Time-domain Skin Effect Simulation of Tubular Conductors Using Fractional Calculus Techniques

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Abstract

This paper describes an original approach consisting on an application of fractional calculus to the time domain simulation of the skin effect in tubular conductors.

Keywords : Diffusion, Skin effect, Fractional differenciator.

1 Introduction

- This study concerns electromagnetic compatibility simulations. The aim is to calculate propagation of perturbations in wires in time the domain using the SABER simulator to solve the network and line propagation equations. Time domain is more convenient to take into account non-linearity in circuits.
- The difficulty, here, consists in the representation of the skin effect along the wires. This issue has been historically treated in the frequency domain

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[1], [2], [3], where it is easy to find the exact solution of impedance, given by the Bessel functions. Then the time domain impulse response is given by its Inverse Fourier Transform.

- Here, we propose the application of fractional calculus theory to the time domain analysis of the skin effect in tubular conductors by using a validated approximation of the frequency domain expression.

- Without considering propagation, as a first step, the fractional calculus is compared to the convolution of the impulse response of the approximated impedance with the input current. This impulse response needs to be causal: we also show it is possible if frequency domain approximations keep characteristics of the real and imaginar part.

2 Physical model: The diffusion equation and impedance in the frequency domain

- Maxwell's equations inside a good conductor (Ohm's law: $\overrightarrow{j} = \sigma \overrightarrow{E}$) are written, supposing that displacement current is negligible compared to conduction current and that there is no charge accumulation.

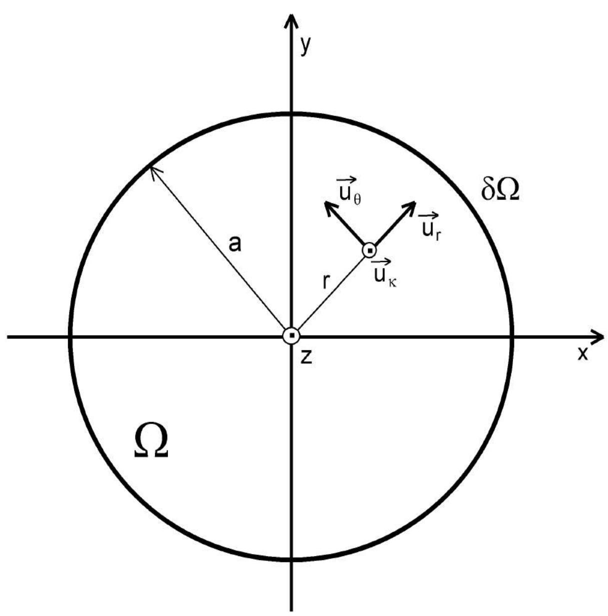

Figure 1. *Cross section of the axisymmetric tubular conductor.*

- For a cylindrical conductor of radius $r = a$ (see Fig. 3), we suppose that the current density has an unique contribution following the axis direction of the conductor. Also, for a cylindrical shape the current is independent of ϕ. Finally we consider no propagation in z so that $\overrightarrow{j} = j_z(r,t)\overrightarrow{u_k}$.

- In order to describe the inner impedance of the wire, we use the diffusion equation for the current density inside the cylindrical conductor given by:

$$\frac{\partial^2 j_z(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial j_z(r,t)}{\partial r} - \sigma\mu_0 \frac{\partial j_z(r,t)}{\partial t} = 0$$

If we perform a Fourier transform over time t the partial differential equation is transformed into an ordinary differential equation:

$$\frac{\mathrm{d}^2 \widehat{j}_z(r,\omega)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\widehat{j}_z(r,\omega)}{\mathrm{d}r} - j\omega\sigma\mu_0 \widehat{j}_z(r,\omega) = 0.$$

The impedance per unit length of the conductor can then be defined as [1], [2],[3],[4]:

$$\widehat{Z} \equiv \frac{\widehat{E}_z(a)}{\widehat{I}} = \frac{\sqrt{-j}}{\sqrt{2}\pi a\sigma\delta} \cdot \frac{J_0(ka)}{J_1(ka)} \tag{1}$$

where J_0 and J_1 are Bessel functions with a complex argument $k = \sqrt{-j\omega\sigma\mu_0}$ and $\delta = \sqrt{\frac{2}{\omega\sigma\mu}}$ is the skin depth.

- Usually, for frequency domain resolution, just the real part of this impedance is used because the imaginary part can be neglected compared to the external inductance of the wire. We will see later that the imaginary part is necessary to preserve causality.

3 Time domain response

- To obtain the time domain evolution of current under skin effect we propose an original approach that has been used in other applications [7], and based on the fractional calculus [6].
- We choose to approximate (1) throughout the whole bandwidth ranging from DC to high frequency with:

$$\widehat{Z}_{app}(\omega) = \frac{1}{\pi a^2\sigma} + \frac{1+j}{2\pi a}\sqrt{\frac{\mu}{2\sigma}}\sqrt{\omega}\,, \qquad \omega \in \ [0,\infty[$$

$$\widehat{Z}_{app}(\omega) = \frac{1}{\pi a^2\sigma} + \frac{1-j}{2\pi a}\sqrt{\frac{\mu}{2\sigma}}\sqrt{|\omega|}\,, \qquad \omega \in \ ]-\infty,0[.$$

Although this expression overestimates the impedance at high frequencies, it's impulse response is real ($\widehat{Z}_{app}(-\omega) = \widehat{Z}^*_{app}(\omega)$) and preserves causality

(Kramers-Kronig relations). The impedance in the two previous equations can be rewritten as:

$$\widehat{Z}_{approx}(\omega) = \frac{1}{\pi a^2 \sigma} + \frac{1}{2\pi a}\sqrt{\frac{\mu}{\sigma}}\sqrt{j\omega}.$$

The time domain response for a causal real function $i(t)$ is then:

$$u(t) = \frac{1}{\pi a^2 \sigma} i(t) + \frac{1}{2\pi a}\sqrt{\frac{\mu}{\sigma}} D^{\frac{1}{2}} i(t) \tag{2}$$

where we define the *fractional derivative* as in [7]:

$$D^{\frac{1}{2}} i(t) \equiv \int_0^t \frac{\mathrm{d}i(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\sqrt{\pi(t-\tau)}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{j\omega}\; \widehat{I}(\omega)\, e^{j\omega t}\, \mathrm{d}\omega.$$

The numerical expression of the fractional derivative can be developed in different ways [7]. We expose here briefly the use of the formal square root of a finite difference operator (Grünwald-Letnikov differintegral).

- Fixing a time step $h > 0$, the values of the $i(t)$ function at regular steps are expressed $i_n = i(nh)$ where $n \in \mathbb{N}$. The finite difference upwind scheme for the first derivative is:

$$\left[\frac{\mathrm{d}i(t)}{\mathrm{d}t}\right]_n \approx \frac{1}{h}(i_n - i_{n-1}) = \left[\frac{1}{h}(I_d - \chi)\, i\right]_k$$

where I_d and δ are identity and delay operators so that $[I_d i]_k = i_k$ and $[\chi i]_k = i_{k-1}$. The definition of the fractional derivative comes then naturally:

$$D^{\frac{1}{2}} \equiv \frac{1}{\sqrt{h}}\sqrt{I_d - \chi} =$$

$$= \frac{1}{\sqrt{h}}\left(1 + \frac{1}{2}(-\chi) + \ldots + \frac{\frac{1}{2}(\frac{1}{2}-1)\ldots(\frac{1}{2}-n+1)}{n!}(-\chi)^n + \ldots\right)$$

$$\left[D^{\frac{1}{2}} i\right]_k = \frac{1}{\sqrt{h}}\sum_{j=0}^{k} \alpha_j i_{k-j}\,.$$

Applying the previous results to (2) it follows:

$$i_k = \frac{1}{\frac{1}{\pi a^2\sigma} + \frac{1}{2\pi a}\sqrt{\frac{\mu}{\sigma}}\frac{1}{\sqrt{h}}\alpha_0}\left(u_k - \frac{1}{2\pi a}\sqrt{\frac{\mu}{\sigma}}\frac{1}{\sqrt{h}}\sum_{j=1}^{k}\alpha_j i_{k-j}\right)\,.$$

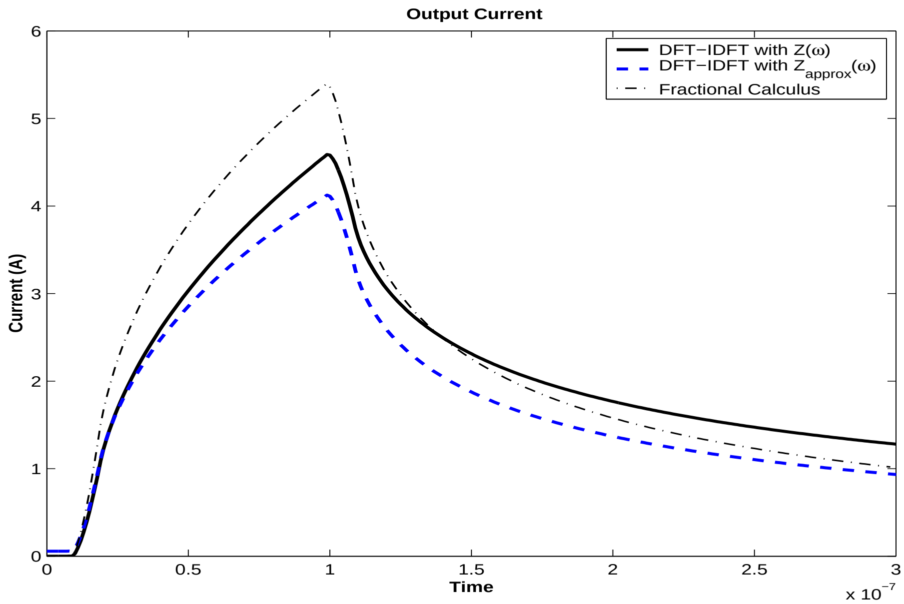

Figure 2. *Current in the wire: comparison with the exact solution, classical approximations and fractional technique.*

4 Simulation Results

• The values correspond to an AWG22 wire, $a = 0.36\ mm$ for the radius of the cylinder, $\sigma = 5.210^7 S/m$ for its electrical conductivity. The excitation signal, $v(t)$, is a pulse of amplitude $A = 1V$ and $t_{rise} = t_{fall} = 1ns$ and pulse width of $80ns$. The time is discretized in regular steps of $h = 1ns$. Output current is calculated at points $i(kh)$ with $k \in \mathbb{N}$ and compared to the the classical DFT plus IDFT of the complete expression of the impedance.

5 Conclusion

In this paper, we demonstrate the application of fractional calculus techniques to the skin effect simulation of wires in the time domain. This numerical method is interesting as all coefficients are calculated *a priori*. Compared to the convolution with the impulse response (numerically computed with Inverse Fourier Transform), the fractional technique also allows to keep an analytic approach in the time domain. It must still be improved in order to minimize the difference between results and it will be integrated into propagation model of wires.

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