Analysis of over-relaxation kinetic schemes: application to the development of stable and second order boundary conditions.

Florence Drui, Emmanuel Franck,
Philippe Helluy, Laurent Navoret

INRIA Tonus / Université de Strasbourg

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CONTEXT

Equations to be solved numerically

- systems of conservation laws:

$$\partial_t \mathbf{W} + \mathrm{div}\,(\mathbf{F}(\mathbf{W})) - \mathrm{div}\,(\mathbf{D}(\mathbf{x},\mathbf{W})\boldsymbol{\nabla}\mathbf{W}) = \mathbf{S}$$

- **example:** the ideal MHD equations

$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - (\nabla \times \mathbf{B}) \times \mathbf{B} &= 0 \\
\partial_t p + \nabla \cdot (\rho \mathbf{u}) + (\gamma - 1) p \nabla \cdot \mathbf{u} &= 0 \\
\partial_t \mathbf{B} - \nabla \times (\mathbf{u} \times \mathbf{B}) &= 0 \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}$$

Challenges

Two characteristics:

1. nonlinear systems of equations,
2. waves associated with different time scales $\Rightarrow$ we want to resolve only some of these scales.

Numerically they imply that:

- **explicit schemes** have very restrictive CFL conditions, due to the fastest time scales,
- **implicit schemes** involve nonlinear operator inversion, expensive matrices storage and inversions.

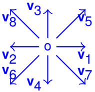
Lattice-Boltzmann and vectorial schemes

Equations to be solved numerically

$$\partial_t \mathbf{W} + \operatorname{div}(\mathbf{F}(\mathbf{W})) - \operatorname{div}(\mathbf{D}(\mathbf{x}, \mathbf{W})\boldsymbol{\nabla}\mathbf{W}) = \mathbf{S}$$

Lattice-Boltzmann methods

- inspired from the kinetic theory,
- discrete set of velocities: choice of a Lattice (**ex:** D2Q9)



- collision step: $f_i(t, \mathbf{x}) = f_i^{eq}(t, \mathbf{x})$
- shift step: $f_i(t + \Delta t, \mathbf{x}) = f_i(t, \mathbf{x} - \mathbf{v}_i \Delta t)$
- macrosopic variables: $\rho = \sum_i f_i$, $\rho\mathbf{u} = \sum_i \mathbf{v}_i f_i, \quad E = \sum_i |\mathbf{v}_i|^2/2 f_i$

$\Rightarrow$ Navier-Stokes equations,

- equivalent kinetic equation with BGK collision operator (LBGK scheme):

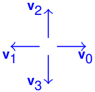
$$\partial_t f_i + \mathbf{v}_i \cdot \boldsymbol{\nabla} f_i = \epsilon_i^{-1}\left(f_i^{eq} - f_i\right)$$

Vectorial kinetic schemes

- generalization of Jin-Xin relaxation scheme,[Jin and Xin, 1995]

$$\begin{cases} \partial_t W + \partial_x Z & = 0 \\ \partial_t Z + \lambda^2 \partial_x W & = \epsilon^{-1}\left(F(W) - Z\right) \end{cases}$$

- **ex:** D2Q4 scheme:



- kinetic equations with relaxation source term:

$$\partial_t f_i + \mathbf{v}_i \cdot \boldsymbol{\nabla} f_i = \epsilon^{-1}(f_i^{eq} - f_i)$$

- macroscopic variables: $\rho = \sum_i f_i$, $\rho u = \sum_i g_i$, $\rho v = \sum_i h_i$...
- properties of this approach are detailed in [Perthame, 1990, Natalini, 1998, Aregba-Driollet and Natalini, 2000, Bouchut, 2004, Chen et al., 1994]

AN EXAMPLE OF LBGK CODE

Vectorial kinetic systems

$$\partial_t \mathbf{f} + \Lambda \nabla \mathbf{f} = \frac{\mathbf{f}^{eq} - \mathbf{f}}{\epsilon}$$

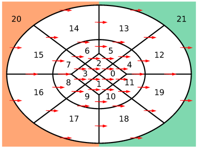
Ideas in the `kirsch` code[Coulette et al., 2018]

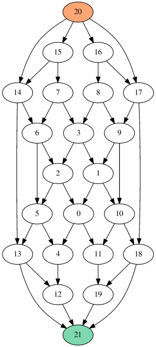
- Time integration:
  - time splitting approach,
  - second-order Crank-Nicolson integration for both steps $\Rightarrow$ implicit relation between $f_{i,j}^{n+1}$ and $f_{i,j}^{n}$, $\Rightarrow$ large time steps,
  - higher time order thanks to composition methods,[Suzuki, 1990]
- transport step: high-order DG on H20 grids,
- two levels of parallelism:

1. parallelization over the kinetic velocities,
2. grid decomposition into macrocells and cartesian sub-grids:
   - the linear system is block-triangular,
   - inversion of matrices in sub-grids (using KLU),
   - graph of macro-cells resolution:

$$f_{i,j}^{n+1} = \frac{\theta\alpha_i}{1+\theta\alpha_i} f_{i,j-1}^{n+1} + \frac{1-(1-\theta)\alpha_i}{1+\theta\alpha_i} f_{i,j}^{n} + \frac{(1-\theta)\alpha_i}{1+\theta\alpha_i} f_{i,j-1}^{n}$$

- task-scheduling programming (StarPU).



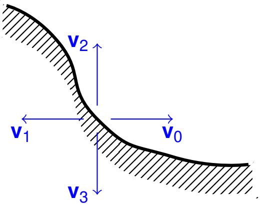


Issue of the boundary conditions

One of the current issues of the LBM approach is the treatment of the boundary conditions:

- **ex:** nD2Q4 scheme,
- here f_1 and f_3 are outgoing quantities
  $\Rightarrow$ no problem,
- f_0 and f_2 are incoming quantities
  $\Rightarrow$ what are their values?
- the macroscopic equation provides only one boundary condition,

$\Rightarrow$ **one relation is missing!**



Objectives of this work

- Study of the second order over-relaxation scheme used to solve nD1Q2 relaxation systems,
- design boundary conditions that preserve the scheme second order.

CONTENTS OF THE PRESENTATION

Relaxation schemes:

- the Jin-Xin relaxation model,
- relation with the vectorial kinetic schemes,
- splitting and composition strategies,
- an example of MHD code.

The over-relaxation scheme:

- derivation from the standard relaxation approach,
- equivalent equation and properties of the over-relaxation scheme,
- stability condition for the 1D transport equation.

Boundary conditions:

- usual boundary conditions for LBM,
- inflow/outflow conditions,
- numerical results.

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THE JIN-XIN RELAXATION SCHEME

Equations to be solved (1D)

$$\partial_t \mathbf{V} + \partial_x \mathbf{F}(\mathbf{V}) = \mathbf{0} \tag{1}$$

recall: nonlinear flux functions $\mathbf{V} \mapsto \mathbf{F}(\mathbf{V})$.

The relaxation model

- approximate systems (1) by systems of linear-flux equations:[Jin and Xin, 1995]

$$\partial_t \mathbf{W}_\varepsilon + \partial_x \mathbf{Z}_\varepsilon = 0, \tag{2}$$

$$\partial_t \mathbf{Z}_\varepsilon + \lambda^2 \partial_x \mathbf{W}_\varepsilon = \frac{1}{\varepsilon}(\mathbf{F}(\mathbf{W}_\varepsilon) - \mathbf{Z}_\varepsilon), \tag{3}$$

- a Chapman-Enskog development gives:
  - at zeroth order in ε: $\mathbf{Z}_\varepsilon = \mathbf{F}(\mathbf{W}_\varepsilon) + O(\varepsilon)$,
  - at first order in ε:

$$\partial_t \mathbf{W}_\varepsilon + \partial_x \mathbf{F}(\mathbf{W}_\varepsilon) = \varepsilon \partial_x \left(\left(\lambda^2 - |\partial \mathbf{F}(\mathbf{W}_\varepsilon)|^2 \right) \partial_x \mathbf{W}_\varepsilon \right) + O(\varepsilon^2) \tag{4}$$

- **consistency** of equation (4) with equation (1),
- **stability** under the subcharacteristic condition: $\lambda > |\partial \mathbf{F}(\mathbf{W}_\varepsilon)|$.

Numerical scheme

Numerical resolution of system (2)-(3) in the $\varepsilon = 0$ limit.

VECTORIAL KINETIC SCHEMES

From Jin-Xin to D1Q2 system

- Jin-Xin relaxation model:

$$\begin{aligned} \partial_t \mathbf{W} + \partial_x \mathbf{Z} &= 0, \\ \partial_t \mathbf{Z} + \lambda^2 \partial_x \mathbf{W} &= \varepsilon^{-1}(\mathbf{F}(\mathbf{W}) - \mathbf{Z}), \end{aligned}$$

- Riemann invariants:

$$\mathbf{f}_+ = \mathbf{W} + \mathbf{Z}/\lambda, \quad \mathbf{f}_- = \mathbf{W} - \mathbf{Z}/\lambda$$

- system for the Riemann invariants (nD1Q2):

$$\begin{aligned} \partial_t \mathbf{f}_+ + \lambda \partial_x \mathbf{f}_+ &= \varepsilon^{-1}(\mathbf{f}_+^{eq} - \mathbf{f}_+), \\ \partial_t \mathbf{f}_- - \lambda \partial_x \mathbf{f}_- &= \varepsilon^{-1}(\mathbf{f}_-^{eq} - \mathbf{f}_-), \end{aligned}$$

- equilibrium functions:

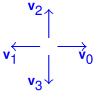
$$\mathbf{f}_+^{eq} = \mathbf{W} + \mathbf{F}(\mathbf{W})/\lambda, \quad \mathbf{f}_-^{eq} = \mathbf{W} - \mathbf{F}(\mathbf{W})/\lambda$$

2D and 3D systems

- Kinetic relaxation systems:

$$\partial_t \mathbf{f} + \Lambda \nabla \mathbf{f} = \frac{1}{\varepsilon}\left(\mathbf{f}^{eq}(\mathbf{W}) - \mathbf{f}\right), \qquad P\mathbf{f} = \mathbf{W}.$$

Example: D2Q4 system



- consistency conditions:
  [Aregba-Driollet and Natalini, 2000]
  [Audusse et al., 2004]

$$\begin{aligned} P\mathbf{f}^{eq}(\mathbf{W}) &= \mathbf{W}, \\ P\Lambda\mathbf{f}^{eq}(\mathbf{W}) &= \mathbf{F}(\mathbf{W}). \end{aligned}$$

$$\Lambda = \mathrm{diag}\left\{ (\lambda, 0), (-\lambda, 0), (0, \lambda), (0, -\lambda) \right\},$$

$$P = (1, 1, 1, 1).$$

Splitting approach and time integration (1)

Kinetic system to be solved

$$\partial_t \mathbf{f} + \Lambda \nabla \mathbf{f} = \frac{1}{\varepsilon} \left(\mathbf{f}^{eq}(\mathbf{W}) - \mathbf{f} \right) ,$$

Operator splitting

- Transport step: $$\partial_t \mathbf{f} + \Lambda \nabla \mathbf{f} = \mathbf{0}, \qquad (5)$$
- Relaxation step: $$\partial_t \mathbf{f} = \frac{1}{\varepsilon} \left(\mathbf{f}^{eq}(\mathbf{W}) - \mathbf{f} \right) , \qquad (6)$$

Transport step over time step h: $T(h)$

Possible numerical schemes:

- exact transport on cartesian grid: $f_i(t+h, \mathbf{x}) = f_i(t, \mathbf{x} - \mathbf{v}_i h)$,
  - $\Rightarrow$ h must be compatible with the grid,
- Semi-Lagrangian schemes: $f_i(t+h, \mathbf{x}) = f_i(t, \mathbf{x} - \mathbf{v}_i h)$
  - $\Rightarrow$ backward SL: interpolation at the foot of the characteristics,
  - $\Rightarrow$ forward SL: projection on the mesh,
- high-order FV or DG schemes,
  - $\Rightarrow$ implicit schemes require matrix inversion.

Source integration: $R(h)$

Possible integrations over time step h:

- exact solution of (6), with $\varepsilon > 0$: $\mathbf{f}(t+h, \mathbf{x}) = \mathbf{f}^{eq} + \exp(-h/\varepsilon)(\mathbf{f}(t, \mathbf{x}) - \mathbf{f}^{eq})$,
  - $\Rightarrow$ $\mathbf{f}^{eq}$ is invariant during the integration,
- projection on the equilibrium ($\epsilon = 0$): $\mathbf{f}(t+h, \mathbf{x}) = \mathbf{f}^{eq}(t, \mathbf{x})$,
  - $\Rightarrow$ provides first-order approximation with the splitting approach,
- Crank-Nicolson integration: $\mathbf{f}(t+h, \mathbf{x}) = (1-\theta)\mathbf{f}^{eq}(t, \mathbf{x}) + \theta \mathbf{f}(t, \mathbf{x})$ with $\theta = (2\varepsilon - h)/(2\varepsilon + h)$,
  - $\Rightarrow$ second-order when $\varepsilon = 0$.

Splitting approach and time integration (2)

Kinetic system to be solved

$$\partial_t \mathbf{f} + \Lambda \nabla \mathbf{f} = \frac{1}{\varepsilon} \left(\mathbf{f}^{eq}(\mathbf{W}) - \mathbf{f} \right) ,$$

Lie and Strang splitting

Lie splitting:
$L(h) = T(h)R(h)$

- first-order splitting

Strang splitting: $S(h) = T(h/2)R(h)T(h/2)$

- second-order splitting with Crank-Nicolson,
- higher-order composition not possible...

Time-symmetry property

Let $P(h)$ be a discrete operator, dependent on time step h,

- **definition of time symmetry:** $P(-h)P(h) = I$ and $P(0) = I$,
- **property:** if $P(h)$ is consistent with a continuous operator $\mathcal{P}$, then it is a second-order consistency,[Hairer et al., 2006, McLachlan and Quispel, 2002]
- $S(h)$ is not time symmetric when $\varepsilon = 0$: $S(0) \neq I$.

Time composition schemes

From a second-order time-symmetric operator $P(h)$, one can build even high-order operators with palindromic composition $Q(h)$:[McLachlan and Quispel, 2002, Hairer et al., 2006, Coulette et al., 2018]

$$Q(h) = P(\gamma_0 h)P(\gamma_1 h)\dots P(\gamma_s h)$$

with $\gamma_i = \gamma_{s-i}$, $0 \leq i \leq s$. **Examples** in: [Suzuki, 1990, Kahan and Li, 1997].

Example of an MHD code

Features of Patapon

- solves the ideal MHD equations,
- nD2Q4 approximation,
- time-symmetric composition with :
  - exact transport step,
  - Crank-Nicolson source integration with $\theta = 0.9$,

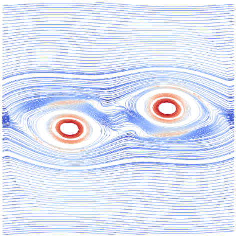
  $\Rightarrow$ numerical resistivity.
- cartesian grid and periodic or Dirichlet BC,
- Python code using PyOpenCL kernels.

Example of simulation: tilt instability

Computation characteristics

- 1024×1024 grid,
- graphic card: Nvidia - 24 GB - 3840 cores
- GPU utilization: 80%
- computation time: 30s (including I/O)

Simulation results



Time: 6.99 s

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Derivation of the over-relaxation scheme

System to be solved

$$\partial_t \mathbf{v} + \partial_x \mathbf{F}(\mathbf{v}) = \mathbf{0} \quad (7)$$

- Two auxiliary sets of variables: **w**, **z**,
- could be considered as relaxation approach with $\varepsilon = 0$...

Transport step

$$\begin{aligned} \partial_t \mathbf{w} + \partial_x \mathbf{z} &= 0, \\ \partial_t \mathbf{z} + \lambda^2 \partial_x \mathbf{w} &= \mathbf{0}, \end{aligned}$$

$\Rightarrow$ exact transport operator:

$$\begin{pmatrix} \mathbf{w}(\cdot, t+h) \\ \mathbf{z}(\cdot, t+h) \end{pmatrix} = T(h) \begin{pmatrix} \mathbf{w}(\cdot, t) \\ \mathbf{z}(\cdot, t) \end{pmatrix},$$

with

$$T(h) := \frac{1}{2} \begin{pmatrix} \tau(h) + \tau(-h) & (\tau(h) - \tau(-h))/\lambda \\ \lambda(\tau(h) - \tau(-h)) & \tau(h) + \tau(-h) \end{pmatrix}$$

and shift operator

$$(\tau(h)\mathbf{v})(x) = \mathbf{v}(x - \lambda h).$$

Over-relaxation step

- Crank-Nicolson ($\varepsilon = 0$):

$$R_0(h) \begin{pmatrix} \mathbf{w} \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \mathbf{w} \\ 2\mathbf{F}(\mathbf{w}) - \mathbf{z} \end{pmatrix}$$

- independent of h : R_0.

Operator splitting

$$S_2(h) := T(\frac{h}{4})\, R_0\, T(\frac{h}{2})\, R_0\, T(\frac{h}{4})$$

EQUIVALENT EQUATION AND SCHEME PROPERTIES

Over-relaxation scheme

$$S_2(h) := T(\frac{h}{4})\, R_0\, T(\frac{h}{2})\, R_0\, T(\frac{h}{4}). \quad (8)$$

Time-symmetric property

- $T(-h)T(h) = I$ and $T(0) = I$
- $R_0 R_0 = I$,

$\Rightarrow$ $S_2(-h)S_2(h) = I$ and $S_2(0) = I$.

We expect a second-order scheme.

Theorem 1[Drui et al., 2018]

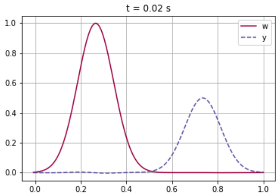
$\mathbf{w}$ and $\mathbf{z}$ being smooth solutions of a time marching algorithm with operator (8), the flux error $\mathbf{y}$ being defined by

$$\mathbf{y} := \mathbf{z} - \mathbf{F}(\mathbf{w}),$$

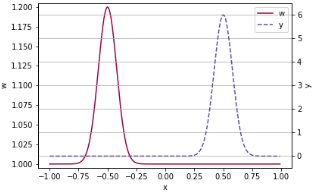
then, up to second order terms in h, $\mathbf{w}$ and $\mathbf{y}$ satisfy:

$$\partial_t \begin{pmatrix} \mathbf{w} \\ \mathbf{y} \end{pmatrix} + \begin{pmatrix} \mathbf{F}'(\mathbf{w}) & 0 \\ 0 & -\mathbf{F}'(\mathbf{w}) \end{pmatrix} \partial_x \begin{pmatrix} \mathbf{w} \\ \mathbf{y} \end{pmatrix} = 0.$$

Example: scalar transport equation:



Example: Euler equations:



Stability for the 1D transport equation (Theorem 2)

1D scalar transport equation

$$\partial_t u + c\partial_x u = 0$$

Equivalent equation

$$\partial_t \begin{pmatrix} w \\ y \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \partial_x \begin{pmatrix} w \\ y \end{pmatrix} + A\Delta t^2 \partial_{xxx} \begin{pmatrix} w \\ y \end{pmatrix} = O(\Delta t^3), \qquad (9)$$

with

$$A = \begin{pmatrix} (\lambda^2 - c^2) & 3c \\ 3c(\lambda^2 - c^2) & -(\lambda^2 - c^2) \end{pmatrix}.$$

Conservation of a convex energy

$$E(t) = \int_\Omega \left((\lambda^2 - c^2)\frac{w^2}{2} + \frac{y^2}{2} \right) = \int_\Omega \left(D \begin{pmatrix} w \\ y \end{pmatrix}, \begin{pmatrix} w \\ y \end{pmatrix} \right), \quad D = \begin{pmatrix} \lambda^2 - c^2 & 0 \\ 0 & 1 \end{pmatrix}$$

- Inserting in (9) reads:

$$\partial_t E(t) + \int_\Omega \left(D \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \partial_x \begin{pmatrix} w \\ y \end{pmatrix}, \begin{pmatrix} w \\ y \end{pmatrix} \right) + \int_\Omega \left(DA\partial_{xxx} \begin{pmatrix} w \\ y \end{pmatrix}, \begin{pmatrix} w \\ y \end{pmatrix} \right) = 0$$

- integration by part gives:

$$\partial_t E(t) = 0.$$

- $E(t) > 0$ if $\lambda > c \Rightarrow$ **subcharacteristic condition**!

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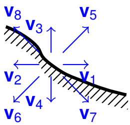
CLASSIC BOUNDARY CONDITIONS IN LBM CODES

The bounce-back condition

- **principle:** an incoming kinetic quantity gets the value of the outgoing kinetic quantity with opposite velocity.[Ziegler, 1993]
- **ex:** D2Q9 scheme:



$$f_1 = f_2, \; f_3 = f_4,$$
$$f_5 = f_6, \; f_8 = f_7.$$

- Navier-Stokes equations: **no-slip** boundary conditions.[Cornubert et al., 1991]

Reflective conditions

- **principle:** the normal velocity only is null [Succi, 2001, Suswaram et al., 2015]
- **ex:** 3D2Q4 scheme for Euler, u is the normal velocity, and

$$\rho = \sum f_i, \quad \rho u = \sum g_i, \quad \rho v = \sum h_i,$$

then

$$\rho u = \lambda(f_0 - f_1) = 0,$$
$$\rho uv = \lambda(g_2 - g_3) = 0,$$
$$\rho uv = \lambda(h_0 - h_1) = 0,$$

provide the relations for the incoming kinetic quantities.

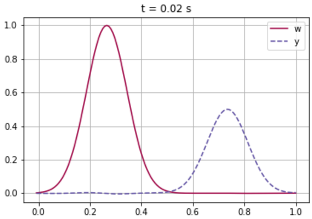
Other approaches

- the bounce-back and reflective conditions are not compatible with an implicit method,
- use of relaxation term towards a Dirichlet condition,[Coulette et al., 2018]
- **ex:** D1Q2 scheme and no-slip condition:

$$\partial_t f_0 + \lambda \partial_x f_0 = \varepsilon^{-1}(f_0^{eq} - f_0) + \tau^{-1}(f_1 - f_0)$$
$$\partial_t f_1 - \lambda \partial_x f_1 = \varepsilon^{-1}(f_1^{eq} - f_1) + \tau^{-1}(f_0 - f_1)$$

Inflow/outflow conditions for the over-relaxation scheme

Case of the 1D transport equation, **but** applies to all incoming waves.



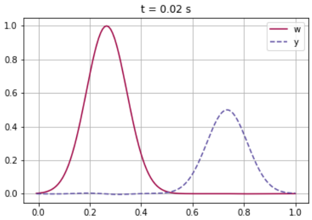
Inflow condition

According to the equivalent equation, w is incoming, y is outgoing.

- imposed condition on w: $$\frac{w_0^n + w_0^{n+1/4}}{2} = v(-c(t_n + \frac{\Delta t}{8})),$$

INFLOW/OUTFLOW CONDITIONS FOR THE OVER-RELAXATION SCHEME

Case of the 1D transport equation, **but** applies to all incoming waves.



Outflow condition (at right boundary)

According to the equivalent equation, w is outgoing, y is incoming. Three strategies:

- **Exact strategy:** imposed condition on w,
$$\frac{w^n_{N+1} + w^{n+1/4}_{N+1}}{2} = v(1 - c(t_n + \frac{\Delta t}{8})).$$
- **Dirichlet strategy:** equilibrium value for z,
$$\frac{z^n_{N+1} + z^{n+1/4}_{N+1}}{2} - c\frac{w^n_{N+1} + w^{n+1/4}_{N+1}}{2} = 0,$$
- **Neumann strategy:** uniform disequilibrium $\partial_x y = 0$,
$$z^{n+1/4}_{N+1} - cw^{n+1/4}_{N+1} = z^{n+1/4}_{N} - cw^{n+1/4}_{N}.$$

NUMERICAL RESULTS - TRANSPORT EQUATION

Equation to be solved

$$\partial_t u + c\partial_x u = 0.$$

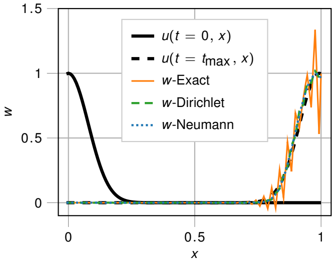
Initial condition:

$$u(x,t) = \exp\left(A(x-\alpha-ct)^2\right), \quad y(x,t) = 0,$$

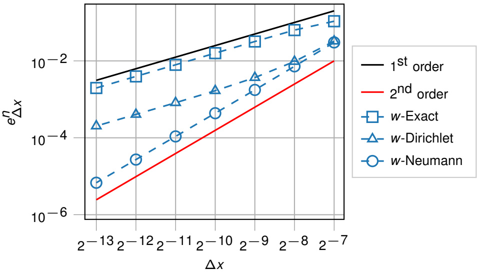
with

$$c = 1, \quad \lambda = 2, \quad t_{\max} = 1, \quad \alpha = 0, \quad \beta = 0 \quad A = -80, \quad \text{and} \quad B = 0.$$

Illustration



Convergence



Numerical results - Euler equations

Equations to be solved (barotropic Euler)

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = \mathbf{0}$$

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho v \end{pmatrix}, \quad \mathbf{f}(\mathbf{u}) = \begin{pmatrix} \rho v \\ (\rho v)^2/\rho + c^2\rho \end{pmatrix}, \quad \rho > 0, \quad c = 10.$$

Initial condition:

$$\rho(x,0) = \rho_0 + C\exp(D(x-x_0)), \quad (\rho v)(x,0) = \rho(x,0)v_0, \quad \rho_0 = 1.0, v_0 = 3c.$$

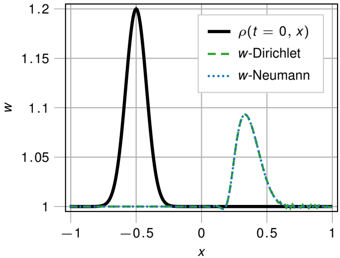
with

$$C = 0.2, \quad D = -80, \quad x_0 = -0.5,$$

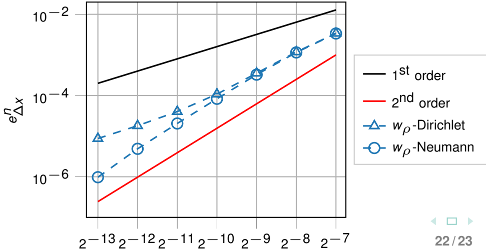
and

$$w_\rho(x,0) = \rho(x,0), \quad w_{\rho v}(x,0) = (\rho v)(x,0), \quad y_\rho(x,0), y_{\rho v}(x,0) = 0,$$

Illustration



Convergence



Conclusion

Objectives:

- solve nonlinear systems of equations with HPC algorithms,
- ensure high-order and stable numerical schemes,
- deal with boundary conditions issues,
- simulate MHD physical problems.

Achievements:

- LBM and vectorial kinetic schemes show good compatibility with HPC (`kirsch` and `patapon` codes),
- time-symmetric and high-order in time schemes have been developed (the basic component being the over-relaxation scheme),
- the analysis of the equivalent equation of the over-relaxation scheme shows how to design compatible boundary conditions,
- 1D second-order boundary conditions have been developed.

Perspectives:

- 2D and 3D boundary conditions,
- analysis and boundary conditions for other kinetic schemes (ex: D1Q3) and LBM approaches.

Thank you for your attention!

Conclusion

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