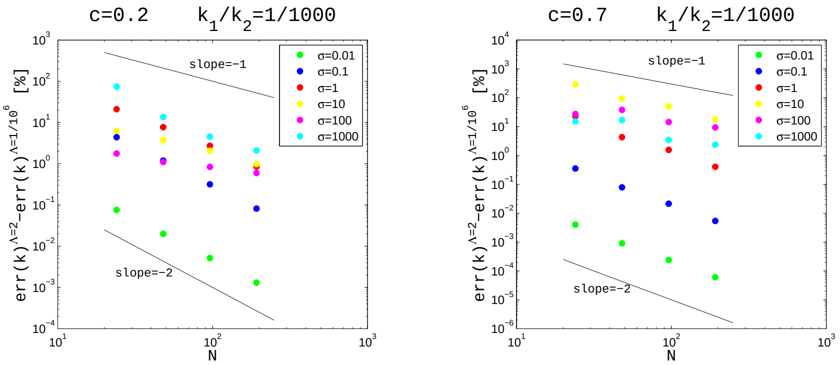
"Stokes-type flow" "Brinkman-type flow" "Darcy-type flow"



Effect of IBF on $u_x(y)$ and $u_x(x)$ for $\frac{k_1}{k_2} = \frac{1}{1000}$



Effect of lattice resolution on accuracy for $\frac{k_1}{k_2} = \frac{1}{1000}$



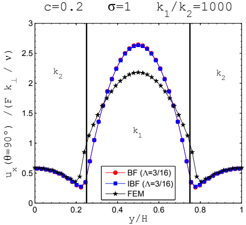
$\forall\, \sigma$ spatial convergence rate between 1$^{\text{st}}$- and 2$^{\text{nd}}$-order

(Data shown for BF model)

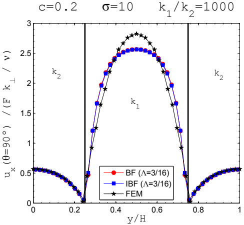
Porous flow across porous cylinder $k_1 \gg k_2$

$u_x(y)$ at vertical ($\theta = 90°$) midplane for $\frac{k_1}{k_2} = 1000$ and $c = 0.2$

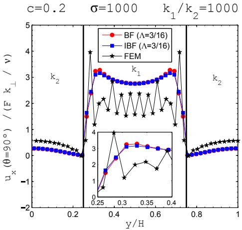
"Stokes-type flow"



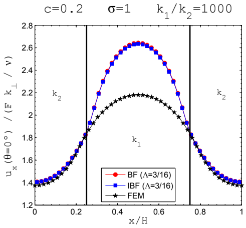
"Brinkman-type flow"

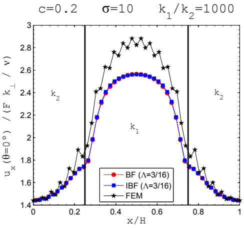


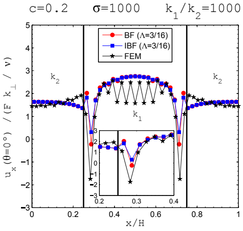
"Darcy-type flow"



$u_x(x)$ at horizontal ($\theta = 0°$) midplane for $\frac{k_1}{k_2} = 1000$ and $c = 0.2$



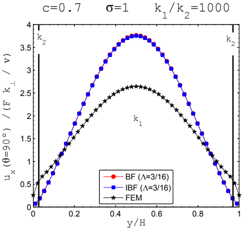




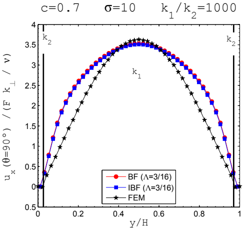
Porous flow across porous cylinder $k_1 \gg k_2$

$u_x(y)$ at vertical ($\theta = 90°$) midplane for $\frac{k_1}{k_2} = 1000$ and $c = 0.7$

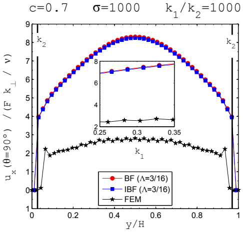
"Stokes-type flow"



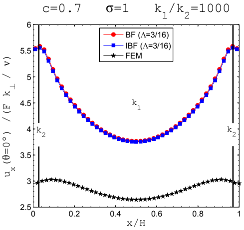
"Brinkman-type flow"

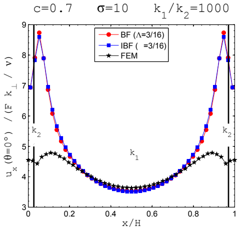


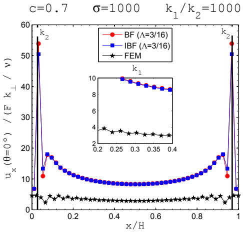
"Darcy-type flow"



$u_x(x)$ at horizontal ($\theta = 0°$) midplane for $\frac{k_1}{k_2} = 1000$ and $c = 0.7$



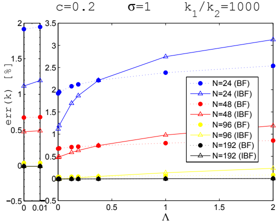




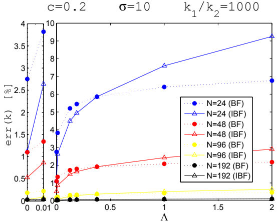
Porous flow across porous cylinder $k_1 \gg k_2$

Effect of Λ on accuracy for $\frac{k_1}{k_2} = 1000$

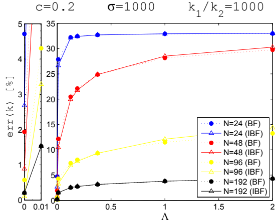
“Stokes-type flow”

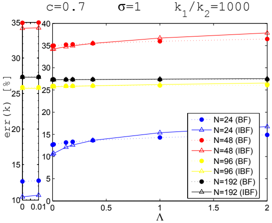


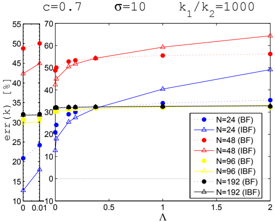
“Brinkman-type flow”

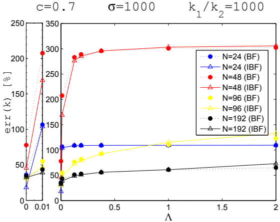


“Darcy-type flow”

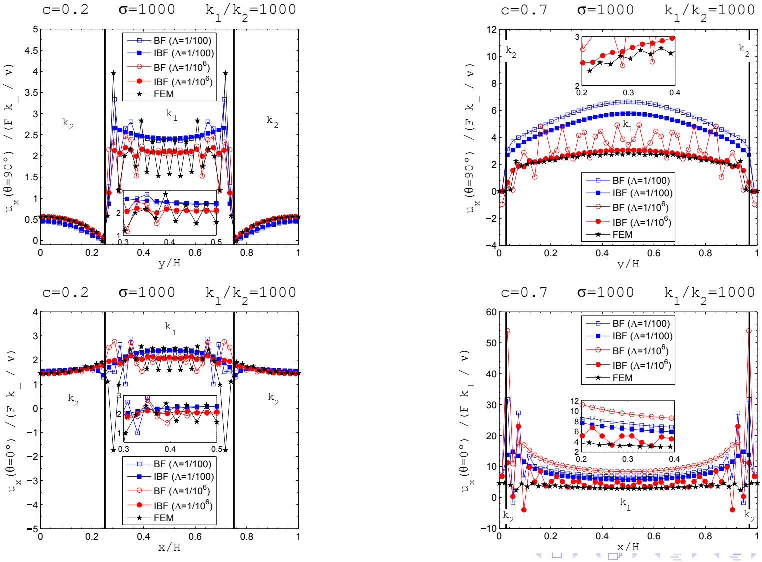




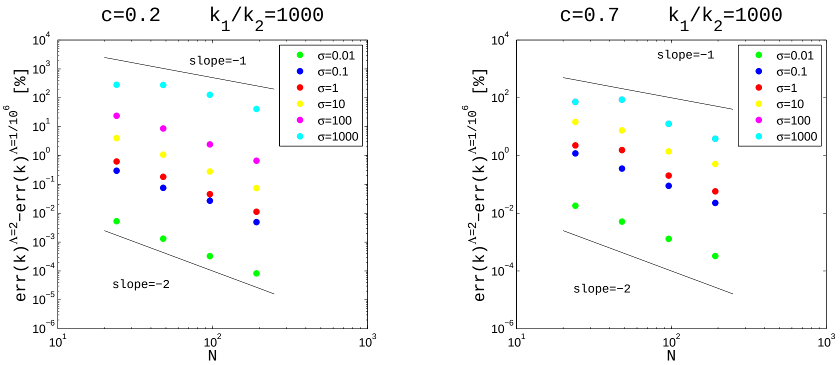




Effect of IBF on $u_x(y)$ and $u_x(x)$ for $\frac{k_1}{k_2} = 1000$



Effect of lattice resolution on accuracy for $\frac{k_1}{k_2} = 1000$



$\forall\, \sigma$ spatial convergence rate between 1^{st}- and 2^{nd}-order

(Data shown for BF model)

Biporous medium: $k_1 \ll k_2$

- Solutions behave, approximately, as case of Brinkman flow around solid cylinder.
- However, with interface playing the role of boundary we recover larger errors (particularly in Brinkman regime).
- TRT velocity solutions are *over-estimated*, except for small Λ
- Small Λ values lead to wiggles velocity solution profiles. They are partially corrected by IBF
- Dependency on lattice resolution indicates spatial convergence rate between $\underline{1^{\text{st}}\text{– and } 2^{\text{nd}}\text{–order}}$

Biporous medium: $k_1 \gg k_2$

- TRT velocity solutions are *significantly over-estimated* in Darcy regime. Solution requires use of small Λ
- However, small Λ values produce *strong* wiggles in velocity profiles. They are successfully corrected by IBF
- Dependency on lattice resolution indicates spatial convergence rate between $\underline{1^{\text{st}}\text{– and } 2^{\text{nd}}\text{–order}}$