Universal vanishing corrections on the position of fronts in the Fisher-KPP class

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The Fisher-KPP equation — a model for reaction-diffusion

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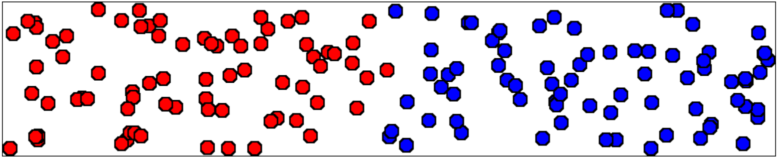
In **mean-field** reaction-diffusion system (chemistry)

A and B diffuse $\qquad A + B \to 2A$



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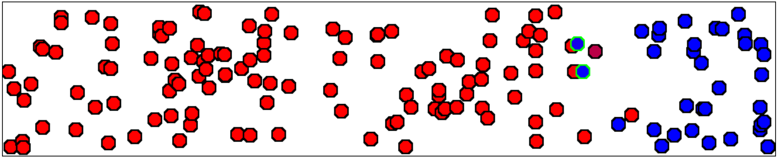
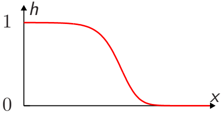


For a **very large** concentration

$$h(x,t) = \begin{pmatrix}\text{proportion of } A \\ \text{around } x \text{ at time } t\end{pmatrix} =$$



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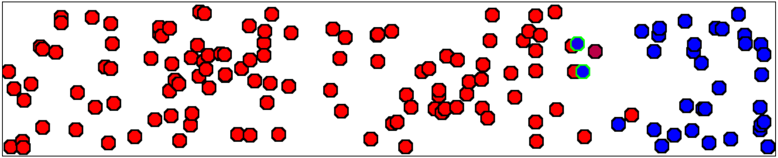

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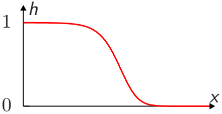

follows Fisher-KPP

Also the mean-field of an evolutionary problem

A and B diffuse $\qquad A$ reproduces faster than B $\qquad$ population size constant

The Fisher-KPP equation — step initial condition

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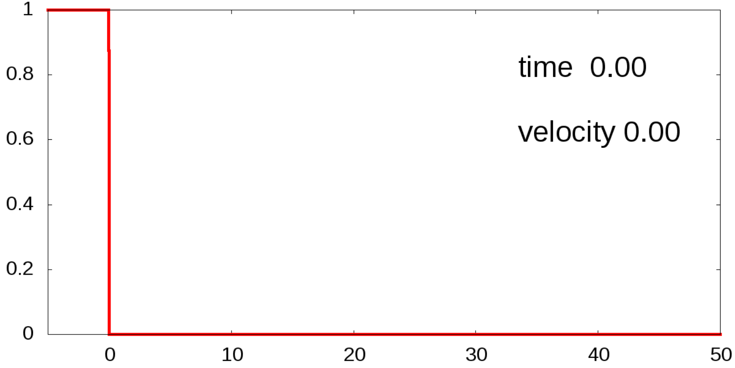
For a step initial condition, $h(x,t)$ as a function of x.



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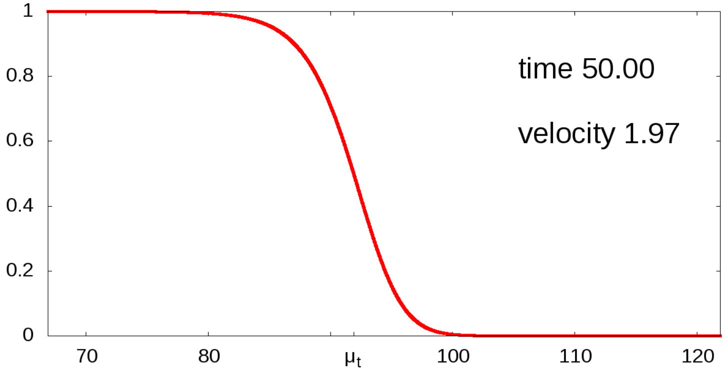

Convergence to a travelling wave: $h(\mu_t + z, t) \to \omega(z)$, $\boxed{\dfrac{\mu_t}{t} \to 2}$

The Fisher-KPP equation — step initial condition

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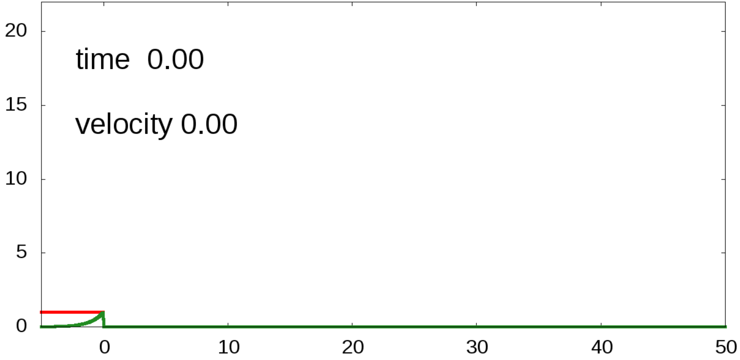
For a step initial condition, $h(x,t)$ and $h(x,t)e^{x-\mu_t}$ as a function of x.



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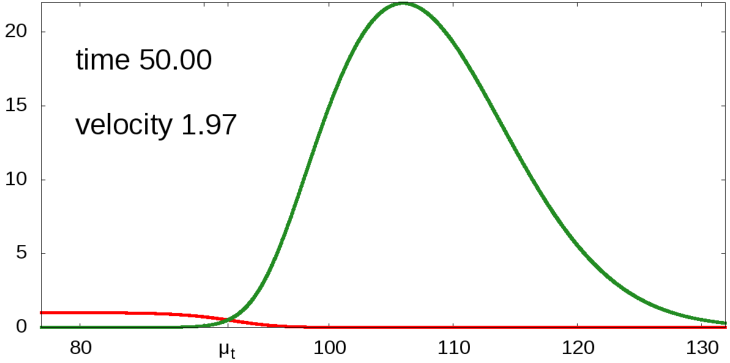

$h(\mu_t + z, t) \to \omega(z)$ with $\omega(z) \sim Aze^{-z}$,

(Scaling regime: $h(\mu_t + z, t)e^z \approx Aze^{-\frac{z^2}{4t}}$)

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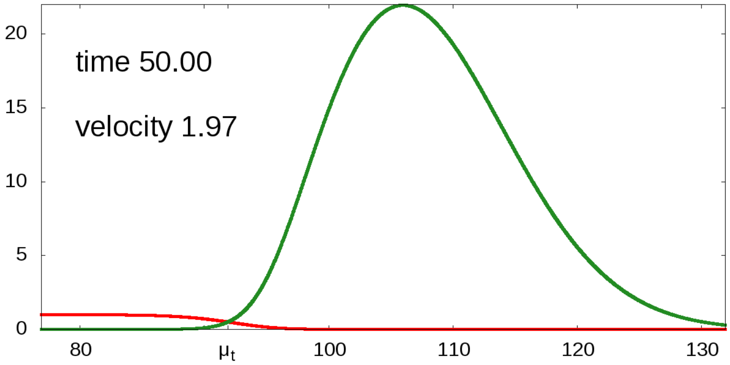

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The Fisher-KPP equation — other initial conditions

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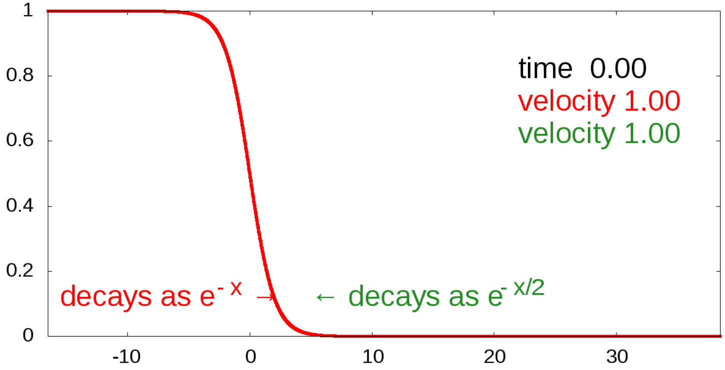
Compare $h(x,t)$ and $h(x,t)$ with initial conditions $\dfrac{1}{1+e^x}$ and $\dfrac{1+.001e^{x/2}}{1+e^x}$



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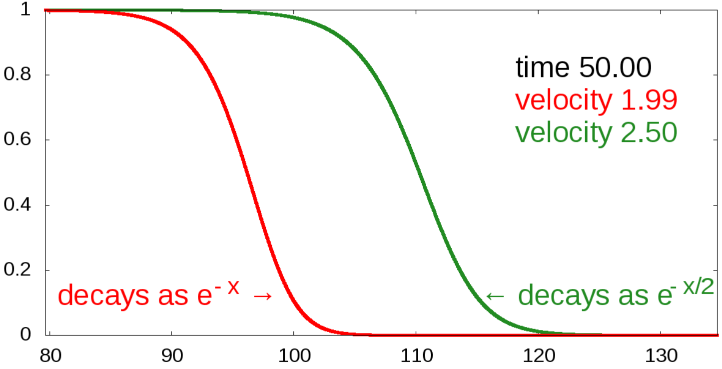

Not the same velocity, not the same travelling wave

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- Velocity v depends only on how h_0 decays at infinity

$$h_0(x) \sim A x^\alpha e^{-\gamma x} \implies \begin{cases} v = 2 & \text{if } \gamma \geq 1, \\ v = \gamma + \frac{1}{\gamma} > 2 & \text{if } \gamma < 1. \end{cases}$$

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  Some other terms require saturation
- The nature of the saturation term is not important

The Fisher-KPP equation — universality

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v_c=2, γ_c=1
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Diffusion, linear growth ($h = 0$ is unstable), saturation ($h = 1$ is stable).

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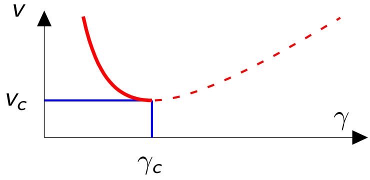

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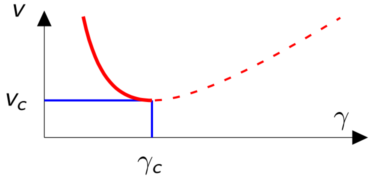

To find $v(\gamma)$:

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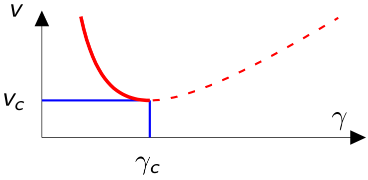

To find $v(\gamma)$: linearise

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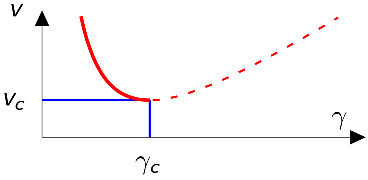

To find $v(\gamma)$: linearise $h(x,t) \propto e^{-\gamma(x-vt)}$

The Fisher-KPP front — precise position

$\partial_t h = \partial_x^2 h + h - h^2$, initial condition h_0, μ_t is the position: $h(\mu_t, t) = \frac{1}{2}$.

Theorem (Bramson 1983)

$$\mu_t = v_c t - \frac{3}{2\gamma_c} \ln t + \text{cste} + o(1) \qquad \text{for large } t$$

if and only if $\int \mathrm{d}x\, h_0(x) x\, e^{\gamma_c x} < \infty$

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Conjecture (Ebert and van Saarloos 2000)

$$\mu_t = v_c t - \frac{3}{2\gamma_c} \ln t + \text{cste} - 3\sqrt{\frac{2\pi}{\gamma_c^5 v''(\gamma_c)}}\, t^{-\frac{1}{2}} + \cdots$$

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Precise position — strategy

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We construct an equation with the simplest possible saturation term

Three approaches

First approach, on the lattice

[Joint work with B. Derrida]

$$\partial_t h(n,t) = \begin{cases} h(n,t) + h(n-1,t) & \text{if } h(n,t) < 1, \\ 0 & \text{if } h(n,t) = 1, \end{cases}$$

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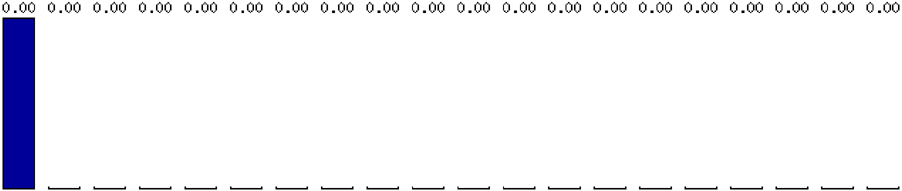

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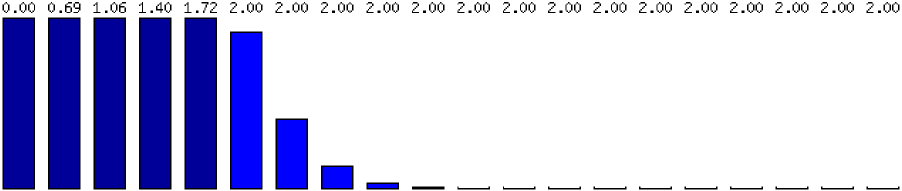

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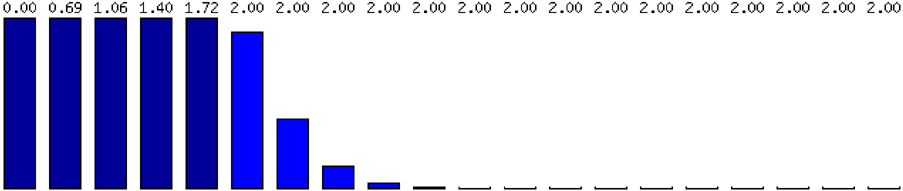

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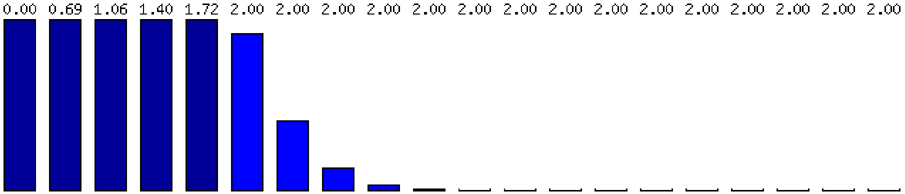

Main result

Assuming $h_0(0) = 0$ and $h_0(n) \searrow$

$$\sum_{n\geq 1} h_0(n) e^{\lambda n} = \frac{1}{e^{\lambda}+1}\left[2\sum_{n\geq 1} e^{\lambda\left(n - \frac{e^{\lambda}+1}{\lambda} t_n\right)} - e^{\lambda}\right]$$

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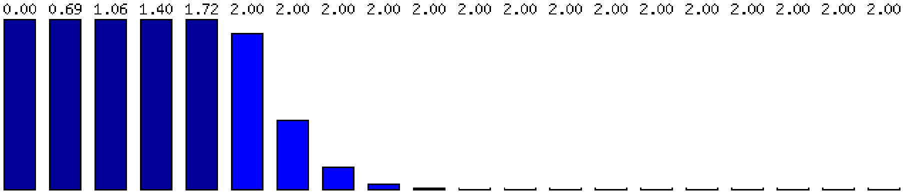

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From there, extrain asymptotic behaviour of t_n as $n \to \infty$.

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Can be solved with same method as First approach. But is it a well-posed problem ?

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What are the μ_t such that $h(\mu_t + z, t) \to \omega(z)$?

Three approaches

First approach, on the lattice

[Joint work with B. Derrida]

$$\partial_t h(n,t) = \begin{cases} h(n,t) + h(n-1,t) & \text{if } h(n,t) < 1, \\ 0 & \text{if } h(n,t) = 1, \end{cases} \qquad t_n = \begin{bmatrix} \text{when } h(n,t) \\ \text{reaches } 1 \end{bmatrix}$$

Third approach, in the continuum

[Joint work with B. Derrida & J. Berestycki]

$$\partial_t h = \partial_x^2 h + h \quad \text{if } x > \mu_t, \qquad h(\mu_t, t) = \alpha, \quad \partial_x h(\mu_t, t) = \beta.$$

Both $h(x,t)$ and μ_t are unknown quantities!

Second approach, in the continuum

[Joint work with J. Berestycki, M. Roberts & S. Harris. Also studied by C. Henderson]

For any given μ_t

$$\begin{cases} \partial_t h = \partial_x^2 h + h & \text{if } x > \mu_t \\ h(\mu_t, t) = 0 \end{cases}$$

if μ_t grows too fast,	$h(\mu_t + z, t) \to 0$
if μ_t grows too slowly,	$h(\mu_t + z, t) \to \infty$
if μ_t grows just right,	$h(\mu_t + z, t) \to \omega(z)$

$$h(x,t) = e^t \int \mathrm{d}y\, h_0(y) \mathbb{E}^y \Big[\delta(B_t - x) \mathbb{1}_{\{B_s > \mu_s, \forall s < t\}} \Big]$$

What are the μ_t such that $h(\mu_t + z, t) \to \omega(z)$? With a fast convergence rate?

Let us get more technical

We focus on

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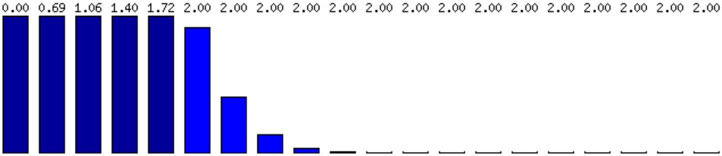

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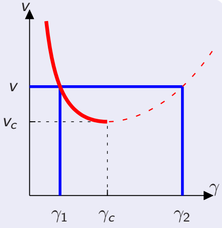

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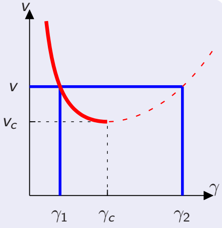

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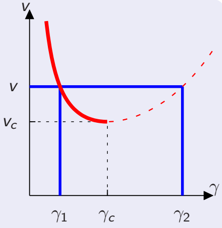

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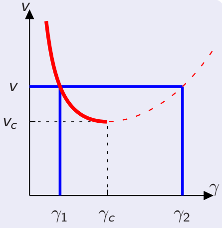

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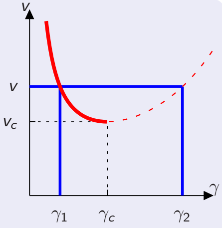

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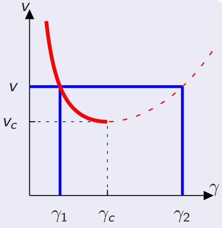

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- one must have $t_n \sim n/v(\gamma)$ to reproduce the singularity in the R.H.S.:

$$\left[n - v(\gamma-\epsilon)\frac{n}{v(\gamma)}\right] = n\epsilon\frac{v'(\gamma)}{v(\gamma)}$$

The case $h_0 = 0$

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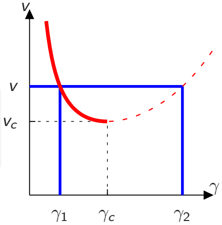

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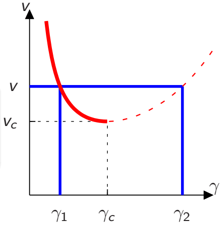

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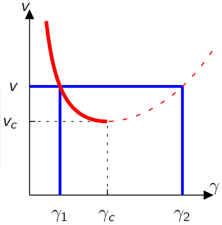

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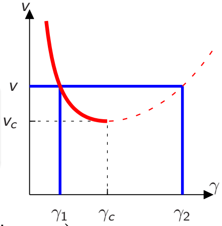

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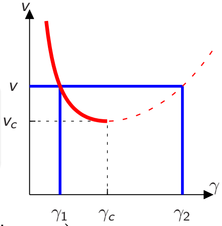

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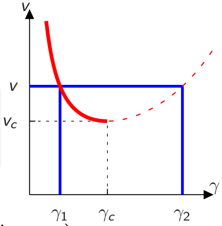

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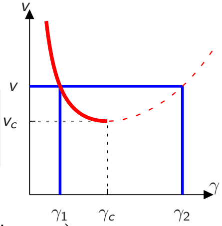

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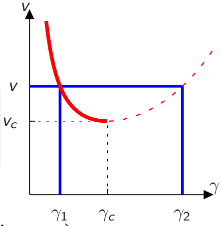

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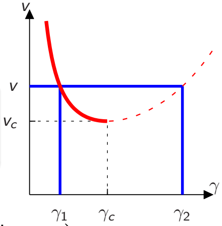

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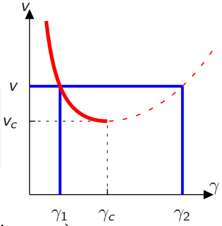

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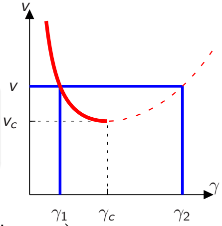

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$$\sum_{n\ge1} e^{-nu} n^{-\alpha} = (\text{nice function of } u) + \begin{cases} \Gamma(1-\alpha)u^{\alpha-1} & (\text{if } \alpha \notin \mathbb{N}^*) \\ \frac{(-1)^\alpha}{(\alpha-1)!}u^{\alpha-1}\ln u & (\text{if } \alpha \in \mathbb{N}^*) \end{cases}$$

Bramson's term

$$\Psi(\lambda) := \sum_{n\geq 1} h_0(n) e^{\lambda n} = \frac{1}{e^\lambda + 1}\left[2\sum_{n\geq 1} e^{\lambda[n - v(\lambda)t_n]} - e^\lambda\right]; \qquad \text{pick } \lambda = \gamma_c - \epsilon$$

If $h_0 = 0$, to get the term of order ϵ right, one must take

$$t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + \mathcal{O}(1)}{\gamma_c v_c}$$

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If $h_0(n) \sim A n^\alpha e^{-\gamma_c n}$

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Then

$$\Psi(\gamma_c - \epsilon) \approx \sum_n A n^\alpha e^{-\epsilon n}$$

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$$\alpha = -3.1: \ \Psi(\gamma_c - \epsilon) = a + b\epsilon + c\epsilon^2 + k\epsilon^{2.1} + \cdots$$

$$\alpha = -3: \ \Psi(\gamma_c - \epsilon) = a + b\epsilon + \epsilon^2 + k\epsilon^2 \ln\epsilon + \cdots$$

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$$\alpha = -1.7: \ \Psi(\gamma_c - \epsilon) = a + k\epsilon^{0.7} + \cdots$$

Bramson's term

$$\Psi(\lambda) := \sum_{n\ge1} h_0(n)e^{\lambda n} = \frac{1}{e^\lambda+1}\Big[2\sum_{n\ge1} e^{\lambda[n-v(\lambda)t_n]} - e^\lambda\Big]; \qquad \text{pick } \lambda=\gamma_c-\epsilon$$

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If $h_0(n)\sim A n^\alpha e^{-\gamma_c n}$

Then

$$\Psi(\gamma_c-\epsilon) \approx \sum_n A n^\alpha e^{-\epsilon n}$$

$\alpha=-3.1:\ \Psi(\gamma_c-\epsilon) = a + b\epsilon + c\epsilon^2 + k\epsilon^{2.1}+\cdots$

$\alpha=-3:\ \Psi(\gamma_c-\epsilon) = a + b\epsilon + \epsilon^2 + k\epsilon^2\ln\epsilon+\cdots$

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Bramson's $\frac{3}{2}\ln t$ term is there if $\alpha<-2$.

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$$\Psi(\lambda) := \sum_{n\geq 1} h_0(n) e^{\lambda n} = \frac{1}{e^\lambda + 1}\left[2\sum_{n\geq 1} e^{\lambda[n-v(\lambda)t_n]} - e^\lambda\right]; \qquad \text{pick } \lambda = \gamma_c - \epsilon$$

If $h_0 = 0$, to get the term of order ϵ right, one must take

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$\alpha = -3.1:\ \Psi(\gamma_c - \epsilon) = a + b\epsilon + c\epsilon^2 + k\epsilon^{2.1} + \cdots$

$\alpha = -3:\ \Psi(\gamma_c - \epsilon) = a + b\epsilon + \epsilon^2 + k\epsilon^2 \ln\epsilon + \cdots$

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$\alpha = -2:\ \Psi(\gamma_c - \epsilon) = a + k\epsilon\ln\epsilon + \cdots$

$\alpha = -1.7:\ \Psi(\gamma_c - \epsilon) = a + k\epsilon^{0.7} + \cdots$

Bramson's $\frac{3}{2}\ln t$ term is there if $\alpha < -2$.
In fact, Bramson's term is there iff $\Psi(\gamma_c - \epsilon) = a + b\epsilon + \cdots$.

Bramson's term

$$\Psi(\lambda) := \sum_{n\geq1} h_0(n) e^{\lambda n} = \frac{1}{e^\lambda + 1}\left[2\sum_{n\geq1} e^{\lambda[n-v(\lambda)t_n]} - e^\lambda\right]; \qquad \text{pick } \lambda = \gamma_c - \epsilon$$

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$\alpha = -3.1$: $\Psi(\gamma_c - \epsilon) = a + b\epsilon + c\epsilon^2 + k\epsilon^{2.1} + \cdots$

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Bramson's $\frac{3}{2}\ln t$ term is there if $\alpha < -2$.
In fact, Bramson's term is there iff $\Psi(\gamma_c - \epsilon) = a + b\epsilon + \cdots$.
Which means that Bramson's term is there iff $\sum_{n\geq1} h_0(n) n < \infty$.

The term from Ebert and van Saarloos

$$\Psi(\lambda) := \sum_{n\ge1} h_0(n)e^{\lambda n} = \frac{1}{e^\lambda + 1}\left[2\sum_{n\ge1} e^{\lambda[n-v(\lambda)t_n]} - e^\lambda\right], \quad \text{and pick } \lambda = \gamma_c - \epsilon$$

Bramson's term to get the ϵ term right iff $\sum_{n\ge1} h_0(n)n < \infty$:

$$t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + \mathcal{O}(1)}{\gamma_c v_c} \quad \Leftrightarrow \quad \mu_t = v_c t - \frac{3}{2\gamma_c}\ln t + \mathcal{O}(1)$$

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For $t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + C}{\gamma_c v_c}$ exactly: $\Psi(\gamma_c - \epsilon) = a + b\epsilon + k\epsilon^2 \ln \epsilon + \cdots$

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Remember:

If $h_0(n) \sim A n^\alpha e^{-\gamma_c n}$

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$$\Psi(\lambda) := \sum_{n\geq 1} h_0(n)e^{\lambda n} = \frac{1}{e^\lambda+1}\left[2\sum_{n\geq 1} e^{\lambda[n-v(\lambda)t_n]} - e^\lambda\right], \quad \text{and pick } \lambda = \gamma_c - \epsilon$$

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- If $\alpha < -3$, there is no $\epsilon^2 \ln\epsilon$ term

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$$t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + \mathcal{O}(1)}{\gamma_c v_c} \quad \Leftrightarrow \quad \mu_t = v_c t - \frac{3}{2\gamma_c}\ln t + \mathcal{O}(1)$$

For $t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + C}{\gamma_c v_c}$ exactly: $\qquad \Psi(\gamma_c - \epsilon) = a + b\epsilon + k\epsilon^2 \ln\epsilon + \cdots$

Remember:
If $h_0(n) \sim A n^\alpha e^{-\gamma_c n}$

$$\alpha = -3.1: \quad \Psi(\gamma_c - \epsilon) = a + b\epsilon + c\epsilon^2 + k\epsilon^{2.1} + \cdots$$

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- If $\alpha < -3$, there is no $\epsilon^2 \ln\epsilon$ term
- The only way to get rid of it is to pick

$$t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + C + \frac{K + o(1)}{\sqrt{n}}}{\gamma_c v_c}$$

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Bramson's term to get the ϵ term right iff $\sum_{n\geq1} h_0(n)n < \infty$:

$$t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + \mathcal{O}(1)}{\gamma_c v_c} \quad \Leftrightarrow \quad \mu_t = v_c t - \frac{3}{2\gamma_c}\ln t + \mathcal{O}(1)$$

For $t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + C}{\gamma_c v_c}$ exactly: $\quad \Psi(\gamma_c - \epsilon) = a + b\epsilon + k\epsilon^2 \ln\epsilon + \cdots$

Remember:
If $h_0(n) \sim An^\alpha e^{-\gamma_c n}$

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- If $\alpha < -3$, there is no $\epsilon^2 \ln\epsilon$ term
- The only way to get rid of it is to pick $$t_n = \frac{n}{v_c} + \frac{\frac{3}{2}\ln n + C + \frac{K + o(1)}{\sqrt{n}}}{\gamma_c v_c}$$
- This is the van Saarloos term, present iff $\Psi(\gamma_c - \epsilon) = a + b\epsilon + o(\epsilon^2 \ln\epsilon)$ (which is nearly the same as $\sum_n h_0(n)e^{\gamma_c n} n^2 < \infty$).

Thank you for listening!

$$\mu_t = \underbrace{v_c t - \frac{3}{2\gamma_c}\ln t + \text{cste}}_{\text{iff } \int \mathrm{d}x\, h_0(x) x\, e^{\gamma_c x} < \infty} - \underbrace{3\sqrt{\frac{2\pi}{\gamma_c^5 v''(\gamma_c)}}\, t^{-\frac{1}{2}}}_{\text{if } \int \mathrm{d}x\, h_0(x) x^2 e^{\gamma_c x} < \infty} + \underbrace{K\frac{\ln t}{t} + \mathcal{O}\left(\frac{1}{t}\right)}_{\text{if } \int \mathrm{d}x\, h_0(x) x^3 e^{\gamma_c x} < \infty}$$

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By matching singularities in a model on the lattice

É. Brunet and B. Derrida, *An exactly solvable travelling wave equation in the Fisher-KPP class*, Journal of Statistical Physics 2015, **161** (4), 801.

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Second approach, by computing some expectation of Brownian paths

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$$\mu_t = \underbrace{v_c t - \frac{3}{2\gamma_c}\ln t + \text{cste}}_{\text{iff } \int \mathrm{d}x\, h_0(x)\, x\, e^{\gamma_c x} < \infty} - \underbrace{3\sqrt{\frac{2\pi}{\gamma_c^5 v''(\gamma_c)}}\, t^{-\frac{1}{2}}}_{\text{if } \int \mathrm{d}x\, h_0(x)\, x^2 e^{\gamma_c x} < \infty} + \underbrace{K\frac{\ln t}{t} + \mathcal{O}\Big(\frac{1}{t}\Big)}_{\text{if } \int \mathrm{d}x\, h_0(x)\, x^3 e^{\gamma_c x} < \infty}$$

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Remark: for a step initial condition, with $\mu_t = 2t + \delta_t$:

Thank you for listening!

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Remark: for a step initial condition, with $\mu_t = 2t + \delta_t$:

$$\forall \epsilon > 0, \quad \int_0^\infty \mathrm{d}t\, e^{-\epsilon^2 t + (1-\epsilon)\delta_t} = 1$$

Thank you for listening!

$$\mu_t = \underbrace{v_c t - \frac{3}{2\gamma_c}\ln t + \text{cste}}_{\text{iff } \int \mathrm{d}x\, h_0(x) x\, e^{\gamma_c x} < \infty} - \underbrace{3\sqrt{\frac{2\pi}{\gamma_c^5 v''(\gamma_c)}}\, t^{-\frac{1}{2}}}_{\text{if } \int \mathrm{d}x\, h_0(x) x^2 e^{\gamma_c x} < \infty} + \underbrace{K\frac{\ln t}{t} + \mathcal{O}\Big(\frac{1}{t}\Big)}_{\text{if } \int \mathrm{d}x\, h_0(x) x^3 e^{\gamma_c x} < \infty}$$

By matching singularities in a model on the lattice

É. Brunet and B. Derrida, *An exactly solvable travelling wave equation in the Fisher-KPP class*, Journal of Statistical Physics 2015, **161** (4), 801.

Second approach, by computing some expectation of Brownian paths

J.Berestycki, É.Brunet, S.C.Harris and M.I.Roberts, *Vanishing corrections for the position in a linear model of FKPP fronts*, Comm. in Mathematical Physics 2017, **349**, 857

Third approach, by matching singularities in a model in the continuum

J.Berestycki, É.Brunet and B. Derrida, *Exact solution and precise asymptotics of a Fisher-KPP type front*, `https://arxiv.org/abs/1705.08416`

Remark: for a step initial condition, with $\mu_t = 2t + \delta_t$:

$$\forall \epsilon > 0,\ \int_0^\infty \mathrm{d}t\, e^{-\epsilon^2 t + (1-\epsilon)\delta_t} = 1 \quad \Longrightarrow \delta_t = -\frac{3}{2}\ln t + a - \frac{3\sqrt{\pi}}{\sqrt{t}} + K\frac{\ln t}{t} + \cdots$$

Bonus: linear FKPP

$$\partial_t h = \partial_x^2 h + h \quad \text{if } x > \mu_t, \qquad h(\mu_t, t) = 0$$

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Write $B_s = \mu_s + \xi_s$ and make a Girsanov transform

$$q(\mu_t + x, t; y) = \mathbb{E}^y_{\text{Bro}}\Big[\delta(\xi_t - x)\mathbb{1}_{\{\xi_s > 0, \forall s < t\}} e^{-\frac{1}{2}\int_0^t \mu_s' \mathrm{d}\xi_s}\Big] e^{-\frac{1}{4}\int_0^t (\mu_s')^2 \mathrm{d}s}$$

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$$\mathbb{E}^y_{\text{Bes}}\Big[e^{-\frac{1}{2}\int_0^t \mu_s' \left(\mathrm{d}\xi_s - \frac{x-y}{t}\mathrm{d}s\right)}\Big|\xi_t = x\Big] \approx \mathbb{E}^y_{\text{Bes}}\Big[e^{-\frac{1}{2}\int_0^\infty \mu_s' \mathrm{d}\xi_s}\Big]$$