Adaptive multiresolution-based lattice Boltzmann schemes and their accuracy analysis *via* the equivalent equations

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CMAP - Ecole polytechnique

5th of May, 2021
Groupe de travail **"Schémas de Boltzmann sur réseau"** - Institut Henri Poincaré

Introduction

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- Time **adaptive mesh** built by **multiresolution**.
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Today[3]

Devising an **asymptotic analysis** for the adaptive LBM-MR scheme.

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Structure of the presentation

Introduction

Lattice Boltzmann schemes

Adaptive LBM-MR method

Equivalent equation analysis on the LBM-MR adaptive scheme

Numerical simulations

Conclusions

Lattice Boltzmann schemes

Lattice Boltzmann schemes: collide and stream

The ingredients

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$$\boldsymbol{m}(t, \boldsymbol{x}) = \boldsymbol{M} \boldsymbol{f}(t, \boldsymbol{x})$$

$$\boldsymbol{f}^\star(t, \boldsymbol{x}) = \boldsymbol{M}^{-1} \Big((\boldsymbol{I} - \boldsymbol{S}) \boldsymbol{m}(t, \boldsymbol{x}) + \boldsymbol{S} \boldsymbol{m}^{\mathrm{eq}}(m^0(t, \boldsymbol{x}), \dots) \Big),$$

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- **Stream**

$$f^\alpha(t+\Delta t,\boldsymbol{x}) = f^{\alpha,\star}(t,\boldsymbol{x}-\boldsymbol{c}_\alpha\Delta x).$$

How to devise the scheme

Most of the schemes follow these principles:

- The discrete velocities are generally isotropic.
- The lines of the matrix $\boldsymbol{M}$ are in general low order polynomials of the discrete velocities, for example $1, X, X^2/2, \ldots,$ [D'Humières, 1992].
- The relaxation matrix $\boldsymbol{S}$ and the equilibria are selected by Chapman-Enskog expansions [Chapman and Cowling, 1991] or using the **equivalent equations** [Dubois, 2008].

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This latter technique is based on the Taylor expansion of the stream phase (we do the 1D for simplicity)

$$f^{\alpha}(t+\Delta t, x) = \sum_{s=0}^{+\infty} \frac{\Delta t^s}{s!} \partial_t^s f^{\alpha}(t,x) = f^{\alpha,\star}(t, x - c_{\alpha}\Delta x) = \sum_{s=0}^{+\infty} \frac{(-c_{\alpha}\Delta x)^s}{s!} \partial_x^s f^{\alpha,\star}(t,x).$$

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We call this formula (especially the right hand side) **target expansion**. Then, one changes the basis with $\boldsymbol{M}$ and identify powers of Δx order by order.

The most simple example of LBM scheme

Probably the most simple LBM scheme is [Graille, 2014]

$$q = 2, \quad c_0 = 1, \quad c_1 = -1, \quad \boldsymbol{M} = \begin{pmatrix} 1 & 1 \\ \lambda & -\lambda \end{pmatrix}, \quad \boldsymbol{S} = \mathrm{diag}(0, s).$$

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Expanding the stream phase

$$f^{\pm} + \frac{\Delta x}{\lambda} \partial_t f^{\pm} + \frac{\Delta x^2}{2\lambda} \partial_{tt} f^{\pm} + O(\Delta x^3) = f^{\pm,\star} \mp \Delta x \partial_x f^{\pm,\star} + \frac{\Delta x^2}{2} \partial_{xx} f^{\pm,\star} + O(\Delta x^3)$$

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Writing the moments: u is the conserved one and v is the non-conserved

$$u + \frac{\Delta x}{\lambda}\partial_t u + \frac{\Delta x^2}{2\lambda}\partial_{tt} u + O(\Delta x^3) = u^\star - \frac{\Delta x}{\lambda}\partial_x v^\star + \frac{\Delta x^2}{2}\partial_{xx} u^\star + O(\Delta x^3),$$

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By conservation of u

$$\partial_t u + \frac{\Delta x}{2}\partial_{tt} u + O(\Delta x^2) = -\partial_x v^\star + \frac{\lambda \Delta x}{2}\partial_{xx} u + O(\Delta x^2),$$

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Equivalent equations

- Leading order Δx^0

$$\partial_t u + \partial_x v^\star = O(\Delta x), \qquad v = v^\star + O(\Delta x) = (1-s)v + s v^{\mathrm{eq}}(u) + O(\Delta x),$$

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Therefore $v = v^{\text{eq}}(u) + O(\Delta x)$ as well as $v^\star = v^{\text{eq}}(u) + O(\Delta x)$. Assuming that $\partial_x v^\star = \partial_x v^{\text{eq}}(u) + O(\Delta x)$, we are consistent with a conservation law with flux $v^{\text{eq}}(u)$. We call this order of development **inertial order**.

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&= v + \frac{\Delta x}{\lambda}\partial_u v^{\mathrm{eq}}(u)\partial_t u + O(\Delta x^2)\\
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This yields

$$\begin{aligned}
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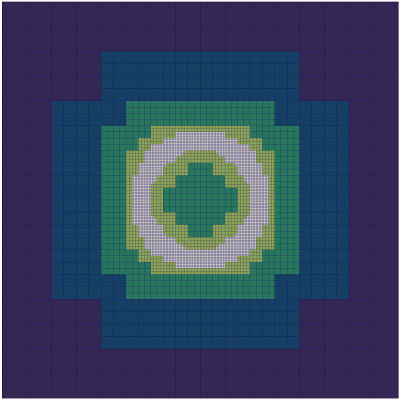
We call this order of development **diffusive order**. For this simple scheme, there are not enough DOF to impose a diffusion structure independently of the hyperbolic structure.

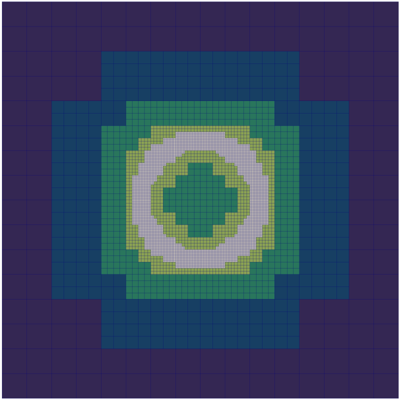
Remark

A key role is played by the **stream** phase which make flux-like terms showing up.

Adaptive LBM-MR method



Consider a bounded domain $\Omega = [0,1]^d$. We can build a **hybrid partition** of such a domain formed by cells at different levels of resolution between $\underline{L}$ and $\overline{L}$.



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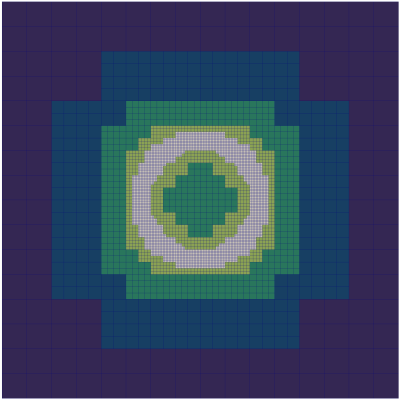
$$C_{\ell,\boldsymbol{k}} = \prod_{a=1}^{d} [2^{-\ell} k_a, 2^{-\ell}(k_a + 1)],$$

for $\ell = \underline{L}, \ldots, \overline{L}$ and $\boldsymbol{k} \in \{0, \ldots, 2^{\Delta \ell} - 1\}$.

Adaptive grids



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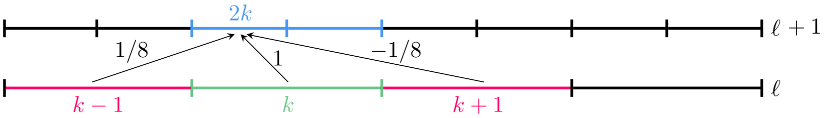
- $\boldsymbol{x}_{\ell,\boldsymbol{k}} := 2^{-\ell}(\boldsymbol{k} + 1/2)$: cell center
- $\Delta x_\ell = 2^{\Delta\ell} \Delta x$: edge length
- $\Delta x = 2^{-\overline{L}}$: finest space-step
- $\Delta\ell = \overline{L} - \ell$: distance between the current level ℓ and the finest level $\overline{L}$

Generate the adaptive grid

Introduce the **prediction operator** [Harten, 1994], [Cohen *et al.*, 2003]



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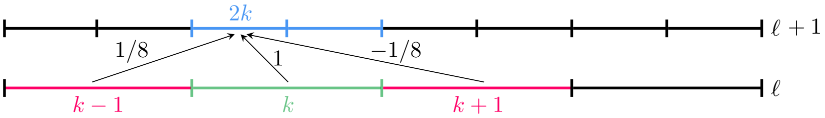

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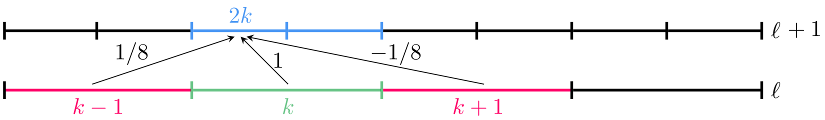

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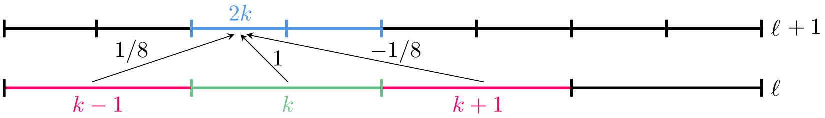

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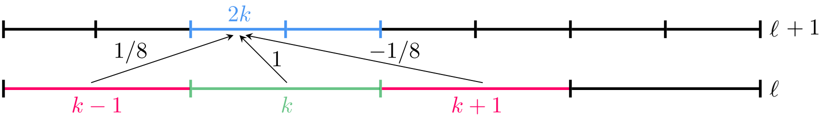

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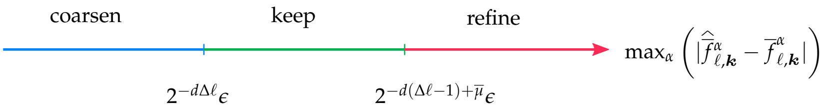
Remark

The prediction operator exactly recovers the average on the cell $C_{\ell+1,2k+\delta}$ when the function f^{α} is polynomial of degree at most $2\gamma + 1$.

Generate the adaptive grid



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$$

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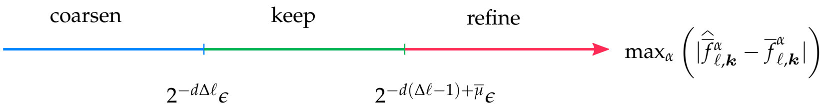

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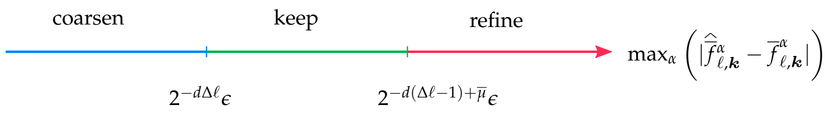
This stretegy grants a control of the additional error introduced by the adaptation by ϵ.



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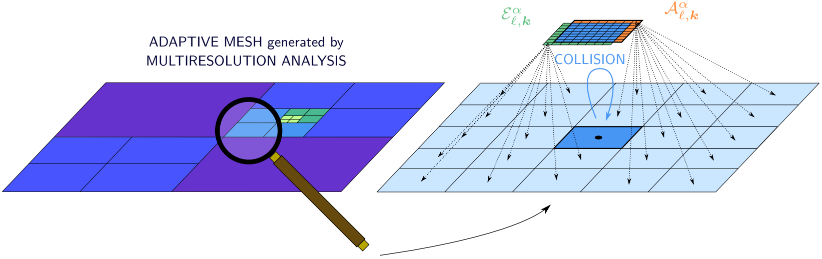
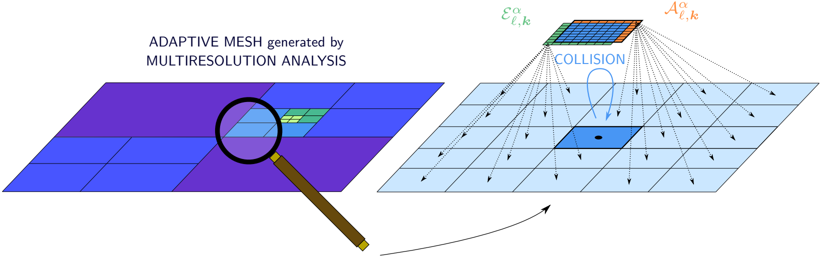
Fixed mesh

In this work, we are not primarly interested by the quality of the whole process in ϵ, which was the subject of previous works. Thus, most of the time, we consider uniform coarsened meshes at level $\underline{L}$.

Adaptive method





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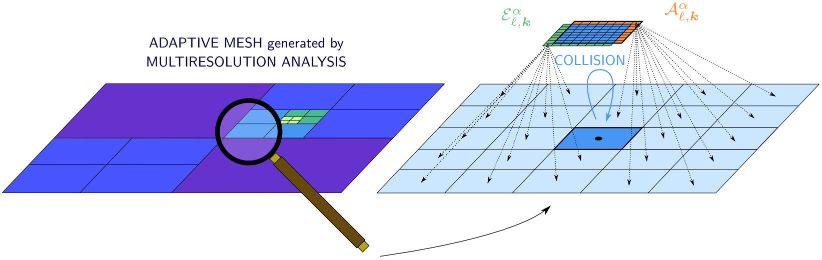

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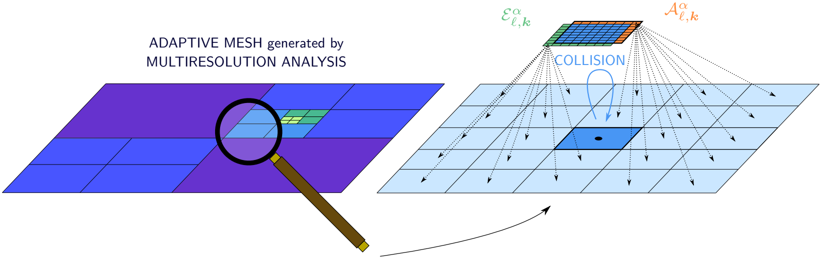
- **Collide** $\overline{\boldsymbol{f}}^{\star}_{\ell,\boldsymbol{k}}(t) = \boldsymbol{M}^{-1}\left((\boldsymbol{I}-\boldsymbol{S})\overline{\boldsymbol{m}}_{\ell,\boldsymbol{k}}(t) + \boldsymbol{S}\boldsymbol{m}^{\text{eq}}(\overline{m}^{0}_{\ell,\boldsymbol{k}}(t),\dots)\right)$.



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- **Stream** $\overline{f}^{\alpha}_{\ell,\boldsymbol{k}}(t+\Delta t) = \overline{f}^{\alpha,\star}_{\ell,\boldsymbol{k}}(t) + \frac{1}{2^{d\Delta\ell}}\left(\sum_{\overline{\boldsymbol{k}} \in \mathcal{E}^{\alpha}_{\ell,\boldsymbol{k}}} \hat{\hat{f}}^{\alpha,\star}_{\overline{L},\overline{\boldsymbol{k}}}(t) - \sum_{\overline{\boldsymbol{k}} \in \mathcal{A}^{\alpha}_{\ell,\boldsymbol{k}}} \hat{\hat{f}}^{\alpha,\star}_{\overline{L},\overline{\boldsymbol{k}}}(t) \right)$,
  where we have taken

$$\mathcal{B}_{\ell,\boldsymbol{k}} = \{\boldsymbol{k}2^{\Delta\ell} + \boldsymbol{\delta} \ : \ \boldsymbol{\delta} \in \{0, \dots, 2^{\Delta\ell} - 1\}^d\},$$

$$\mathcal{E}^{\alpha}_{\ell,\boldsymbol{k}} = (\mathcal{B}_{\ell,\boldsymbol{k}} - \boldsymbol{c}_{\alpha}) \setminus \mathcal{B}_{\ell,\boldsymbol{k}}, \qquad \mathcal{A}^{\alpha}_{\ell,\boldsymbol{k}} = \mathcal{B}_{\ell,\boldsymbol{k}} \setminus (\mathcal{B}_{\ell,\boldsymbol{k}} - \boldsymbol{c}_{\alpha}).$$

In the figure, $\boldsymbol{c}_{\alpha} = (1, 1)$. Why is it interesting???

We consider the non-isothermal Euler system with the well-known Lax-Liu problem [Lax and Liu, 1998] simulated using a vectorial D2Q4 scheme[4]:

[4] **Bellotti, Gouarin, Graille, Massot** - *Multidimensional fully adaptive lattice Boltzmann methods with error control based on multiresolution analysis* - Submitted to JCP - 2021 - `https://arxiv.org/abs/2103.02903`.

Example of result - Navier Stokes

We consider the von Karman vortex shedding simulated using a D2Q9 scheme[5]:

[5] **Bellotti, Gouarin, Graille, Massot** - *Multidimensional fully adaptive lattice Boltzmann methods with error control based on multiresolution analysis* - Submitted to JCP - 2021 - `https://arxiv.org/abs/2103.02903`.

Equivalent equation analysis on the LBM-MR adaptive scheme

Target expansion

Remark

We analyze the **stream phase** without taking the different models for the collision phase into account. This is totally justified as long as the equilibria are **linear** but we shall numerically verify that the study applies to non-linear situations.

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We want to find the **maximum order of accuracy** of our adaptive strategies according to the size of the prediction stencil γ. We adopt the point of view of Finite Differences [Leveque, 2002]. When considered at the finest level $\overline{L}$

$$f^{\alpha}(t+\Delta t, x_{\overline{L},k}) = f^{\alpha,\star}(t, x_{\overline{L},k-c_{\alpha}}) = f^{\alpha,\star}(t, x_{\overline{L},k} - c_{\alpha}\Delta x).$$

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Thus we can apply a Taylor expansion to both sides of the equation, yielding

$$\sum_{s=0}^{+\infty} \frac{\Delta t^s}{s!} \partial_t^s f^{\alpha}(t, x_{\overline{L},k}) = \sum_{s=0}^{+\infty} \frac{(-c_{\alpha}\Delta x)^s}{s!} \partial_x^s f^{\alpha,\star}(t, x_{\overline{L},k})$$

$$= f^{\alpha,\star} - \underbrace{c_{\alpha}\Delta x \partial_x f^{\alpha,\star}}_{\text{Inertial term}} + \underbrace{\frac{c_{\alpha}^2 \Delta x^2}{2} \partial_{xx} f^{\alpha,\star}}_{\text{Diffusive term}} - \underbrace{\frac{c_{\alpha}^3 \Delta x^3}{6} \partial_x^3 f^{\alpha,\star}}_{\text{Dispersive term}} + \dots,$$

The right hand side is called **target expansion**. Indeed, the left hand side shall always be the same because the time-step Δt is fixed by the finest mesh.

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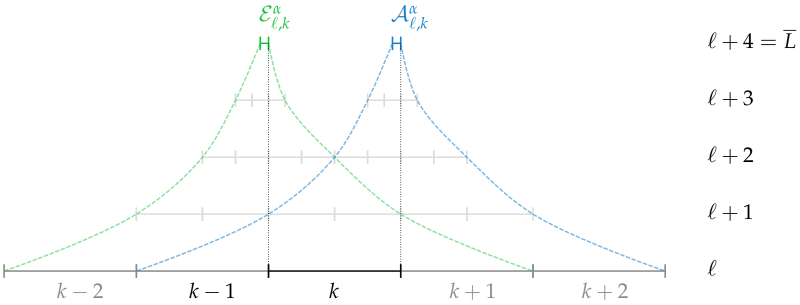
How to analyze our scheme? Assume, without loss of generality, that $\max_{\alpha} |c_{\alpha}| \leq 2$ and $\gamma \leq 1$.



With a set of weights $(C^{\alpha}_{\Delta\ell,m})^{m=+2}_{m=-2} \subset \mathbb{R}$

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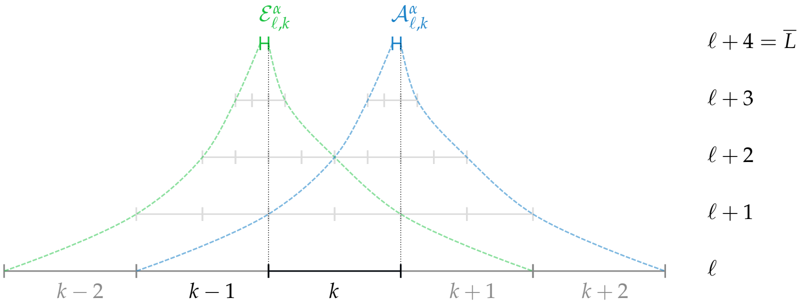
Recursion flattening



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The advantage is that the pseudo-flux term can be developed using Taylor expansions adopting a Finite Difference point of view.

Expansion of the LBM-MR scheme

$$\sum_{s=0}^{+\infty} \frac{\Delta t^s}{s!} \partial_t^s f^{\alpha}(t, x_{\ell,k}) = f^{\alpha,\star}(t, x_{\ell,k}) + \sum_{s=0}^{+\infty} \left(\frac{(\Delta x_{\ell})^s}{2^{\Delta \ell} s!} \left(\sum_{m=-2}^{+2} m^s C^{\alpha}_{\Delta \ell, m} \right) \partial_x^s f^{\alpha,\star}(t, x_{\ell,k}) \right),$$

$$= f^{\alpha,\star}(t, x_{\ell,k}) + \sum_{s=0}^{+\infty} \left(\frac{2^{\Delta \ell (s-1)} (\Delta x)^s}{s!} \left(\sum_{m=-2}^{+2} m^s C^{\alpha}_{\Delta \ell, m} \right) \partial_x^s f^{\alpha,\star}(t, x_{\ell,k}) \right),$$

$$= \left(1 + \frac{1}{2^{\Delta \ell}} \sum_{m=-2}^{+2} C^{\alpha}_{\Delta \ell, m} \right) f^{\alpha,\star} + \overbrace{\left(\sum_{m=-2}^{+2} m C^{\alpha}_{\Delta \ell, m} \right) \Delta x \partial_x f^{\alpha,\star}}^{\text{Inertial term}}$$

$$+ \underbrace{\left(2^{\Delta \ell} \sum_{m=-2}^{+2} m^2 C^{\alpha}_{\Delta \ell, m} \right) \frac{\Delta x^2}{2} \partial_{xx} f^{\alpha,\star}}_{\text{Diffusive term}} + \underbrace{\left(2^{2\Delta \ell} \sum_{m=-2}^{+2} m^3 C^{\alpha}_{\Delta \ell, m} \right) \frac{\Delta x^3}{6} \partial_x^3 f^{\alpha,\star}}_{\text{Dispersive term}} + \dots$$

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The goal of this game is to match as much terms as possible of the target expansion: approximated physics and stability conditions as close as possible to that of the reference scheme at level $\overline{L}$ for the adaptive scheme at the local level of refinement ℓ. These conditions are checked locally: we request them for any possible level.

$$\sum_{m=-2}^{+2} C^{\alpha}_{\Delta\ell,m} = 0, \qquad \text{and} \qquad \sum_{m=-2}^{+2} m^s C^{\alpha}_{\Delta\ell,m} = \frac{(-c_\alpha)^s}{2^{\Delta\ell(s-1)}}, \quad \text{for} \quad s \in \{1, 2, 3, \dots\} = \mathbb{N}^\star,$$

...of course for every α and for every $\Delta\ell$!!!

Apply the expansion to some scheme

In this presentation, we consider three numerical schemes:

- The **Haar scheme**: LBM-MR with $\gamma = 0$, thus

$$\widehat{\overline{f}}^{\alpha}_{\ell+1,2k+\delta} = \overline{f}^{\alpha}_{\ell,k}, \qquad (\textit{talis pater, qualis filius})_{\text{Abælardus}}.$$

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- The first **non-trivial wavelet scheme**: LBM-MR with $\gamma = 1$, thus

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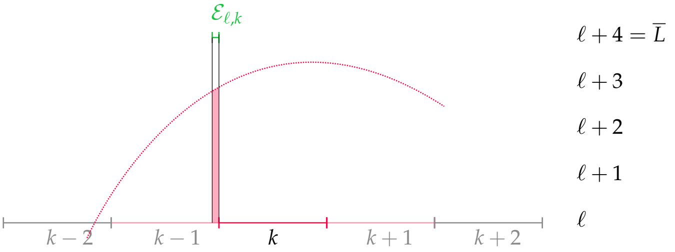
- The **Lax-Wendroff scheme** by [Fakhari *et al.*, 2014]

$$\overline{f}^{\alpha}_{\ell,\boldsymbol{k}}(t+\Delta t) = \left(1 - \frac{1}{4^{\Delta\ell}}\right)\overline{f}^{\alpha,\star}_{\ell,\boldsymbol{k}}(t) + \frac{1}{2^{\Delta\ell+1}}\left(1+\frac{1}{2^{\Delta\ell}}\right)\overline{f}^{\alpha,\star}_{\ell,\boldsymbol{k}-\boldsymbol{c}_{\alpha}/|\boldsymbol{c}_{\alpha}|_2}(t) - \frac{1}{2^{\Delta\ell+1}}\left(1-\frac{1}{2^{\Delta\ell}}\right)\overline{f}^{\alpha,\star}_{\ell,\boldsymbol{k}+\boldsymbol{c}_{\alpha}/|\boldsymbol{c}_{\alpha}|_2}(t).$$

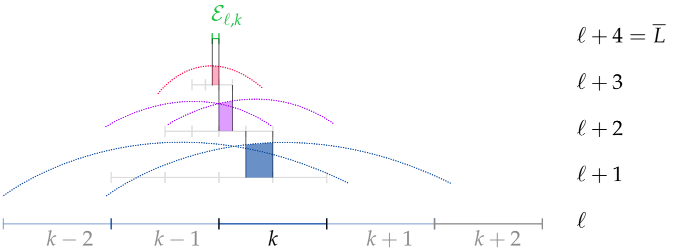
This is not a multiresolution scheme: we consider it for comparison purposes.

More details on the schemes

Lax Wendroff



LBM-MR for $\gamma = 1$



The Haar scheme $\gamma = 0$

Proposition (Match for $\gamma = 0$)

Let $d = 1$, $\gamma = 0$ and $\Delta\ell \geq 0$, then the flattened coefficients of the advection phase read

$$C^{\alpha}_{\Delta\ell,0} = -|c_{\alpha}|, \qquad C^{\alpha}_{\Delta\ell,-c_{\alpha}/|c_{\alpha}|} = |c_{\alpha}|,$$

and those not listed are equal to zero. Therefore, the adaptive stream phase matches that of the reference scheme up to order $s = 1$. This also writes

$$\sum_{m=-2}^{+2} C^{\alpha}_{\Delta\ell,m} = 0, \qquad \underbrace{\sum_{m=-2}^{+2} m C^{\alpha}_{\Delta\ell,m} = -c_{\alpha}}_{\textit{Inertial term}}, \qquad \underbrace{\sum_{m=-2}^{+2} m^2 C^{\alpha}_{\Delta\ell,m} = \frac{(-c_{\alpha})^2}{2^{\Delta\ell}}}_{\textit{Diffusive term}}.$$

The non-trivial scheme $\gamma = 1$

Proposition (Match for $\gamma = 1$)

Let $d = 1$, $\gamma = 1$ and $\Delta\ell > 0$, then the flattened weights of the stream phase are given by the recurrence relations

$$\begin{pmatrix} C^{\alpha}_{\Delta\ell,-2} \\ C^{\alpha}_{\Delta\ell,-1} \\ C^{\alpha}_{\Delta\ell,0} \\ C^{\alpha}_{\Delta\ell,1} \\ C^{\alpha}_{\Delta\ell,2} \end{pmatrix} = \begin{pmatrix} 0 & -1/8 & 0 & 0 & 0 \\ 2 & 9/8 & 0 & -1/8 & 0 \\ 0 & 9/8 & 2 & 9/8 & 0 \\ 0 & -1/8 & 0 & 9/8 & 2 \\ 0 & 0 & 0 & -1/8 & 0 \end{pmatrix} \begin{pmatrix} C^{\alpha}_{\Delta\ell-1,-2} \\ C^{\alpha}_{\Delta\ell-1,-1} \\ C^{\alpha}_{\Delta\ell-1,0} \\ C^{\alpha}_{\Delta\ell-1,1} \\ C^{\alpha}_{\Delta\ell-1,2} \end{pmatrix},$$

where the initialization is given by $C^{\alpha}_{0,-c_\alpha} = 1$ and $C^{\alpha}_{0,0} = -1$ and the remaining terms set to zero. Therefore, the adaptive stream phase matches that of the reference scheme up to order $s = 3$. This also writes

$$\sum_{m=-2}^{+2} C^{\alpha}_{\Delta\ell,m} = 0, \qquad \underbrace{\sum_{m=-2}^{+2} m C^{\alpha}_{\Delta\ell,m} = -c_\alpha}_{\text{Inertial term}},$$

$$\underbrace{\sum_{m=-2}^{+2} m^2 C^{\alpha}_{\Delta\ell,m} = \frac{c_\alpha^2}{2^{\Delta\ell}}}_{\text{Diffusive term}}, \qquad \underbrace{\sum_{m=-2}^{+2} m^3 C^{\alpha}_{\Delta\ell,m} = -\frac{c_\alpha^3}{4^{\Delta\ell}}}_{\text{Dispersive term}}, \qquad \underbrace{\sum_{m=-2}^{+2} m^4 C^{\alpha}_{\Delta\ell,m} = \frac{c_\alpha^4}{8^{\Delta\ell}}}_{4^{th}\text{-order term (biLaplacian)}}.$$

(The last equation and its label are struck through on the slide.)

The non-trivial scheme $\gamma = 1$ - Proof

Assume to know the coefficients of the flattened advection for level $\ell+1$ (for $\Delta\ell - 1$).
We have

$$\sum_{\overline{k}\in\mathcal{E}^{\alpha}_{\ell,k}} \widehat{\widehat{f}}^{\alpha,\star}_{\overline{L},\overline{k}} - \sum_{\overline{k}\in\mathcal{A}^{\alpha}_{\ell,k}} \widehat{\widehat{f}}^{\alpha,\star}_{\overline{L},\overline{k}} = \Bigg(\overbrace{\sum_{\overline{k}\in\mathcal{E}^{\alpha}_{\ell+1,2k}} \widehat{\widehat{f}}^{\alpha,\star}_{\overline{L},\overline{k}}}^{\text{Ingoing left son}} - \overbrace{\sum_{\overline{k}\in\mathcal{A}^{\alpha}_{\ell+1,2k}} \widehat{\widehat{f}}^{\alpha,\star}_{\overline{L},\overline{k}}}^{\text{Outgoing left son}}\Bigg) + \Bigg(\overbrace{\sum_{\overline{k}\in\mathcal{E}^{\alpha}_{\ell+1,2k+1}} \widehat{\widehat{f}}^{\alpha,\star}_{\overline{L},\overline{k}}}^{\text{Ingoing right son}} - \overbrace{\sum_{\overline{k}\in\mathcal{A}^{\alpha}_{\ell+1,2k+1}} \widehat{\widehat{f}}^{\alpha,\star}_{\overline{L},\overline{k}}}^{\text{Outgoing right son}}\Bigg),$$

$$= \sum_{m=-2}^{+2} C^{\alpha}_{\Delta\ell-1,m} \widehat{\overline{f}}^{\alpha,\star}_{\ell+1,2k+m} + \sum_{m=-2}^{+2} C^{\alpha}_{\Delta\ell-1,m} \widehat{\overline{f}}^{\alpha,\star}_{\ell+1,2k+1+m},$$

$$= \sum_{m=-2}^{+2} C^{\alpha}_{\Delta\ell-1,m} \widehat{\overline{f}}^{\alpha,\star}_{\ell+1,2k+m} + \sum_{m=-1}^{+3} C^{\alpha}_{\Delta\ell-1,m-1} \widehat{\overline{f}}^{\alpha,\star}_{\ell+1,2k+m}$$

$$= \sum_{m=-2}^{+3} \tilde{C}^{\alpha}_{\Delta\ell-1,m} \widehat{\overline{f}}^{\alpha,\star}_{\ell+1,2k+m},$$

with

$$\tilde{C}^{\alpha}_{\Delta\ell-1,m} = \begin{cases} C^{\alpha}_{\Delta\ell-1,-2}, & m = -2, \\ C^{\alpha}_{\Delta\ell-1,m} + C^{\alpha}_{\Delta\ell-1,m-1}, & m = -1, 0, 1, 2, \\ C^{\alpha}_{\Delta\ell-1,2}, & m = 3. \end{cases}$$

The non-trivial scheme $\gamma = 1$ - Proof

Using the prediction operator

$$\sum_{m=-2}^{+3} \tilde{C}^{\alpha}_{\Delta\ell-1,m} \widehat{\overline{f}}^{\alpha,\star}_{\ell+1,2k+m} = \tilde{C}^{\alpha}_{\Delta\ell-1,-2}\left(f_{\ell,k-1} + \frac{1}{8}f_{\ell,k-2} - \frac{1}{8}f_{\ell,k}\right) + \tilde{C}^{\alpha,}_{\Delta\ell-1,-1}\left(f_{\ell,k-1} - \frac{1}{8}f_{\ell,k-2} + \frac{1}{8}f_{\ell,k}\right)$$
$$+ \tilde{C}^{\alpha}_{\Delta\ell-1,0}\left(f_{\ell,k} + \frac{1}{8}f_{\ell,k-1} - \frac{1}{8}f_{\ell,k+1}\right) + \tilde{C}^{\alpha}_{\Delta\ell-1,1}\left(f_{\ell,k} - \frac{1}{8}f_{\ell,k-1} + \frac{1}{8}f_{\ell,k+1}\right)$$
$$+ \tilde{C}^{\alpha}_{\Delta\ell-1,2}\left(f_{\ell,k+1} + \frac{1}{8}f_{\ell,k} - \frac{1}{8}f_{\ell,k+2}\right) + \tilde{C}^{\alpha}_{\Delta\ell-1,3}\left(f_{\ell,k+1} - \frac{1}{8}f_{\ell,k} + \frac{1}{8}f_{\ell,k+2}\right),$$

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so that after tedious computations, we arrive at

$$\sum_{m=-2}^{+3} \tilde{C}^{\alpha}_{\Delta\ell-1,m} \widehat{\overline{f}}^{\alpha,\star}_{\ell+1,2k+m} = \left(-\frac{1}{8}C^{\alpha}_{\Delta\ell-1,-1}\right)\overline{f}^{\alpha,\star}_{\ell,k-2} + \left(2C^{\alpha}_{\Delta\ell-1,-2} + \frac{9}{8}C^{\alpha}_{\Delta\ell-1,-1} - \frac{1}{8}C^{\alpha}_{\Delta\ell-1,1}\right)\overline{f}^{\alpha,\star}_{\ell,k-1}$$
$$+ \left(\frac{9}{8}C^{\alpha}_{\Delta\ell-1,-1} + 2C^{\alpha}_{\Delta\ell-1,0} + \frac{9}{8}C^{\alpha}_{\Delta\ell-1,1}\right)\overline{f}^{\alpha,\star}_{\ell,k}$$
$$+ \left(-\frac{1}{8}C^{\alpha}_{\Delta\ell-1,-1} + \frac{9}{8}C^{\alpha}_{\Delta\ell-1,1} + 2C^{\alpha}_{\Delta\ell-1,2}\right)\overline{f}^{\alpha,\star}_{\ell,k+1} + \left(-\frac{1}{8}C^{\alpha}_{\Delta\ell-1,1}\right)\overline{f}^{\alpha,\star}_{\ell,k+2},$$

concluding the first part of the proof.

The non-trivial scheme $\gamma = 1$ - Proof

Using the prediction operator

$$\sum_{m=-2}^{+3} \tilde{C}^{\alpha}_{\Delta\ell-1,m} \widehat{\overline{f}}^{\alpha,\star}_{\ell+1,2k+m} = \tilde{C}^{\alpha}_{\Delta\ell-1,-2}\left(f_{\ell,k-1} + \frac{1}{8}f_{\ell,k-2} - \frac{1}{8}f_{\ell,k}\right) + \tilde{C}^{\alpha,}_{\Delta\ell-1,-1}\left(f_{\ell,k-1} - \frac{1}{8}f_{\ell,k-2} + \frac{1}{8}f_{\ell,k}\right)$$
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concluding the first part of the proof. Then, let us proceed by recurrence: for $\Delta\ell = 0$ the thesis trivially holds.

The non-trivial scheme $\gamma = 1$ - Proof

Using the prediction operator

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concluding the first part of the proof. Then, let us proceed by recurrence: for $\Delta\ell = 0$ the thesis trivially holds. Assume that it holds for $\Delta\ell - 1$.

- $\sum_{m=-2}^{+2} C^{\alpha}_{\Delta\ell,m} = \cdots = 2\sum_{m=-2}^{+2} C^{\alpha}_{\Delta\ell-1,m} = 0.$
- $\sum_{m=-2}^{+2} mC^{\alpha}_{\Delta\ell,m} = \cdots = \sum_{m=-2}^{+2} mC^{\alpha}_{\Delta\ell-1,m} = -c_{\alpha}.$
- $\sum_{m=-2}^{+2} m^2C^{\alpha}_{\Delta\ell,m} = \cdots = \frac{1}{2}\sum_{m=-2}^{+2} m^2C^{\alpha}_{\Delta\ell-1,m} = \frac{1}{2}\frac{c_{\alpha}^2}{2^{\Delta\ell-1}} = \frac{c_{\alpha}^2}{2^{\Delta\ell}}.$
- $\sum_{m=-2}^{+2} m^3C^{\alpha}_{\Delta\ell,m} = \cdots = \frac{1}{4}\sum_{m=-2}^{+2} m^3C^{\alpha}_{\Delta\ell-1,m} = -\frac{1}{4}\frac{c_{\alpha}^3}{4^{\Delta\ell-1}} = -\frac{c_{\alpha}^3}{4^{\Delta\ell}},$

that concludes the proof.

Proposition (Match for Lax-Wendroff)

Let $d = 1$ and $\Delta\ell \geq 0$, then the flattened coefficients of the advection phase are given by

$$C^{\alpha}_{\Delta\ell,0} = -\frac{|c_\alpha|^2}{2^{\Delta\ell}}, \quad C^{\alpha}_{\Delta\ell,-c_\alpha/|c_\alpha|} = \frac{|c_\alpha|}{2}\left(1 + \frac{|c_\alpha|}{2^{\Delta\ell}}\right), \quad C^{\alpha}_{\Delta\ell,c_\alpha/|c_\alpha|} = -\frac{|c_\alpha|}{2}\left(1 - \frac{|c_\alpha|}{2^{\Delta\ell}}\right).$$

Therefore, the adaptive stream phase matches that of the reference scheme up to order $s = 2$. This also writes

$$\sum_{m=-2}^{+2} C^{\alpha}_{\Delta\ell,m} = 0, \qquad \underbrace{\sum_{m=-2}^{+2} m C^{\alpha}_{\Delta\ell,m} = -c_\alpha,}_{\text{Inertial term}}$$

$$\underbrace{\sum_{m=-2}^{+2} m^2 C^{\alpha}_{\Delta\ell,m} = \frac{c_\alpha^2}{2^{\Delta\ell}},}_{\text{Diffusive term}} \qquad \underbrace{\sum_{m=-2}^{+2} m^3 C^{\alpha}_{\Delta\ell,m} = -\frac{c_\alpha^3}{4^{\Delta\ell}}}_{\text{Dispersive term}}$$

(The dispersive term equation is struck through on the slide.)

Conclusion on the schemes

The previous analysis shows that:

- A multiresolution scheme matches until $s = 2\gamma + 1$.
- All the schemes match the **inertial term**.
- Only the scheme for $\gamma = 1$ and Lax-Wendroff match the **diffusive term**.
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Therefore:

$$\partial_t u + \underbrace{\nabla \cdot (\boldsymbol{\varphi}(u))}_{\substack{\gamma=0 \\ \gamma=1 \\ \text{Lax-Wendroff}}} - \underbrace{\nabla \cdot (\boldsymbol{D} \nabla u)}_{\substack{\gamma=1 \\ \text{Lax-Wendroff}}} = \underbrace{\text{H.O.Ts}}_{\gamma=1}.$$

- The scheme for $\gamma = 0$ is almost **unusable** in practice.
- The Lax-Wendroff scheme is the **minimal scheme** for real applications (Navier-Stokes, etc...), because we also control diffusion. Still, it can threaten stability.
- The scheme for $\gamma \geq 1$ is the "**best**". It also keeps 3^{rd} order term, so better control on the stability.

Numerical simulations

Points of emphasis

The previous analysis was valid for

- Smooth solutions.
- In the limit of small Δx_ℓ for every $\ell = \underline{L}, \dots, \overline{L}$.

The aim of the following numerical simulations is to **validate the previous approach** by showing that it provides a useful tool to *a priori* study the behavior of the adaptive scheme.

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- $\mathrm{E}_{\mathrm{ref}}$: error of the reference scheme (at $\overline{L}$) *vs.* exact solution.
- $\mathrm{E}_{\mathrm{adap}}^{\overline{L}}$: error of the adaptive scheme (at $\underline{L}$) *vs.* exact solution at level $\overline{L}$, using the reconstruction operator.
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and the plan is to make

$$\mathrm{D}_{\mathrm{adap}} \ll \mathrm{E}_{\mathrm{ref}}, \qquad \Rightarrow \qquad \mathrm{E}_{\mathrm{adap}}^{\overline{L}} \approx \mathrm{E}_{\mathrm{ref}},$$

regardless the fact that it converges or not for $\Delta x \to 0$.

Points of emphasis

Remark (bis)

We are not interested in evaluating the quality of the multiresolution adaptation with respect to the parameter ϵ: we consider a uniform mesh at the lowest resolution $\underline{L}$. Remember that the match property is **uniform** in ℓ.

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We are not interested in evaluating the quality of the multiresolution adaptation with respect to the parameter ϵ: we consider a uniform mesh at the lowest resolution $\underline{L}$. Remember that the match property is **uniform** in ℓ. This also provides a **worst case scenario** to undoubtedly prove the resilience of our numerical strategy.

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We are not interested in evaluating the quality of the multiresolution adaptation with respect to the parameter ϵ: we consider a uniform mesh at the lowest resolution $\underline{L}$. Remember that the match property is **uniform** in ℓ. This also provides a **worst case scenario** to undoubtedly prove the resilience of our numerical strategy. Similar scenarios can happen

- when the mesh is updated using some **stiff** variable [Fakhari *et al.*, 2016] and [N'Guessan *et al.*, 2019] but we still want to achieve a good accuracy in the coarsely meshed areas for the non-stiff variables.
- a fixed adapted mesh is used: [Filippova and Hänel, 1998] and many others.

1D Linear advection equation

- The aim of this test case is to validate our analysis in a case where:
  - **Convergent** reference scheme: $\mathrm{E}_{\mathrm{ref}} \to 0$ as $\Delta x \to 0$, see [DELLACHERIE, 2014], [CAETANO *et al.*, 2019].
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$$\begin{cases} \partial_t u + \partial_x (Vu) = 0, \\ u(t=0,x) = \frac{1}{(4\pi\nu t_0)^{1/2}} \exp\left(-\frac{|x|^2}{4\nu t_0}\right), \end{cases}$$

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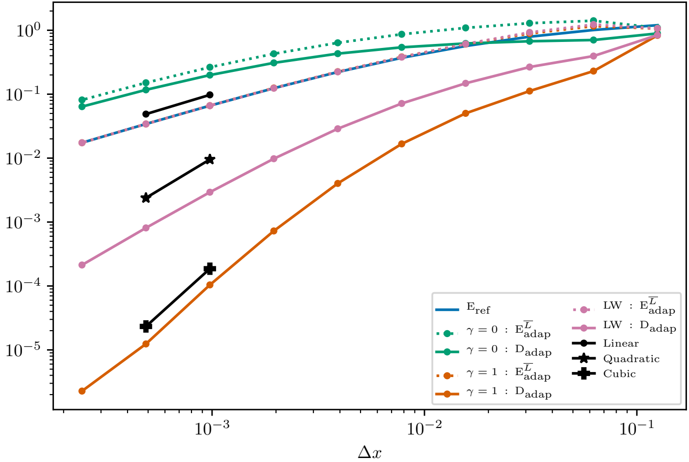
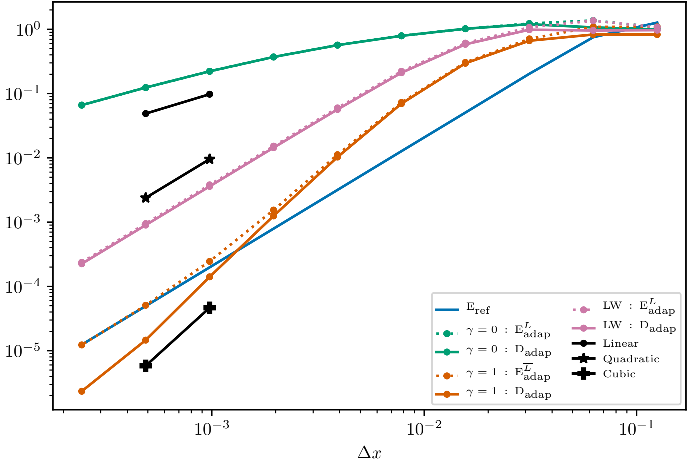
- The scheme is the D1Q2 given at the beginning of the presentation. We fix the level distance $\Delta\ell_{\min}$ and we increase $\overline{L}$ (thus reduce Δx).

1D Linear advection equation: $\Delta\ell_{\text{min}} = 2$ and $s = 1$



We have also tested $\Delta\ell_{\text{min}} = 6$ having similar results. Remember: $E^{\overline{L}}_{\text{adap}} \leq E_{\text{ref}} + D_{\text{adap}}$.



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1D Linear advection diffusion equation

- The aim of this test case is to validate our analysis in a case where:
  - **Not convergent** reference scheme as $\Delta x \to 0$, but a **richer** structure.
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- We consider a D1Q3 scheme with velocities $c_0 = 0, c_1 = 1$ and $c_2 = -1$ with change of basis and relaxation matrix given by

$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \lambda & -\lambda \\ 0 & \lambda^2/2 & \lambda^2/2 \end{pmatrix}, \qquad \boldsymbol{S} = \text{diag}(0, s_v, s_w).$$

  With equilibria and relaxation parameters:

$$m^{1,\text{eq}} = Vm^0, \qquad m^{2,\text{eq}} = \kappa m^0$$

$$s_v = (1/2 + \lambda\nu/(\Delta x(2\kappa - V^2)))^{-1}, \qquad s_w = 1.$$

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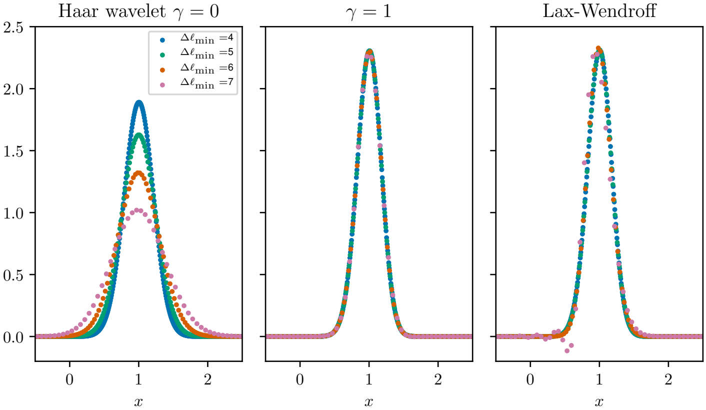
  We fix the maximal level $\overline{L}$ and we decrease the minimum level $\underline{L}$ (we increase $\Delta\ell_{\mathrm{min}}$.

1D Linear advection diffusion equation: $\overline{L} = 11$

	Haar $\gamma = 0$		$\gamma = 1$		Lax-Wendroff	
$\Delta\ell_{\min}$	$E^{\overline{L}}_{adap}$	D_{adap}	$E^{\overline{L}}_{adap}$	D_{adap}	$E^{\overline{L}}_{adap}$	D_{adap}
0	1.94e-02	0.00e+00	1.94e-02	0.00e+00	1.94e-02	0.00e+00
1	2.30e-02	1.55e-02	1.94e-02	7.88e-07	1.94e-02	3.63e-05
2	4.68e-02	4.52e-02	1.94e-02	3.41e-06	1.92e-02	1.82e-04
3	9.92e-02	9.94e-02	1.94e-02	1.31e-05	1.87e-02	7.63e-04
4	1.91e-01	1.92e-01	1.94e-02	5.40e-05	1.65e-02	3.09e-03
5	3.33e-01	3.34e-01	1.93e-02	2.78e-04	8.32e-03	1.24e-02
6	5.24e-01	5.26e-01	1.84e-02	1.74e-03	3.16e-02	5.03e-02
7	7.47e-01	7.48e-01	1.07e-02	1.89e-02	1.96e-01	2.15e-01



1D viscous Burgers equation

- The aim of this test case is to validate our analysis in a case where:
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$$\begin{cases} \partial_t u + \partial_x (u^2/2) - \nu \partial_{xx} u = 0, \\ u(t=0, x) = \frac{1}{(4\pi\nu t_0)^{1/2}} \exp\left(-\frac{x^2}{4\nu t_0}\right), \end{cases}$$

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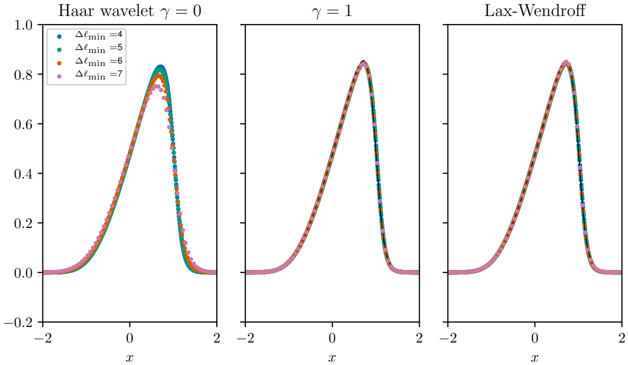
- The scheme the D1Q3 with

$$m^{1,\text{eq}} = (m^0)^2/2, \qquad m^{2,\text{eq}} = (m^0)^3/6 + \kappa m^0/2,$$
$$s_v = (1/2 + \lambda\nu/(\Delta x \kappa))^{-1}, \qquad s_w = 1.$$

Again, we fix the maximal level $\overline{L}$ and we decrease the minimum level $\underline{L}$ (we increase $\Delta \ell_{\text{min}}$.

1D viscous Burgers equation: large diffusion

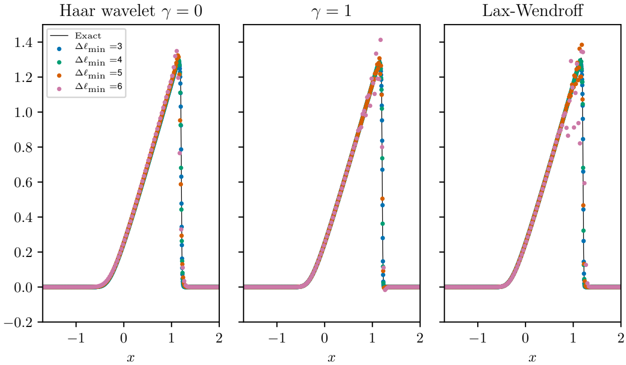
	Haar $\gamma = 0$		$\gamma = 1$		Lax-Wendroff	
$\Delta\ell_{\mathrm{min}}$	$\mathrm{E}^{\overline{L}}_{\mathrm{adap}}$	$\mathrm{D}_{\mathrm{adap}}$	$\mathrm{E}^{\overline{L}}_{\mathrm{adap}}$	$\mathrm{D}_{\mathrm{adap}}$	$\mathrm{E}^{\overline{L}}_{\mathrm{adap}}$	$\mathrm{D}_{\mathrm{adap}}$
0	1.23e-02	0.00e+00	1.23e-02	0.00e+00	1.23e-02	0.00e+00
1	1.24e-02	9.99e-04	1.23e-02	1.88e-07	1.23e-02	1.60e-06
2	1.27e-02	2.99e-03	1.23e-02	9.34e-07	1.23e-02	8.02e-06
3	1.41e-02	6.95e-03	1.23e-02	3.89e-06	1.23e-02	3.37e-05
4	1.94e-02	1.48e-02	1.23e-02	1.57e-05	1.22e-02	1.36e-04
5	3.25e-02	3.00e-02	1.23e-02	6.30e-05	1.19e-02	5.48e-04
6	6.03e-02	5.90e-02	1.23e-02	2.60e-04	1.09e-02	2.20e-03
7	1.13e-01	1.12e-01	1.22e-02	1.18e-03	8.62e-03	9.08e-03



Coherent with the theoretical analysis (smooth solution).

1D viscous Burgers equation: small diffusion

	Haar $\gamma = 0$		$\gamma = 1$		Lax-Wendroff	
$\Delta\ell_{\text{min}}$	$E^{\overline{L}}_{\text{adap}}$	D_{adap}	$E^{\overline{L}}_{\text{adap}}$	D_{adap}	$E^{\overline{L}}_{\text{adap}}$	D_{adap}
0	5.31e-03	0.00e+00	5.31e-03	0.00e+00	5.31e-03	0.00e+00
1	4.96e-03	1.16e-03	5.31e-03	3.47e-06	5.29e-03	2.72e-05
2	4.61e-03	3.41e-03	5.31e-03	2.34e-05	5.22e-03	1.38e-04
3	6.78e-03	7.76e-03	5.30e-03	1.41e-04	4.92e-03	6.17e-04
4	1.47e-02	1.64e-02	5.31e-03	8.63e-04	4.58e-03	3.54e-03
5	3.20e-02	3.34e-02	6.14e-03	6.08e-03	1.48e-02	1.70e-02
6	6.49e-02	6.57e-02	3.36e-02	3.37e-02	1.05e-01	1.04e-01
7	1.24e-01	1.25e-01	2.45e-01	2.42e-01	8.18e-01	8.19e-01



The theoretical analysis cannot predict this (singular solution): need for mesh adaptation.

The scheme we use is the D2Q9 with velocities given by

$$c_\alpha = \begin{cases} (0,0), & \alpha = 0, \\ \left(\cos\left(\frac{\pi}{2}(\alpha-1)\right), \sin\left(\frac{\pi}{2}(\alpha-1)\right)\right), & \alpha = 1,2,3,4, \\ \left(\cos\left(\frac{\pi}{2}(\alpha-5)+\frac{\pi}{4}\right), \sin\left(\frac{\pi}{2}(\alpha-5)+\frac{\pi}{4}\right)\right), & \alpha = 5,6,7,8, \end{cases}$$

with the moments by [Lallemand and Luo, 2000] relaxing with $\boldsymbol{S} = \mathrm{diag}(0, s, s, 1, 1, 1, 1, 1, 1)$

$$\boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \lambda & 0 & -\lambda & 0 & \lambda & -\lambda & -\lambda & \lambda \\ 0 & 0 & \lambda & 0 & -\lambda & \lambda & \lambda & -\lambda & -\lambda \\ -4\lambda^2 & -\lambda^2 & -\lambda^2 & -\lambda^2 & -\lambda^2 & 2\lambda^2 & 2\lambda^2 & 2\lambda^2 & 2\lambda^2 \\ 0 & -2\lambda^3 & 0 & 2\lambda^3 & 0 & \lambda^3 & -\lambda^3 & -\lambda^3 & \lambda^3 \\ 0 & 0 & -2\lambda^3 & 0 & 2\lambda^3 & \lambda^3 & \lambda^3 & -\lambda^3 & -\lambda^3 \\ 4\lambda^4 & -2\lambda^4 & -2\lambda^4 & -2\lambda^4 & -2\lambda^4 & \lambda^4 & \lambda^4 & \lambda^4 & \lambda^4 \\ 0 & \lambda^2 & -\lambda^2 & \lambda^2 & -\lambda^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda^2 & -\lambda^2 & \lambda^2 & -\lambda^2 \end{pmatrix},$$

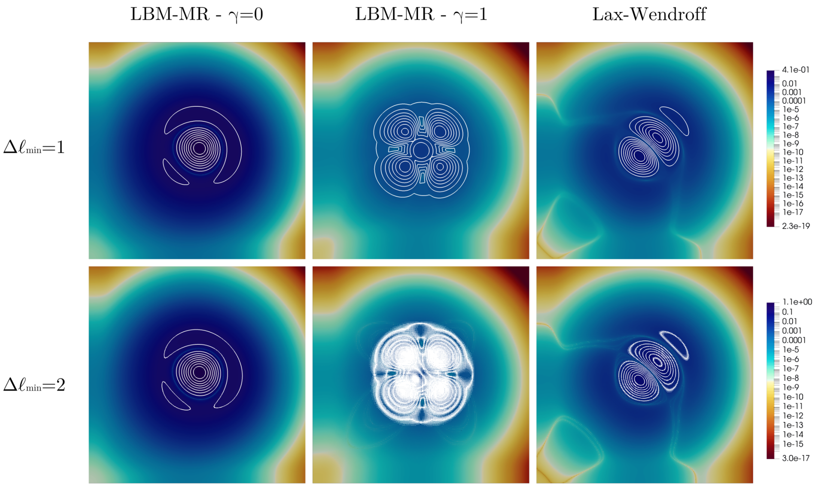
with $s = (1/2 + 3\nu/(\lambda\Delta x))^{-1}$ to enforce the diffusivity. The equilibria are based on the second-order expansion of the Maxwellian

$$m^{1,\mathrm{eq}} = V_x m^0, \quad m^{2,\mathrm{eq}} = V_y m^0, \quad m^{3,\mathrm{eq}} = (-2\lambda^2 + 3|\boldsymbol{V}|^2) m^0,$$
$$m^{4,\mathrm{eq}} = -\lambda^2 V_x m^0, \quad m^{5,\mathrm{eq}} = -\lambda^2 V_y m^0, \quad m^{6,\mathrm{eq}} = (\lambda^4 - 3\lambda^2|\boldsymbol{V}|^2) m^0,$$
$$m^{7,\mathrm{eq}} = (V_x^2 - V_y^2) m^0, \quad m^{8,\mathrm{eq}} = V_x V_y m^0.$$

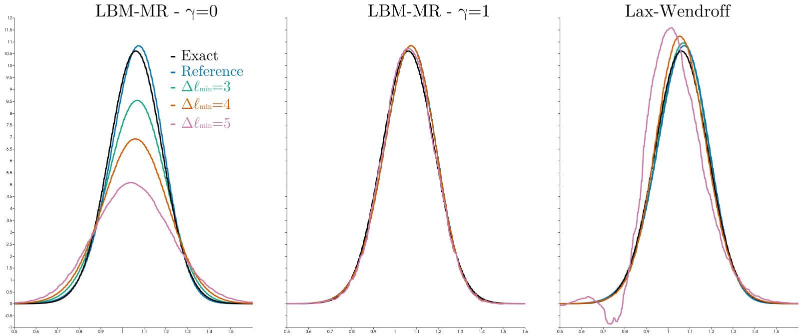
Same kind of tests than in 1D: prove that our analysis extends to 2D to quite "rich" models.

2D Linear advection equation: $\overline{L} = 9$

Spatial behavior of D_{adap} (in logarithmic scale) and contours:



	Haar $\gamma = 0$		$\gamma = 1$		Lax-Wendroff	
$\Delta\ell_{\text{min}}$	$E^{\overline{L}}_{\text{adap}}$	D_{adap}	$E^{\overline{L}}_{\text{adap}}$	D_{adap}	$E^{\overline{L}}_{\text{adap}}$	D_{adap}
0	4.86e-02	0.00e+00	4.86e-02	0.00e+00	4.86e-02	0.00e+00
1	4.61e-02	2.79e-02	4.86e-02	9.42e-05	4.80e-02	8.20e-04
2	7.58e-02	8.06e-02	4.87e-02	3.89e-04	4.56e-02	4.09e-03
3	1.64e-01	1.75e-01	4.87e-02	1.62e-03	3.71e-02	1.71e-02
4	3.16e-01	3.29e-01	4.82e-02	7.49e-03	4.01e-02	6.90e-02
5	5.38e-01	5.51e-01	4.99e-02	4.94e-02	2.39e-01	2.82e-01
6	8.16e-01	8.26e-01	4.74e-01	5.14e-01	1.00e+00	1.04e+00

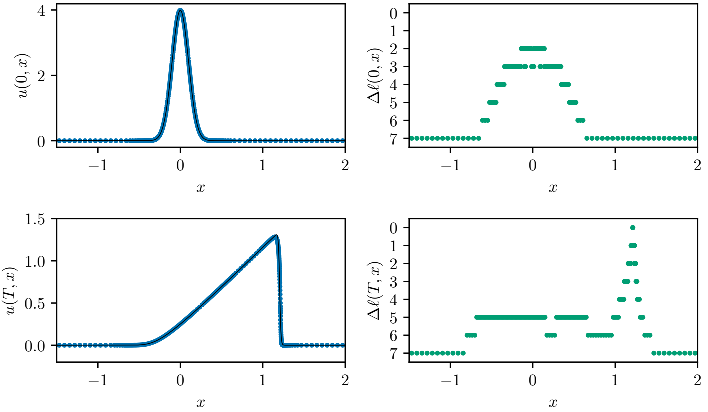


Again on the 1D viscous Burgers equation: small diffusion

What could be the effect of **mesh adaptation** with multiresolution?

Again on the 1D viscous Burgers equation: small diffusion

What could be the effect of **mesh adaptation** with multiresolution? Is the adaptive scheme accurate enough to allow, even if the initial mesh is quite coarsened with respect to the finest level $\overline{L}$, to progressively refine the mesh when steep gradients occur.



For singular solutions, a dynamic refinement algorithm is actually needed.

Conclusions

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What has been done (theoretically)

- Analysis based on the **equivalent equations** [Dubois, 2008] for the LBM-MR schemes.
- Find the **maximal order of compliance** of the adaptive scheme with the desired physics, depending on the prediction stencil γ.

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Conclusions (stream)

- **Good agreement** between the empirical behavior and the asymptotic analysis.
    - The Lax-Wendroff scheme [Fakhari *et al.*, 2014]: minimal setting to use most of the LBM schemes. Unpredictable dispersive behaviors: threat to the stability.
    - The Haar scheme $\gamma = 0$ is almost unusable: it modifies the diffusive terms.
    - The LBM-MR scheme for $\gamma \geq 1$: most reliable of the analyzed schemes, both in terms of consistency and stability.
- If the solution is **singular**: adaptive mesh adaptation needed!

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Conclusions (collision) [BONUS - QUESTIONS]

- Our leaves collision is a good choice: accuracy is only marginally affected.
- More refined collision strategy have to be especially needed and carefully optimized.

An interesting question

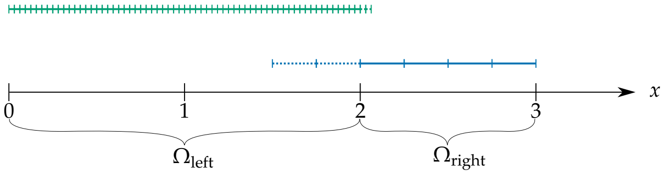
During this presentation, we received an interesting question:

What happens to a **wave passing through a fixed level jump**? Do we expect large spurious reflected waves?

We answer it in

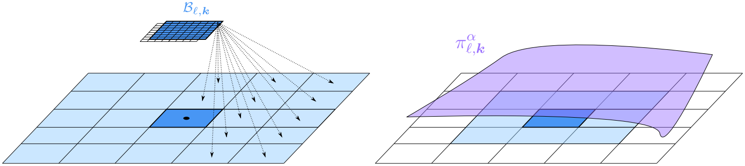
Bellotti, Gouarin, Graille, Massot - *Does the multiresolution lattice Boltzmann method allow to deal with waves passing through mesh jumps?* - Submitted to Comptes Rendus Mathématique - 2021 - `https://arxiv.org/abs/2105.12609` and `https://hal.archives-ouvertes.fr/hal-03235133v1`

The setting looks like:



Thank you for your attention!
Looking forward to receiving your questions!

Alternative collision approaches [BONUS]



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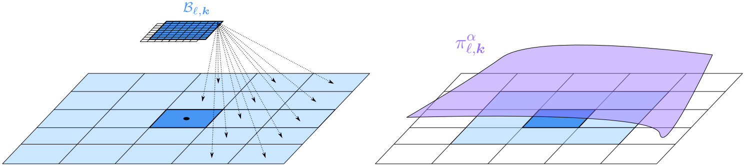
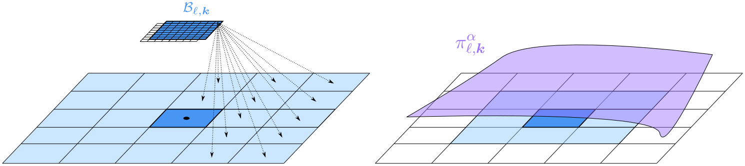


- **Reconstructed collision**

$$\overline{\boldsymbol{f}}^{\star}_{\ell,\boldsymbol{k}}(t) = \boldsymbol{M}^{-1}\left((\boldsymbol{I}-\boldsymbol{S})\overline{\boldsymbol{m}}_{\ell,\boldsymbol{k}}(t) + \frac{\boldsymbol{S}}{2^{d\Delta\ell}}\sum_{\overline{\boldsymbol{k}}\in\mathcal{B}_{\ell,\boldsymbol{k}}}\boldsymbol{m}^{\text{eq}}(\widehat{\overline{m}}^{0}_{L,\overline{\boldsymbol{k}}}(t),\dots)\right).$$



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- **Predict-and-quadrate collision**, following [Hovhannisyan and Müller, 2010]

$$\overline{\boldsymbol{f}}^{\star}_{\ell,\boldsymbol{k}}(t) = \boldsymbol{M}^{-1}\left((\boldsymbol{I}-\boldsymbol{S})\overline{\boldsymbol{m}}_{\ell,\boldsymbol{k}}(t) + \frac{\boldsymbol{S}}{|C_{\ell,\boldsymbol{k}}|} \sum_{i=1}^{N} \tilde{w}_i \boldsymbol{m}^{\text{eq}}(\pi^{0}_{\ell,\boldsymbol{k}}(t,\tilde{\boldsymbol{x}}_i),\dots) \right).$$

1D viscous Burgers equation: large and small diffusion [BONUS]

The stream phase is the LBM-MR scheme for $\gamma = 1$, which has proved to be the most reliable stream phase we analyzed.

	Leaves		Reconstructed		Predict-and-quadrate	
$\Delta\ell_{\text{min}}$	$E^{\overline{L}}_{\text{adap}}$	D_{adap}	$E^{\overline{L}}_{\text{adap}}$	D_{adap}	$E^{\overline{L}}_{\text{adap}}$	D_{adap}
0	1.23e-02	0.00e+00	1.23e-02	0.00e+00	1.23e-02	5.18e-08
1	1.23e-02	1.88e-07	1.23e-02	1.14e-07	1.23e-02	1.27e-07
2	1.23e-02	9.34e-07	1.23e-02	5.70e-07	1.23e-02	5.76e-07
3	1.23e-02	3.89e-06	1.23e-02	2.40e-06	1.23e-02	2.41e-06
4	1.23e-02	1.57e-05	1.23e-02	9.78e-06	1.23e-02	9.79e-06
5	1.23e-02	6.30e-05	1.23e-02	4.06e-05	1.23e-02	4.06e-05
6	1.23e-02	2.60e-04	1.23e-02	1.86e-04	1.23e-02	1.86e-04
7	1.22e-02	1.18e-03	1.23e-02	9.97e-04	1.23e-02	9.98e-04

	Leaves		Reconstructed		Predict-and-quadrate	
$\Delta\ell_{\text{min}}$	$E^{\overline{L}}_{\text{adap}}$	D_{adap}	$E^{\overline{L}}_{\text{adap}}$	D_{adap}	$E^{\overline{L}}_{\text{adap}}$	D_{adap}
0	5.31e-03	0.00e+00	5.31e-03	0.00e+00	5.31e-03	1.19e-06
1	5.31e-03	3.47e-06	5.31e-03	2.79e-06	5.31e-03	3.02e-06
2	5.31e-03	2.34e-05	5.31e-03	2.28e-05	5.31e-03	2.29e-05
3	5.30e-03	1.41e-04	5.28e-03	1.43e-04	5.28e-03	1.43e-04
4	5.31e-03	8.63e-04	5.27e-03	8.93e-04	5.27e-03	8.93e-04
5	6.14e-03	6.08e-03	5.83e-03	5.73e-03	5.84e-03	5.76e-03
6	3.36e-02	3.37e-02	3.11e-02	3.14e-02	3.12e-02	3.15e-02
7	2.45e-01	2.42e-01	2.27e-01	2.23e-01	2.22e-01	2.19e-01