Coupling of two Discretization Schemes for the Lattice Boltzmann Equation

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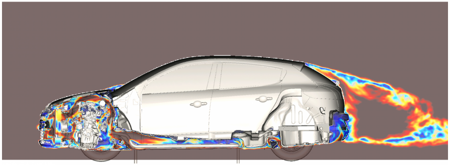
Computationa**L** methods with **I**ntensive **M**ultiphysics **B**oltzmann solver

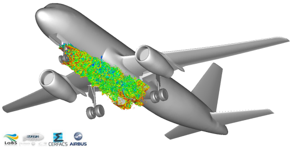
Partners

Industrial	Academic
Airbus	AMU
CS	ECL
Renault	UPS
...	...

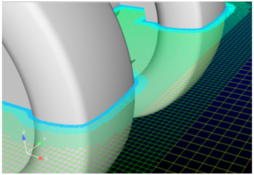
ProLB

- **ONERA** is involved with a number of departments
- Thesis: Mesh refinement

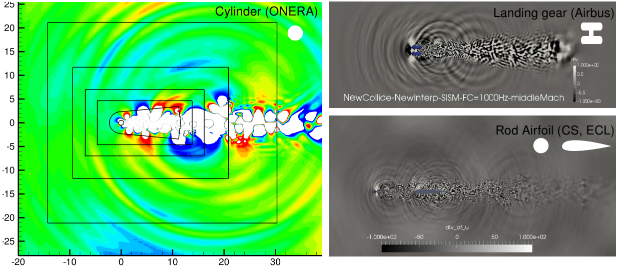




Numerical noise generation at the interface



Mesh refinement in classical LB algorithm by **factor 2** (or multiple of 2)
$\rightarrow$ aeroacoustic solutions may be contaminated



Solution approach: Coupling of classical algorithm with finite volume formulation

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Discretization in space and time of the DVBE

$$\frac{\partial f_\alpha}{\partial t} + \xi_\alpha \frac{\partial f_\alpha}{\partial x} = \Omega\left(f_\alpha\right) = \frac{1}{\tau}\left(f_\alpha^{eq} - f_\alpha\right), \quad 0 \leq \alpha \leq q-1 \tag{1}$$

1. Method of characteristics (classical stream-collide)
2. Finite volume method

$$\frac{df_\alpha}{ds} = \frac{\partial f_\alpha}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial f_\alpha}{\partial x}\frac{\partial x}{\partial s} = -\frac{1}{\tau}\left[f_\alpha - f_\alpha^{eq}\right] \quad \text{with} \quad f_\alpha = f_\alpha(x(s), t(s)) \tag{2}$$

True for $\frac{\partial t}{\partial s} = 1$ and $\frac{\partial x}{\partial s} = \xi_\alpha$. Thus $t(s) = t(0) + s$ and $x(s) = x(0) + \xi_\alpha s$.
Integrating from $s = 0$ to $s = \Delta t$, we obtain

$$f_\alpha\left(x + \xi_\alpha \Delta t, t + \Delta t\right) - f_\alpha(x, t) = -\frac{1}{\tau}\int_0^{\Delta t}\left[f_\alpha(x + \xi_\alpha s, t + s) - f_\alpha^{eq}(x + \xi_\alpha s, t + s)\right] ds \tag{3}$$

with $\xi_\alpha = c \cdot e_\alpha$, c is the lattice speed $\Delta x / \Delta t$, usually set to 1.

Discretization in space and time of the DVBE

Solving the integral on the right hand side with the trapezoidal rule yields

$$f_\alpha(x+\xi_\alpha\Delta t, t+\Delta t) - f_\alpha(x,t) =$$
$$-\frac{\Delta t}{2\tau}\left[f_\alpha(x+\xi_\alpha\Delta t, t+\Delta t) - f_\alpha^{eq}(x+\xi_\alpha\Delta t, t+\Delta t) + f_\alpha(x,t) - f_\alpha^{eq}(x,t)\right] + \mathcal{O}(\Delta t)^3 \tag{4}$$

Making equation (4) explicit and applying following change of variable

$$g_\alpha = f_\alpha + \frac{\Delta t}{2\tau}\left(f_\alpha - f_\alpha^{eq}\right), \tag{5}$$

we obtain

$$g_\alpha(x+\xi_\alpha\Delta t, t+\Delta t) = g_\alpha(x,t) - \frac{\Delta t}{\tau_g}\left(g_\alpha(x,t) - g_\alpha^{eq}(x,t)\right) \tag{6}$$

which is normally solved in a two-step *stream-collide* algorithm:

Collision:

$$\hat{g}_\alpha(x,t) = g_\alpha(x,t) - \frac{\Delta t}{\tau_g}\left(g_\alpha(x,t) - g_\alpha^{eq}(x,t)\right)$$

Stream:

$$g_\alpha(x+\xi_\alpha\Delta t, t+\Delta t) = \hat{g}_\alpha(x,t)$$

Discretization in space and time of the DVBE

1. Method of characteristics (classical stream-collide)
2. Finite volume method

In order to derive the finite volume formulation of the BE we depart from the DVLBE(1) and integrate over the volume $d\mathcal{V}$.

$$\int_V \frac{\partial f_\alpha}{\partial t} dV + \int_V \xi_\alpha \frac{\partial f_\alpha}{\partial x} dV = \int_V -\frac{1}{\tau}\left[f_\alpha - f_\alpha^{eq}\right] dV \quad (7)$$

Shrestha et al. [1] showed that it is possible to apply the same semi-implicit treatment of the collision operator to the finite volume formulation:

$$g_\alpha(x, t+\Delta t) = \underbrace{g_\alpha(x,t) - \frac{\Delta t}{\tau_g}\left(g_\alpha(x,t) - g_\alpha^{eq}(x,t)\right)}_{\hat{g}_\alpha(x,t)=\text{post-collision}} - \boxed{\Delta t \frac{S_\beta}{V}\xi_\alpha} \cdot n_\beta \left[f_\alpha(x_\beta, t+\Delta t/2)\right] \quad (8)$$

For the evaluation of the surface fluxes we chose a 2^{nd} order scheme in **time** (Heun predictor-corrector, DUGKS) and 2^{nd} or higher order scheme in **space** (centred, QUICK)

Discretization in space and time of the DVBE

Evaluation of $F^{t+\frac{1}{2}} = \xi_\alpha \cdot n_\beta \left[f_\alpha(x_\beta, t + \Delta t/2) \right]$

1. *Heun* predictor-corrector scheme
2. DUGKS

$$F^{t+\frac{1}{2}} = \xi_\alpha \cdot n_\beta \left[\frac{f^*_\alpha(x_\beta, t + \Delta t)}{2} + \frac{f_\alpha(x_\beta, t)}{2} \right] \tag{9}$$

with $f_\alpha = g_\alpha + \frac{1}{2\tilde{\tau}_g} \left(f^{eq}_\alpha - g_\alpha \right)$ and $f_\alpha = \hat{g}_\alpha + \frac{1}{2(\tilde{\tau}_g - 1)} \left(\hat{g}_\alpha - f^{eq}_\alpha \right)$ respectively

Three step (advect-collide) algorithm:

Step 1: Collision $\rightarrow$ $\hat{g}_\alpha(x, t)$

Step 2: Prediction $\rightarrow$ $g^*_\alpha(x, t + \Delta t)$, implies conversion $\hat{g} \rightarrow f$

Step 3: Correction $g_\alpha(x, t + \Delta t) = \hat{g}_\alpha(x, t) - F^{t+\frac{1}{2}}$, i. c. $g \rightarrow f$

Step 4: Step 1

Discretization in space and time of the DVBE

Evaluation of $F^{t+\frac{1}{2}} = \xi_\alpha \cdot n_\beta \left[f_\alpha(x_\beta, t + \Delta t/2)\right]$

1. *Heun* predictor-corrector scheme
2. DUGKS

$$f_\alpha(x_\beta, t + \Delta t/2) = g^f_\alpha(x_\beta, t + h) + \frac{\Delta t^f}{2\tau^f_g}\left(f^{eq}_\alpha(x_\beta, t + h) - g^f_\alpha(x_\beta, t + h)\right)$$

with $h = \Delta t/2$ being the timestep of the stream-collide algorithm on a refined lattice.

$$g^f(x_\beta, t + h) = \hat{g}^f(x_\beta - \xi_\alpha h, t) \tag{10}$$

where the term on the RHS is approximated by

$$\hat{g}^f(x_\beta - \xi_\alpha h, t) = \hat{g}^f(x_\beta, t) - \xi_\alpha \cdot \sigma_\beta \tag{11}$$

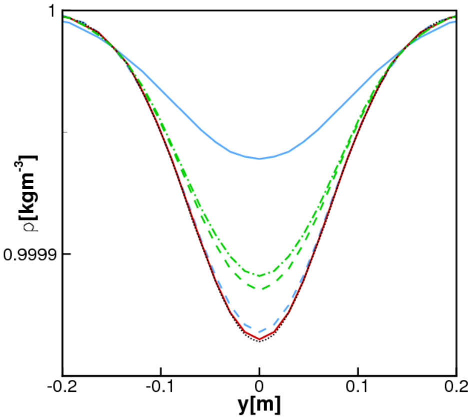
$\hat{g}^f(x_\beta, t)$ is obtained from the pre-collision coarse grained functions $g(x, t)$ (approximated with QUICK or centred scheme) and

$$\sigma_\beta = \frac{\hat{g}^f_\alpha(x + \Delta x, t) - \hat{g}^f_\alpha(x, t)}{\Delta x} \tag{12}$$

Single step algorithm!

Comparison of the two schemes

scheme	streaming	finite volume				
		DUGKS	*Heun* predictor-corrector		simple Euler
CFL	1	0.5	0.5	1	0.125
flux approximation	-	centred, QUICK	centred	QUICK	QUICK



Solution of vortex after 5000 iterations:

- ··· analytic
- — stream
- — Heun QUICK $CFL = 1$
- – – Heun centred $CFL = 0.5$
- – – DUGKS centred $CFL = 0.5$
- – · – DUGKS QUICK $CFL = 0.5$

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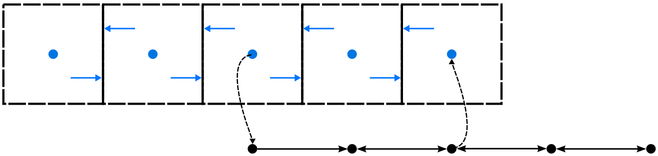
Algorithm

- In order to be congruent in the time step, we choose to couple the stream algorithm with the QUICK-Heun finite volume scheme that is stable at CFL=1 (in combination with a 7 point explicit filter).
- First results have shown that in order to couple the two algorithms, a multi-grid approach with a certain interface of at least $2\Delta x$ is required.



Sketch of the algorithm coupling on a uniform mesh

Algorithm

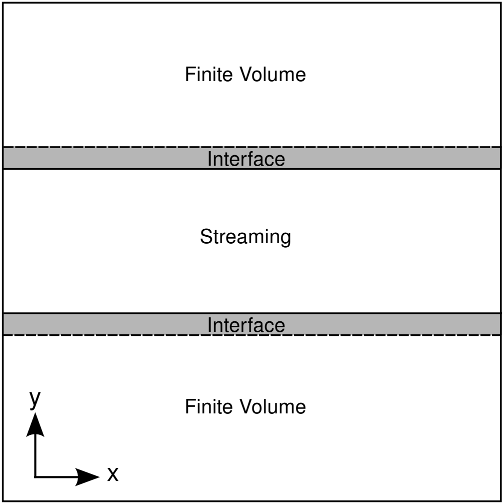
Streaming + *Heun* predictor corrector:

- **Step 1:** Collision in every node of the domain
- **Step 2a:** Memory shift in the domain where streaming is used
- **Step 2b:** Prediction of $g_\alpha^*(x, t + \Delta t)$ with a simple Euler scheme
- **Step 3:** Correction of $g_\alpha(x, t + \Delta t)$ with *Heun* scheme
- **Step 4:** Mutual transfer of information in the transition nodes
- **Step 5:** Step 1

blue: two step algorithm, red: additional steps

Results of two test cases are presented in the following.

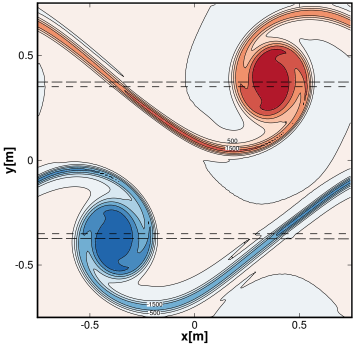
Numerical setup



mesh

- in-house LBM code
- periodic BC in X and Y
- $N = 101$, $N = 201$ and $N = 401$ respectively

Double Shear Layer



vorticity

$$u(x, y, t_0) = U_0 tanh(\frac{y - L/4}{d_0}) tanh(\frac{3/4L - y}{d_0})$$

$$v(x, y, t_0) = U_0 a_0 sin(2\pi(x/L + 0.25))$$

$$\rho(x, y, t_0) = \rho_0$$

with $d_0 = 6\Delta x$ and $a_0 = 0.01\rho_0$

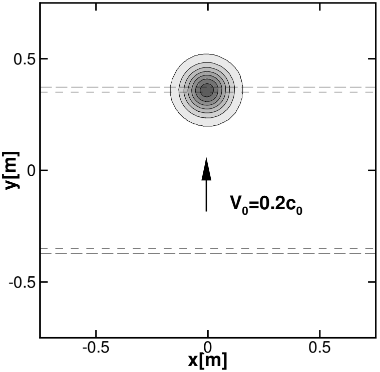
Vortex

$$u(x, y, t_0) = -\frac{\Gamma}{R^2} y \times exp\left(\frac{1}{2}\left(1 - \frac{r^2}{2R^2}\right)\right)$$

$$v(x, y, t_0) = V_0 + \frac{\Gamma}{R^2} x \times exp\left(\frac{1}{2}\left(1 - \frac{r^2}{2R^2}\right)\right)$$

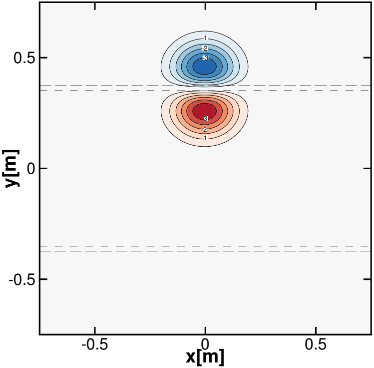
$$\rho(x, y, t_0) = \rho_0 - \frac{\Gamma^2}{2c_0^2}\rho_0 \times exp\left(1 - \frac{r^2}{R^2}\right)$$

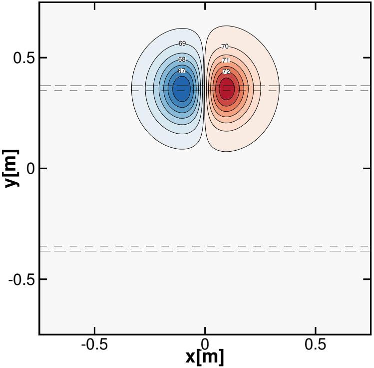
with $M = 0.2$, $\Gamma = 0.1\,V_0$ and $R = 6\Delta x$
for N=101.



density

Vortex

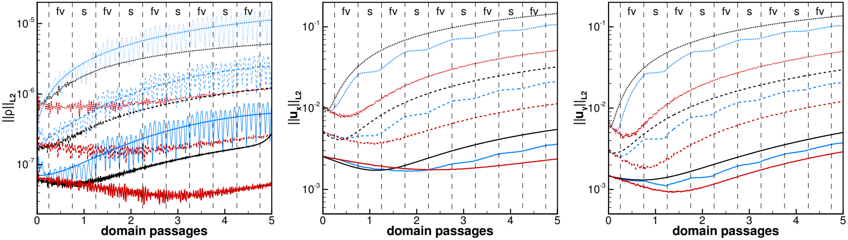




u *v*

velocity

Vortex

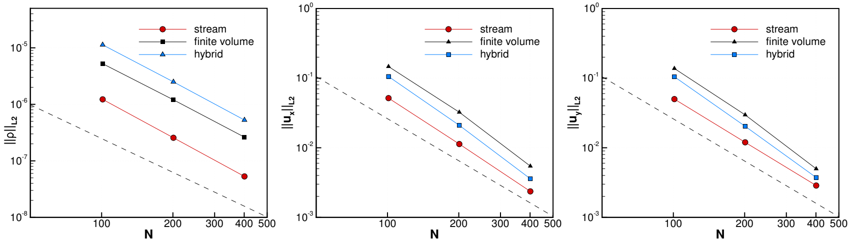


$||\rho||_{L_2}$ $||u_x||_{L_2}$ $||u_y||_{L_2}$

— stream, — finite volume, — hybrid

· · · N101, - - - N201, — N401

Vortex



$||\rho||_{L_2}$ $||u_x||_{L_2}$ $||u_y||_{L_2}$

L_2 of macroscopic variables for varying mesh size N after 5 domain passages:
— stream, — finite volume, — hybrid

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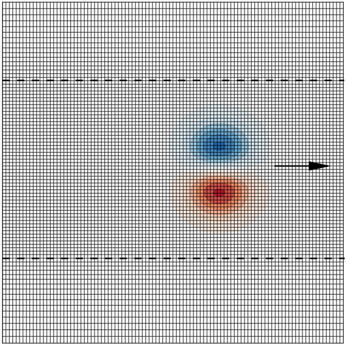
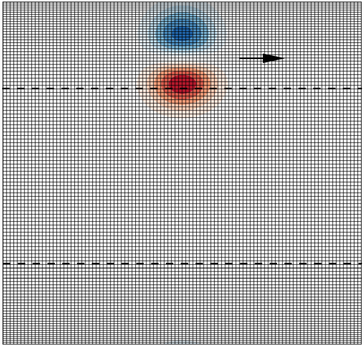
Conclusion and application

- **First study** to show that it is possible to **couple** the streaming step in a LBM algorithm with a finite volume formulation
- This is of interest for regions with anisotropic flows



Coarse grading: $\Delta y \rightarrow 2\Delta y$



Fine grading: $\Delta y \rightarrow 0.5\Delta y$

Mesh adaptation according to local flow features

Thank you for your attention!

References I

[1] K. Shrestha G. Mompean, E. Calzavarini. *Finite-volume versus streaming-based lattice Boltzmann algorithm for fluid-dynamics simulations: A one-to-one accuracy and performance study*. Physical Review E 93, 2016.

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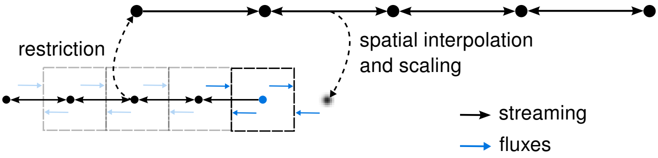

Sketch of the initially proposed algorithm for a non-uniform mesh