Multiple Relaxation-Time Lattice Boltzmann Model for advection-diffusion equations and its application to radar image processing.

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Jun 23, 2021 : Group of work – Lattice Boltzmann Scheme

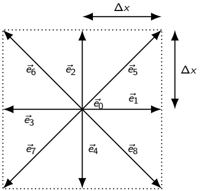
Table of Contents

Table of Contents

Let $\vec{x} \in \mathbb{R}^2$ and $t \in \mathbb{R}^+$ be the spatial position and the time respectively. The Boltzmann equation (without force term):

$$\frac{\partial}{\partial t} f(\vec{x}, t) + \vec{v}^T \cdot \vec{\nabla} f(\vec{x}, t) = \mathcal{Q}(f),$$
$$f(\vec{x}, \lambda\vec{e}_i, t + \Delta t) = f^*(\vec{x} - \lambda\vec{e}_i\Delta t, \lambda\vec{e}_i, t), \qquad (1)$$

where $\lambda = \frac{\Delta x}{\Delta t}$ is the numerical lattice velocity and f^* the density distribution after collision. Let $\vec{v}_i = \lambda\vec{e}_i$ be the discrete velocity vector. The dynamic is divided in two steps: streaming and collision.



$D2Q9$ LB scheme and elementary direction vector $\vec{e}_i$, $\forall i \in \{0, \ldots, 8\}$.

Tools to recover the equivalent PDEs simulated by the LB scheme:

Taylor Expansion [1, 2] and **Moments space** [3]

F. Dubois, “Une introduction au schéma de Boltzmann sur réseau,” in *ESAIM: proceedings*, vol. 18, pp. 181–215, EDP Sciences, 2007.

F. Dubois, “Third order equivalent equation of lattice Boltzmann scheme,” *Discrete & Continuous Dynamical Systems-A*, vol. 23, no. 1&2, p. 221, 2009.

D. d’Humières, “Generalized lattice-Boltzmann equations,” in *Rarefied Gas Dynamics: Theory and Simulations*, vol. 159, pp. 450–458, AIAA Progress in Aeronautics and Astronautics, 1992.

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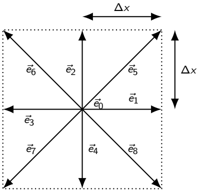

$D2Q9$ LB scheme and elementary direction vector $\vec{e}_i$, $\forall i \in \{0, \ldots, 8\}$.

1- **Taylor expansion** of (1) at third order writes:

$$f_i(\vec{x}, t) + \Delta t \frac{\partial}{\partial t} f_i(\vec{x}, t) + \frac{\Delta t^2}{2} \frac{\partial^2}{\partial t^2} f_i(\vec{x}, t) + \frac{\Delta t^3}{6} \frac{\partial^3}{\partial t^3} f_i(\vec{x}, t) + \mathcal{O}(\Delta t^4)$$
$$= f_i^*(\vec{x}, t) - \Delta t \vec{v}_i^T \cdot \vec{\nabla} f_i^*(\vec{x}, t) + \frac{\Delta t^2}{2} \vec{v}_i^T \cdot \boldsymbol{H}\left(f_i^*(\vec{x}, t)\right) \cdot \vec{v}_i$$
$$- \frac{\Delta t^3}{6} \vec{v}_i^T \cdot \vec{\nabla} \left(\vec{v}_i^T \cdot \boldsymbol{H}\left(f_i^*(\vec{x}, t)\right) \cdot \vec{v}_i \right) + \mathcal{O}(\lambda^4 \Delta t^4),$$

Acoustic scale $\Longleftrightarrow \frac{\Delta x}{\Delta t} = ¢ \Longrightarrow \mathcal{O}(\lambda^n \Delta t^n) = \mathcal{O}(\Delta x^n) = \mathcal{O}(\Delta t^n)$.

Let $\vec{x} \in \mathbb{R}^2$ and $t \in \mathbb{R}^+$ be the spatial position and the time respectively. The Boltzmann equation (without force term):

$$\frac{\partial}{\partial t} f(\vec{x}, t) + \vec{v}^T \cdot \vec{\nabla} f(\vec{x}, t) = \mathcal{Q}(f),$$
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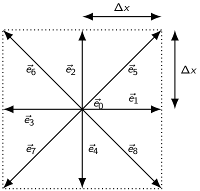
where $\lambda = \frac{\Delta x}{\Delta t}$ is the numerical lattice velocity and f^* the density distribution after collision. Let $\vec{v}_i = \lambda\vec{e}_i$ be the discrete velocity vector. The dynamic is divided in two steps: streaming and collision.



$D2Q9$ LB scheme and elementary direction vector $\vec{e}_i, \ \forall i \in \{0, \dots, 8\}$.

2- **Moments space** and **moments vector** $\vec{m}$ defined by

$$\vec{m}(\vec{x}, t) = \boldsymbol{M}\vec{f}(\vec{x}, \vec{v}, t) \quad \Longleftrightarrow \quad \vec{f}(\vec{x}, \vec{v}, t) = \boldsymbol{M}^{-1}\vec{m}(\vec{x}, t),$$

where $\boldsymbol{M}$ is the invertible transformation matrix.
The collision step in the moment space writes:

$$m_k^* = (1 - s_k)m_k + s_k m_k^{eq}, \quad \forall k \in \{1, 2, \dots, 8\},$$

where s_k is the relaxation time and m_k^{eq} the equilibrium moment as function of the conserved variable T.

Let $\vec{x} \in \mathbb{R}^2$ and $t \in \mathbb{R}^+$ be the spatial position and the time respectively. The Boltzmann equation (without force term):

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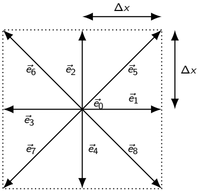

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$$\vec{m}(\vec{x}, t) = \boldsymbol{M} \vec{f}(\vec{x}, \vec{v}, t) \iff \vec{f}(\vec{x}, \vec{v}, t) = \boldsymbol{M}^{-1} \vec{m}(\vec{x}, t),$$

where $\boldsymbol{M}$ is the invertible transformation matrix.

The construction of $\boldsymbol{M}$ is linked with the physical moment used to recover the equivalent PDEs.

moment	T	j_x	j_y	E	p_{xx}	p_{xy}	q_x	q_y	χ
equilibrium	1	$\lambda ¢_{j_x}$	$\lambda ¢_{j_y}$	$\lambda^2 ¢_{\mathrm{E}}$	$\lambda^2 ¢_{\mathrm{p}_{xx}}$	$\lambda^2 ¢_{\mathrm{p}_{xy}}$	$\lambda^3 ¢_{j_x} ¢_{q_x}$	$\lambda^3 ¢_{j_y} ¢_{q_y}$	$\lambda^4 ¢_{\chi}$

Introduction of $D2Q9$ LB scheme

Let $\vec{x} \in \mathbb{R}^2$ and $t \in \mathbb{R}^+$ be the spatial position and the time respectively. The Boltzmann equation (without force term):

$$\frac{\partial}{\partial t} f(\vec{x}, t) + \vec{v}^T \cdot \vec{\nabla} f(\vec{x}, t) = \mathcal{Q}(f),$$

$$f(\vec{x}, \lambda\vec{e}_i, t + \Delta t) = f^*(\vec{x} - \lambda\vec{e}_i\Delta t, \lambda\vec{e}_i, t), \qquad (1)$$

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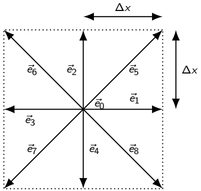

$D2Q9$ LB scheme and elementary direction vector $\vec{e}_i$, $\forall i \in \{0, \dots, 8\}$.

The PDE to be simulated by the LB scheme (1):

$$\frac{\partial}{\partial t} T(\vec{x}, t) + \nabla \cdot (\vec{w}(\vec{x}) T(\vec{x}, t)) - \kappa \Delta T(\vec{x}, t) = 0.$$

A diffusion and non-constant advection problem

Table of Contents

The moment m_k and the density distribution f_i verify

$$m_k = m_k^* + \mathcal{O}(\Delta t) = m_k^{eq} + \mathcal{O}(\Delta t) \quad \text{and}$$
$$f_i = f_i^* + \mathcal{O}(\Delta t) = f_i^{eq} + \mathcal{O}(\Delta t).$$

moment	T	j_x	j_y	E	p_{xx}	p_{xy}
equilibrium	1	$\lambda ¢_{j_x}$	$\lambda ¢_{j_y}$	$\lambda^2 ¢_{\mathrm{E}}$	$\lambda^2 ¢_{\mathsf{p}_{xx}}$	$\lambda^2 ¢_{\mathsf{p}_{xy}}$

The conserved variable T verify:

$$\mathcal{O}(\Delta t) = \frac{\partial}{\partial t} T(\vec{x}, t) + \lambda \nabla \cdot (\vec{w}(\vec{x}) T(\vec{x}, t)) .$$

moment	T	j_x	j_y	E	p_{xx}	p_{xy}
equilibrium	1	$\lambda w_x(\vec{x})$	$\lambda w_y(\vec{x})$	$\lambda^2 ¢_{\mathrm{E}}$	$\lambda^2 ¢_{\mathrm{p}_{xx}}$	$\lambda^2 ¢_{\mathrm{p}_{xy}}$

Taylor expansion for advection-diffusion problems

Second-order

The conserved variable T verify:

$$\mathcal{O}\left(\Delta t^2\right) = \frac{\partial}{\partial t} T(\vec{x},t) + \lambda \nabla\cdot \left(\vec{w}(\vec{x}) T(\vec{x},t)\right)$$

$$- \Delta t \lambda^2 \text{¢}'_{\mathrm{E}} \sigma_1 \Delta T(\vec{x},t) \boxed{-\Delta t \lambda^2 \sigma_1 \nabla\cdot \left[T(\vec{x},t) \boldsymbol{J}\left(\vec{w}(\vec{x})\right) \cdot \vec{w}(\vec{x}) \right]},$$

where $\sigma_k = \frac{1}{s_k} - \frac{1}{2}$ are coefficient introduced by Hénon [4].The additional term arises from the non-constant advection vector $\vec{w}(\vec{x})$ in $\frac{\partial^2}{\partial t^2} T(\vec{x},t)$ calculation.

moment	T	j_x	j_y	E	p_{xx}	p_{xy}
equilibrium	1	$\lambda w_x(\vec{x})$	$\lambda w_y(\vec{x})$	$\lambda^2\left(\text{¢}'_{\mathrm{E}} + \frac{\lVert\vec{w}(\vec{x})\rVert^2}{2}\right)$	$\lambda^2\left(w_x(\vec{x})^2 - w_y(\vec{x})^2\right)$	$\lambda^2 w_x(\vec{x}) w_y(\vec{x})$

M. Hénon, "Viscosity of a lattice gas," *Complex Systems*, vol. 1, no. 4, pp. 763–789, 1987.

The conserved variable T verify:

$$\mathcal{O}\left(\Delta t^2\right) = \frac{\partial}{\partial t} T(\vec{x}, t) + \lambda \nabla \cdot \left(\vec{w}\, T(\vec{x}, t)\right) - \Delta t \lambda^2 ¢'_{\mathrm{E}} \sigma_1 \Delta T(\vec{x}, t),$$

where $\sigma_k = \frac{1}{s_k} - \frac{1}{2}$ are coefficient introduced by Hénon [4].The **constant advection case** permits to recover the PDE without additional term.

moment	T	j_x	j_y	E	p_{xx}	p_{xy}
equilibrium	1	λw_x	λw_y	$\lambda^2 \left(¢'_{\mathrm{E}} + \frac{\Vert\vec{w}\Vert^2}{2}\right)$	$\lambda^2 \left(w_x^2 - w_y^2\right)$	$\lambda^2 w_x w_y$

M. Hénon, "Viscosity of a lattice gas," *Complex Systems*, vol. 1, no. 4, pp. 763–789, 1987.

For constant advection case, the conserved variable T verify:

$$\mathcal{O}\left(\Delta t^3\right) = \frac{\partial}{\partial t} T(\vec{x}, t) + \lambda \nabla \cdot \left(\vec{w}\, T(\vec{x}, t)\right) - \Delta t \lambda^2 \not{c}'_{\mathrm{E}} \sigma_1 \Delta T(\vec{x}, t).$$

where $\sigma_k = \frac{1}{s_k} - \frac{1}{2}$.

Third order for constant advection case

Equivalent PDE

For constant advection case, the conserved variable T verify:

$$
\begin{aligned}
\mathcal{O}\left(\Delta t^3\right) = & \frac{\partial}{\partial t} T(\vec{x},t) + \lambda \nabla\cdot\left(\vec{w}\, T(\vec{x},t)\right) - \Delta t \lambda^2 ¢'_{\mathrm{E}} \sigma_1 \Delta T(\vec{x},t) \\
& - \Delta t^2 \lambda^3 ¢'_{\mathrm{E}} 2 \left[\sigma_1^2 - \frac{1}{12}\right] \vec{w}^T \cdot \vec{\nabla}\left(\Delta\left(T\right)\right) \\
& + \Delta t^2 \left[\sigma_1\sigma_3 - \frac{1}{12}\right] \begin{pmatrix} w_x \left[¢_{q_x} - \lambda^3 ¢'_{\mathrm{E}} - \lambda^3 \frac{\|\vec{w}\|^2}{2}\right] \\ w_y \left[¢_{q_y} - \lambda^3 ¢'_{\mathrm{E}} - \lambda^3 \frac{\|\vec{w}\|^2}{2}\right] \end{pmatrix}^T \cdot \vec{\nabla}\left(\Delta\left(T\right)\right) \\
& + \Delta t^2 \left[\sigma_1\sigma_4 - \frac{1}{12}\right] \begin{pmatrix} w_x \left[\frac{\lambda^3}{2} w_y^2 - \frac{\lambda^3}{2} w_x^2 + \lambda^3 - ¢_{q_x}\right] \\ w_y \left[\frac{\lambda^3}{2} w_y^2 - \frac{\lambda^3}{2} w_x^2 - \lambda^3 + ¢_{q_y}\right] \end{pmatrix}^T \cdot \vec{\nabla}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) T \\
& + 2\Delta t^2 \left[\sigma_1\sigma_5 - \frac{1}{12}\right] \begin{pmatrix} w_y \left[2¢_{q_y} - \lambda^3 - \lambda^3 \vec{w}_x^2\right] \\ w_x \left[2¢_{q_x} - \lambda^3 - \lambda^3 \vec{w}_y^2\right] \end{pmatrix}^T \cdot \vec{\nabla}\left(\frac{\partial^2}{\partial x \partial y} T\right).
\end{aligned}
$$

where $\sigma_k = \frac{1}{s_k} - \frac{1}{2}$.

Third order for constant advection case

Equivalent PDE

For constant advection case, the conserved variable T verify:

$$
\begin{aligned}
\mathcal{O}\left(\Delta t^3\right) = &\frac{\partial}{\partial t} T(\vec{x},t) + \lambda \nabla \cdot \left(\vec{w}\, T(\vec{x},t)\right) - \Delta t \lambda^2 \text{¢}'_{\mathrm{E}} \sigma_1 \Delta T(\vec{x},t) \\
& - \Delta t^2 \lambda^3 \text{¢}'_{\mathrm{E}} 2 \left[\sigma_1^2 - \frac{1}{12}\right] \vec{w}^{\,T} \cdot \vec{\nabla}\left(\Delta\left(T\right)\right) \\
& + \Delta t^2 \left[\sigma_1 \sigma_3 - \frac{1}{12}\right] \begin{pmatrix} w_x \left[\text{¢}_{q_x} - \lambda^3 \text{¢}'_{\mathrm{E}} - \lambda^3 \frac{\|\vec{w}\|^2}{2}\right] \\ w_y \left[\text{¢}_{q_y} - \lambda^3 \text{¢}'_{\mathrm{E}} - \lambda^3 \frac{\|\vec{w}\|^2}{2}\right] \end{pmatrix}^T \cdot \vec{\nabla}\left(\Delta\left(T\right)\right) \\
& + \Delta t^2 \left[\sigma_1 \sigma_4 - \frac{1}{12}\right] \begin{pmatrix} w_x \left[\frac{\lambda^3}{2} w_y^2 - \frac{\lambda^3}{2} w_x^2 + \lambda^3 - \text{¢}_{q_x}\right] \\ w_y \left[\frac{\lambda^3}{2} w_y^2 - \frac{\lambda^3}{2} w_x^2 - \lambda^3 + \text{¢}_{q_y}\right] \end{pmatrix}^T \cdot \vec{\nabla}\left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) T \\
& + 2\Delta t^2 \left[\sigma_1 \sigma_5 - \frac{1}{12}\right] \begin{pmatrix} w_y \left[2\text{¢}_{q_y} - \lambda^3 - \lambda^3 \vec{w}_x^2\right] \\ w_x \left[2\text{¢}_{q_x} - \lambda^3 - \lambda^3 \vec{w}_y^2\right] \end{pmatrix}^T \cdot \vec{\nabla}\left(\frac{\partial^2}{\partial x \partial y} T\right).
\end{aligned}
$$

where $\sigma_k = \frac{1}{s_k} - \frac{1}{2}$. The proposed set of relaxation time (with MRT hypothesis):

$$
\sigma_1 = \sigma_3 = \sigma_4 = \sigma_5 = \frac{1}{\sqrt{12}} \; \boldsymbol{(\sigma_6 \text{ and } \sigma_8 \text{ free})}.
$$

Initial condition: $T(\vec{x},0) = \sin\left(2\pi \vec{k}^T \cdot \vec{x}\right)$, $\forall \vec{x} \in \Omega$;

Analytic solution: $T^{th}(\vec{x},t) = \sin\left(2\pi \vec{k}^T \cdot (\vec{x} - \vec{w}t)\right) \mathrm{e}^{-\|2\pi\vec{k}\|\kappa t}$, $\forall \vec{x} \in \Omega$, $\forall t > 0$;

Boundaries Conditions: Periodic for all boundaries (avoid boundary accuracy);

Physical variables: $\kappa = 2{\cdot}10^{-2}$ and $\vec{w} = \left(10^{-1}, -5{\cdot}10^{-2}\right)^T$;

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Physical variables: $\kappa = 2{\cdot}10^{-2}$ and $\vec{w} = \left(10^{-1}, -5{\cdot}10^{-2}\right)^T$;

LB variable: $\lambda = 5{\cdot}10^3$, $\Delta x = \frac{1}{\ell\cdot 10^2}$, $\forall\ell\in\{1,2,\ldots,10\}$, $\Delta t = \frac{\Delta x}{\lambda}$, $\cent'_{\mathrm{E}} = \frac{-\kappa}{\sigma_1\Delta t\lambda^2}$, $\cent_{q_x} = \cent_{q_y} = \cent_{\chi} = 0$, $s_6 = 2$ and $s_8 = 1.2$;

Error betwenn numerical and analytic solution:

$$Err\left(T^{LB}-T^{th}\right) = \sqrt{\Delta x^2\sum_{\vec{x}\in\mathcal{L}}\left(T^{LB}(\vec{x})-T^{th}(\vec{x})\right)^2}.$$

Third order for constant advection case

Numerical Validation of third order accuracy

Initial condition: $T(\vec{x},0) = \sin\left(2\pi \vec{k}^T \cdot \vec{x}\right)$, $\forall \vec{x} \in \Omega$;

Analytic solution: $T^{th}(\vec{x},t) = \sin\left(2\pi \vec{k}^T \cdot (\vec{x} - \vec{w}t)\right) \mathrm{e}^{-\|2\pi\vec{k}\|\kappa t}$, $\forall \vec{x} \in \Omega$, $\forall t > 0$;

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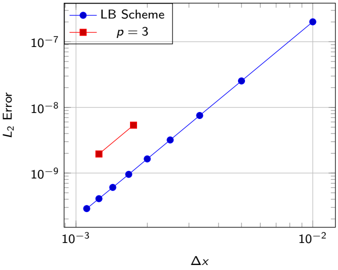
LB variable: $\lambda = 5{\cdot}10^3$, $\Delta x = \frac{1}{\ell \cdot 10^2}$, $\forall \ell \in \{1, 2, \ldots, 10\}$, $\Delta t = \frac{\Delta x}{\lambda}$, $\cent'_{\mathrm{E}} = \frac{-\kappa}{\sigma_1 \Delta t \lambda^2}$, $\cent_{q_x} = \cent_{q_y} = \cent_{\chi} = 0$, $s_6 = 2$ and $s_8 = 1.2$;



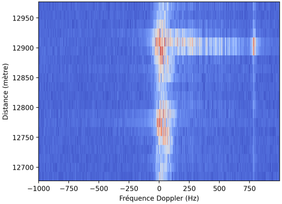
LB scheme: $p \simeq 2.98$

Error betwenn numerical and analytic solution:

$$Err\left(T^{LB} - T^{th}\right) = \sqrt{\Delta x^2 \sum_{\vec{x} \in \mathcal{L}} \left(T^{LB}(\vec{x}) - T^{th}(\vec{x})\right)^2}.$$

Table of Contents

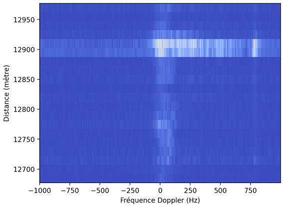
Marine radar images: low contrast, weak contours and a strong interference noise (in the Range-Doppler Map: a Fourier domain);



Raw marine radar images (in Range-Doppler Map) with a target at 12 900m.

Marine radar images: low contrast, weak contours and a strong interference noise (in the Range-Doppler Map: a Fourier domain);

First noise extraction: by image/signal processing, the signal of interest still contains noise and may lose clarity;



Marine radar images after first noise extraction (in Range-Doppler Map) with a target at $12\,900\mathrm{m}$.

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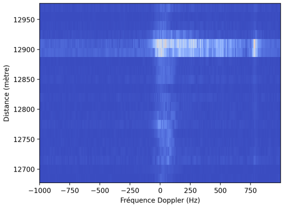
The LB scheme goals: **enhance** the remaining signal + **reduce the noise** arising from the image processing.



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Enhancement: provided by an advection term driven by the remaining information gradient pointing to the maxima ($\vec{w}(\vec{x})$);

Noise reduction: provided by the Cahn-Hilliard energy [5]:

- diffusion term $\left(\kappa = \varepsilon \frac{\mu}{¢_W}\right)$;
- a double well potential (the force term).

S. M. Allen and J. W. Cahn, "A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening," *Acta metallurgica*, vol. 27, no. 6, pp. 1085–1095, 1979.

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Boundaries condition: left and right: periodic; top and bottom: homogeneous Neumann.

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The LB scheme simulates the PDE

$$\frac{\partial}{\partial t} T(\vec{x}, t) + \nabla \cdot (\vec{w}(\vec{x}) T(\vec{x}, t)) - \overbrace{\varepsilon \frac{\mu}{\cent_W}}^{\kappa} \Delta T(\vec{x}, t) = \overbrace{-\frac{\mu}{\varepsilon \cent_W} W'(T)}^{\text{force term}},$$

where the double well potential $W(x) = 0.5x^2 (1 - x)^2$ and

$$\cent_W = \int_0^1 W(x)\, \mathrm{d}x \simeq \frac{1}{60}.$$

LB variables: $\Delta x = 10^{-1}$, $\Delta t = 10^{-2}$, $¢'_{\mathrm{E}} = \frac{-\kappa}{\sigma_1 \Delta t \lambda^2}$, $¢_{q_x} = ¢_{q_y} = 10^{-3}$, $¢_{\chi} = 0$, $\vec{c}_{\vec{j}} = \vec{w}$ and $s_6 = s_8 = 1$.

The temporal iterations are stopped when the relative error

$$\frac{\| T^{LB}(\vec{x}, t + \Delta t) - T^{LB}(\vec{x}, t) \|_{L^2}}{\| T^{LB}(\vec{x}, t + \Delta t) \|_{L^2}} \leq \text{tol.}$$

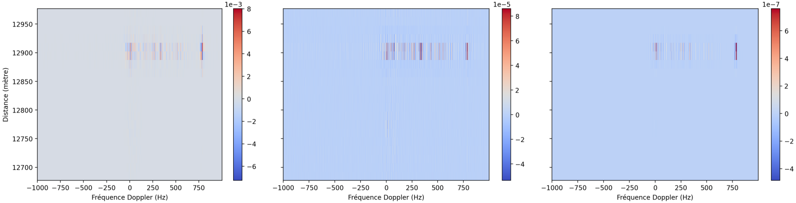
LB variables: $\Delta x = 10^{-1}$, $\Delta t = 10^{-2}$, $¢'_{\mathrm{E}} = \frac{-\kappa}{\sigma_1 \Delta t \lambda^2}$, $¢_{q_x} = ¢_{q_y} = 10^{-3}$, $¢_{\chi} = 0$, $\vec{c}_{\vec{j}} = \vec{w}$ and $s_6 = s_8 = 1$.

- Additional term: lowest numerical influence in the temporal evolution of the temperature $T \rightarrow$ negligible;

$\lambda \nabla \cdot (\vec{w}(\vec{x}) T(\vec{x}, t))$

$-\Delta t \lambda^2 ¢'_{\mathrm{E}} \sigma_1 \Delta T(\vec{x}, t)$

$-\Delta t \lambda^2 \sigma_1 \nabla \cdot [T(\vec{x}, t) \boldsymbol{J}(\vec{w}(\vec{x})) \cdot \vec{w}(\vec{x})]$

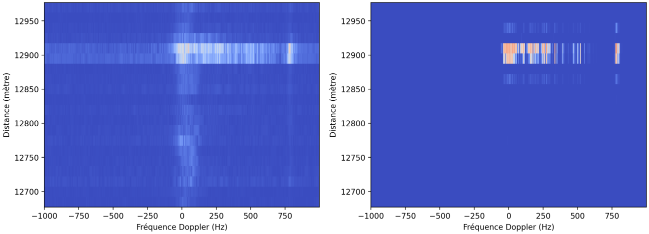


(a) Advection term. (b) Diffusion term. (c) Additional term.

Figure: Terms of the equivalent PDE at second order after scheme convergence, induced by the non-constant advection of an advection-diffusion equation.

LB variables: $\Delta x = 10^{-1}$, $\Delta t = 10^{-2}$, $\text{¢}'_{\mathrm{E}} = \frac{-\kappa}{\sigma_1 \Delta t \lambda^2}$, $\text{¢}_{q_x} = \text{¢}_{q_y} = 10^{-3}$, $\text{¢}_{\chi} = 0$, $\vec{c}_{\vec{j}} = \vec{w}$ and $s_6 = s_8 = 1$.

- Additional term: lowest numerical influence in the temporal evolution of the temperature $T \rightarrow$ negligible;
- Previous setting for relaxation time to suppress certain second order terms;

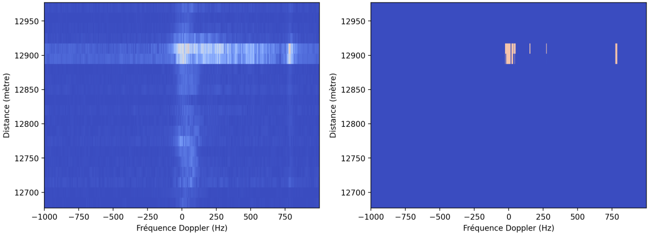


(a) Initial condition. (b) Temperature after scheme convergence.

Figure: Temperature T after scheme convergence, seen in the RDM and following an advection-diffusion LB scheme with non-constant advection.

LB variables: $\Delta x = 10^{-1}$, $\Delta t = 10^{-2}$, $¢'_{\mathrm{E}} = \frac{-\kappa}{\sigma_1 \Delta t \lambda^2}$, $¢_{q_x} = ¢_{q_y} = 10^{-3}$, $¢_{\chi} = 0$, $\vec{c}_{\vec{j}} = \vec{w}$ and $s_6 = s_8 = 1$.

- Additional term: lowest numerical influence in the temporal evolution of the temperature $T \rightarrow$ negligible;
- Previous setting for relaxation time to suppress certain second order terms;
  - Result improvement by correction of s_1.



(a) Initial condition. (b) Temperature after scheme convergence.

Figure: Temperature T after scheme convergence, seen in the RDM and following an advection-diffusion LB scheme with non-constant advection.

SRT or BGK: few LB parameters but lack of stability [6, 7];

T. Gebäck and A. Heintz, "A lattice Boltzmann method for the advection-diffusion equation with Neumann boundary conditions," *Communications in Computational Physics*, vol. 15, no. 2, pp. 487–505, 2014.

L. Li, R. Mei, and J. F. Klausner, "Lattice Boltzmann Models for the convection-diffusion equation: D2Q5 vs D2Q9," *International Journal of Heat and Mass Transfer*, vol. 108, pp. 41–62, 2017.

SRT or BGK: few LB parameters but lack of stability [6, 7];

MRT: significant number of parameters. Higher order calculations drive the choice of certain parameters;

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SRT or BGK: few LB parameters but lack of stability [6, 7];

MRT: significant number of parameters. Higher order calculations drive the choice of certain parameters;

For the stability study, the relative error:

$$\frac{\|T^{LB}(\vec{x}, t+\Delta t) - T^{LB}(\vec{x}, t)\|_{L^2}}{\|T^{LB}(\vec{x}, t+\Delta t)\|_{L^2}},$$

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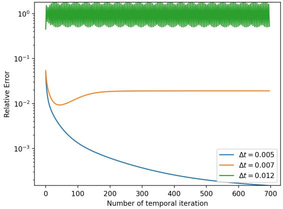
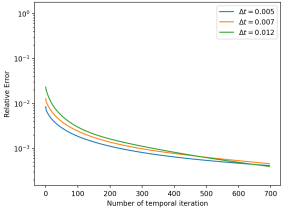


SRT LB Scheme



MRT LB Scheme

1. A set of relaxation time to have third order accuracy for advection-diffusion problem;
2. Simulated PDE of a diffusion and non constant advection problem (MRT-LB scheme to $D2Q9$ lattice);
   - The additional term may negligible up to the context;
3. Efficient signal enhancement (real time) for marine radar images;

- F. Dubois, "Une introduction au schéma de Boltzmann sur réseau," in ESAIM: *proceedings*, vol. 18, pp. 181–215, EDP Sciences, 2007.
- F. Dubois, "Third order equivalent equation of lattice Boltzmann scheme," *Discrete & Continuous Dynamical Systems-A*, vol. 23, no. 1&2, p. 221, 2009.
- D. d'Humières, "Generalized lattice-Boltzmann equations," in *Rarefied Gas Dynamics: Theory and Simulations*, vol. 159, pp. 450–458, AIAA Progress in Aeronautics and Astronautics, 1992.
- M. Hénon, "Viscosity of a lattice gas," *Complex Systems*, vol. 1, no. 4, pp. 763–789, 1987.
- S. M. Allen and J. W. Cahn, "A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening," *Acta metallurgica*, vol. 27, no. 6, pp. 1085–1095, 1979.
- T. Gebäck and A. Heintz, "A lattice Boltzmann method for the advection-diffusion equation with Neumann boundary conditions," *Communications in Computational Physics*, vol. 15, no. 2, pp. 487–505, 2014.
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$$m_k + \Delta t \frac{\partial}{\partial t} m_k + \frac{\Delta t^2}{2} \frac{\partial^2}{\partial t^2} m_k^{eq} = m_k^* - \Delta t \sum_{i=0}^{8} M_{k,i} \vec{v}_i^T \cdot \vec{\nabla} f_i^* + \frac{\Delta t^2}{2} \sum_{i=0}^{8} M_{k,i} \vec{v}_i^T \cdot \boldsymbol{H} \left(f_i^{eq} \right) \cdot \vec{v}_i + \mathcal{O}(\Delta t^3).$$

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For $k = 0$, the moment $m_0 = m_0^* = m_0^{eq} = T$.

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For $k = 0$, the moment $m_0 = m_0^* = m_0^{eq} = T$.

$$T + \Delta t \frac{\partial}{\partial t} T + \frac{\Delta t^2}{2} \frac{\partial^2}{\partial t^2} T = T - \Delta t \sum_{i=0}^{8} \vec{v}_i^T . \vec{\nabla} f_i^* + \frac{\Delta t^2}{2} \sum_{i=0}^{8} \vec{v}_i^T . \boldsymbol{H} \left(f_i^{eq} \right) . \vec{v}_i + \mathcal{O}(\Delta t^3)$$

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The term $\sum_{i=0}^{8} \vec{v}_i^T . \vec{\nabla} f_i^*$ is decomposed by the formula

$$\sum_{i=0}^{8} z_i f_i(\vec{x}, t) = \sum_{i,k=0}^{8} \frac{\langle M_{k,i}, z_i \rangle}{\| M_{k,i} \|^2} m_k .$$

$$\mathcal{O}(\Delta t^3) = \Delta t \frac{\partial}{\partial t} T + \Delta t \nabla \cdot \vec{j}^{*} - \frac{\Delta t^2}{2} \left[\sum_{i=0}^{8} \vec{v}_i^{T} \cdot \boldsymbol{H} \left(f_i^{eq} \right) \cdot \vec{v}_i - \frac{\partial^2}{\partial t^2} T \right]$$

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The use of previous approximation of non-conserved moment (for $\vec{j}^*$) leads to

$$\mathcal{O}(\Delta t^3) = \Delta t \frac{\partial}{\partial t} T + \Delta t \nabla \cdot \vec{j}^{eq} - \Delta t^2 \left(\frac{1}{s_1} - \frac{1}{2} \right) \left[\sum_{i=0}^{8} \vec{v}_i^T \cdot \boldsymbol{H} \left(f_i^{eq} \right) \cdot \vec{v}_i - \frac{\partial^2}{\partial t^2} T \right].$$

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For second order terms and using the same decomposition formula, one obtains

$$\sum_i \vec{v}_i^T \cdot \boldsymbol{H} \left(f_i^{eq} \right) \cdot \vec{v}_i = \lambda^2 \left[\frac{\partial^2}{\partial x^2} \left(¢_{\mathrm{E}} + \frac{1}{2} ¢_{\mathrm{p}_{xx}} \right) T + 2 \frac{\partial^2}{\partial x \partial y} \left(¢_{\mathrm{p}_{xy}} T \right) + \frac{\partial^2}{\partial y^2} \left(¢_{\mathrm{E}} - \frac{1}{2} ¢_{\mathrm{p}_{xx}} \right) T \right]$$

$$\mathcal{O}(\Delta t^3) = \Delta t \frac{\partial}{\partial t} T + \Delta t \nabla \cdot \vec{j}^* - \frac{\Delta t^2}{2} \left[\sum_{i=0}^{8} \vec{v}_i^T \cdot \boldsymbol{H} \left(f_i^{eq} \right) \cdot \vec{v}_i - \frac{\partial^2}{\partial t^2} T \right]$$

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For second derivative in time, one obtains

$$\begin{aligned} \frac{\partial^2}{\partial t^2} T &= -\lambda \nabla \cdot \left(\vec{w}(\vec{x}) \frac{\partial}{\partial t} T \right) + \mathcal{O}(\Delta t) = \lambda^2 \nabla \cdot \left[\vec{w}(\vec{x}) \nabla \cdot \left(\vec{w}(\vec{x}) T(\vec{x}, t) \right) \right] + \mathcal{O}(\Delta t) \\ &= \lambda^2 \left(\frac{\partial^2}{\partial x^2} \left(w_x^2(\vec{x}) T(\vec{x}, t) \right) + 2 \frac{\partial^2}{\partial x \partial y} \left(w_x(\vec{x}) w_y(\vec{x}) T(\vec{x}, t) \right) \right. \\ &\quad \left. + \frac{\partial^2}{\partial y^2} \left(w_y^2(\vec{x}) T(\vec{x}, t) \right) - \nabla \cdot \left[T(\vec{x}, t) \boldsymbol{J} \left(\vec{w}(\vec{x}) \right) \cdot \vec{w}(\vec{x}) \right] \right) + \mathcal{O}(\Delta t). \end{aligned}$$

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Therefore, the following system has to be solved:

$$\begin{cases} ¢_{\mathrm{E}} + \frac{1}{2}¢_{\mathrm{p}xx} - w_x^2 & = \alpha \\ ¢_{\mathrm{p}xy} - w_x w_y & = 0 \\ ¢_{\mathrm{E}} - \frac{1}{2}¢_{\mathrm{p}xx} - w_y^2 & = \alpha \end{cases} \iff \begin{cases} ¢_{\mathrm{E}} & = \alpha + \frac{\|\vec{w}\|^2}{2} = ¢'_{\mathrm{E}} + \frac{\|\vec{w}\|^2}{2} \\ ¢_{\mathrm{p}xy} & = w_x w_y \\ ¢_{\mathrm{p}xx} & = w_x^2 - w_y^2 \end{cases}.$$

■

For second order terms and using the same decomposition formula, one obtains

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&= \lambda^2 \left(\frac{\partial^2}{\partial x^2}\left(w_x^2(\vec{x}) T(\vec{x}, t) \right) + 2\frac{\partial^2}{\partial x \partial y}\left(w_x(\vec{x}) w_y(\vec{x}) T(\vec{x}, t) \right) \right. \\
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■

$$\mathcal{O}\left(\Delta t^2\right) = \frac{\partial}{\partial t} T(\vec{x}, t) + \lambda \nabla \cdot \left(\vec{w}(\vec{x}) T(\vec{x}, t) \right) - \Delta t \lambda^2 \cent'_{\mathrm{E}} \sigma_1 \Delta T(\vec{x}, t) - \Delta t \lambda^2 \sigma_1 \nabla \cdot \left[T(\vec{x}, t) \boldsymbol{J}\left(\vec{w}(\vec{x}) \right) \cdot \vec{w}(\vec{x}) \right]$$