Conservation Laws Invariant for Galileo Group; Cemracs Preliminary resuls

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Abstract. We propose a notion of hyperbolic system of conservation laws invariant for the Galileo group of transformations. We show that with natural physical and mathematical hypotheses, such a system conducts to the gas dynamics equations or to exotic systems that are detailed in this contribution to Cemracs 99.

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Contents

1. General framework

Let t and x two real variables. We denote by $\mathbb{R}^2$ the set of line vectors : $(t,\, x) \in \mathbb{R}^2$ and $\mathbb{R}^{2,\,\mathrm{t}} \equiv \mathbb{R} \times^{\mathrm{t}} \mathbb{R}$ the set of column vectors that are the transposed from line vectors :

(1.1) $$\begin{pmatrix} t \\ x \end{pmatrix} \in \mathbb{R}^{2,\,\mathrm{t}} .$$

For $v \in \mathbb{R}$, we denote by y_v the **special Galileo transform** defined by the relations

(1.2) $$y_v \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ x - vt \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

and we introduce also the **space symmetry** q defined by the conditions :

(1.3) $$q \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} t \\ -x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} .$$

Proposition 1. Relation between the generators.
We have the following relations between the operators y_v and q :

(1.4) $$\forall\, v,\, w \in \mathbb{R}, \qquad y_v \bullet y_w = y_{v+w}$$

(1.5) $$q^2 = \mathrm{Id}$$

(1.6) $$\forall\, v \in \mathbb{R}, \qquad y_v \bullet q \bullet y_v = q$$

where Id denotes the identity matrix.

Definition 1. Galileo group.
We denote by $GL_2(\mathbb{R})$ the group of two by two invertible matrices with real coefficients. By definition, the Galileo group G is the subgroup of $GL_2(\mathbb{R})$

generated by the matrices y_v for $v \in \mathbb{R}$, the matrix q and the relations (1.4), (1.5) and (1.6).

Definition 2. Space of states.
Let m be a positive integer. The space of states Ω is an open convex cone included in $\mathbb{R}^{m,\,\mathrm{t}}$:

$$(1.7) \qquad \Omega \subset \mathbb{R}^{m,\,\mathrm{t}} \qquad \text{and} \qquad \forall\, W \in \Omega\,, \quad \forall\, \lambda > 0\,, \quad \lambda\, W \in \Omega\,.$$

Hypothesis 1.
The Galileo group operates linearly on the space of states.
For $g \in G$ and $W \in \Omega$ the action $g \bullet W$ of g on the state W is well defined as a **linear** operation of the group G on the set Ω :

$$(1.8) \qquad \forall\, g \in G, \quad \forall\, W \in \Omega, \qquad g \bullet W \in \Omega$$

$$(1.9) \qquad \forall\, g,\, g' \in G, \quad \forall\, W \in \Omega, \qquad (g\, g') \bullet W \,=\, g \bullet (g' \bullet W)\,.$$

From an algebraic point of view, there exists an m by m set of matrices $Y(v)$ and a matrix R with m lines and m columns such that

$$(1.10) \qquad \forall\, v \in \mathbb{R}, \quad \forall\, W \in \Omega, \qquad Y(v) \bullet W \in \Omega$$

$$(1.11) \qquad \forall\, W \in \Omega, \qquad R \bullet W \in \Omega\,.$$

Then the relations (1.4), (1.5) and (1.6) between the generators and the hypotheses (1.8) and (1.9) can be written in the vocabulary of the m by m matrices $Y(v)$ and R :

$$(1.12) \qquad \forall\, v,\, w \in \mathbb{R}, \qquad Y(v) \bullet Y(w) \,=\, Y(v + w)$$

$$(1.13) \qquad R^2 \,=\, \mathrm{Id}$$

$$(1.14) \qquad \forall\, v \in \mathbb{R}, \qquad Y(v) \bullet R \bullet Y(v) \,=\, R\,.$$

2. Conservation laws

We introduce a regular flux function $\Omega \ni W \longmapsto f(W) \in \mathbb{R}^{m,\,\mathrm{t}}$ and we consider the associated **system of conservation laws** in one space dimension :

$$(2.1) \qquad \frac{\partial}{\partial t} W(t,x) \,+\, \frac{\partial}{\partial x} f(W(t,x)) \,=\, 0$$

where the unknown function $[0, +\infty[\times \mathbb{R} \ni (t,x) \longmapsto W(t,x) \in \Omega$ takes its values inside the convex open cone Ω. We suppose that the system (2.1) of conservation laws admits a **mathematical entropy** (Friedrichs and Lax [FL71]) $\eta :\ \Omega \ni W \longmapsto \eta(W) \in \mathbb{R}$ which is here supposed to be a regular (of $\mathcal{C}^2$ class), **strictly convex** function admitting an associated entropy flux $\xi(\bullet)$. Recall that a pair of (mathematical entropy, entropy flux) allows any regular solution of system (2.1) to satisfy an additional conservation law (Godunov [Go61]) :

$$\frac{\partial}{\partial t}\,\eta(W(t,x)) + \frac{\partial}{\partial x}\big[\,\xi(W(t,x))\big] \,=\, 0 \qquad (2.2)$$

and that a weak entropy solution of system (2.1) is by definition (see e.g. Godlewski-Raviart [GR96]) a weak solution that satisfies also the inequality

$$\frac{\partial}{\partial t}\,\eta(W(t,x)) + \frac{\partial}{\partial x}\big[\,\xi(W(t,x))\big] \,\leq\, 0 \qquad (2.3)$$

in the sense of distributions.

• We introduce the Frechet derivative $\mathrm{d}\eta(W)$ of the entropy and its associated partial derivatives $\Omega \ni W \longmapsto \varphi(W) \in \Phi$ that takes its values in some set $\Phi \subset \mathbb{R}^m$ and defines the so-called **entropy variables** :

$$\forall\, W \in \Omega\,, \quad \forall r \in \mathbb{R}^{m,\,\mathrm{t}}\,, \qquad \mathrm{d}\eta(W) \bullet r \,=\, \varphi(W) \bullet r\,. \qquad (2.4)$$

We consider also the dual function of the entropy : $\Phi \ni \varphi \longmapsto \eta^\star(\varphi) \in \mathbb{R}$ in the sense proposed by Moreau [Mo66], that is

$$\eta^\star(\varphi) \,=\, \sup_{W \in \Omega} \big(\,\varphi \bullet W \,-\, \eta(W)\big)\,. \qquad (2.5)$$

This dual function satisfies

$$\mathrm{d}\eta^\star(\varphi) \,=\, \mathrm{d}\varphi \bullet W(\varphi)\,, \quad \text{where } \varphi \,=\, \mathrm{d}\eta(W(\varphi))\,. \qquad (2.6)$$

Hypothesis 2. Entropy flux and velocity.
There exists a regular (of $\mathcal{C}^1$ class) function $u \,:\, \Omega \ni W \longmapsto u(W) \in \mathbb{R}$ which allows to write the entropy flux $\xi(\bullet)$ associated to the strictly convex entropy $\eta(\bullet)$ under the form

$$\forall\, W \in \Omega\,, \qquad \xi(W) \,=\, \eta(W)\, u(W)\,. \qquad (2.7)$$

• For the study of gas dynamics in Lagrangian coordinates, Després [De98] has developed the general case where the function $\Omega \ni W \longmapsto u(W) \in \mathbb{R}$ is identically null. Hypothesis 2 is very little restrictive because we can define $u(W)$ by the relation $u(W) = \frac{\xi(W)}{\eta(W)}$ as soon as the state W is such that the entropy $\eta(W)$ is not null. We observe also that field $u(\bullet)$ is homogeneous to the ratio of space variable x over time variable t. For this reason, we have introduced in [Du99] the following

Definition 3. Velocity of the state W.
When hypothesis 2 is satisfied, we call velocity of the state W the scalar $u(W)$ defined according to the relation (2.7).

Definition 4. Thermodynamic flux.
When the hypothesis 2 is satisfied, the thermodynamic flux $j : \Omega \ni W \longmapsto j(W) \in \mathbb{R}^{m,\,\mathrm{t}}$ is defined according to the relation

(2.8) $$j(W) \;\equiv\; f(W) \,-\, u(W)\, W \,.$$

The hypothesis 2 is satisfied if and only if we have the following relation between the thermodynamic flux $j(\bullet)$ and the velocity $u(\bullet)$:

(2.9) $$\forall\, W \in \Omega\,, \qquad \varphi(W) \bullet \mathrm{d}j(W) \;+\; \eta^\star(\varphi(W))\; \mathrm{d}u(W) \;=\; 0\,.$$

Proof of Proposition 2.
It is sufficient to express the classical compatibility relation (see e.g. [GR96]) between the entropy flux, the derivative of entropy and the derivative of the flux function, i.e. $\mathrm{d}(\eta\, u) \;\equiv\; \varphi \bullet \mathrm{d}f\,;$ this relation is what is usefull to deduce the relation (2.2) from the original system (2.1) of conservation laws. We derive the relation (2.8). Then we get $\mathrm{d}f(W) \;=\; (\mathrm{d}u(W))\, W \,+ u(W)\, \mathrm{d}W \,+\, \mathrm{d}j(W)$ and

$$\begin{aligned}
\varphi \bullet \mathrm{d}f \;-\; \mathrm{d}(\eta\, u) \;=& \\
=\; & (\mathrm{d}u(W))\, \varphi \bullet W \;+\; u(W)\, \varphi \bullet \mathrm{d}W \;+\; \varphi \bullet \mathrm{d}j(W) \;-\; (\varphi\, u(W) \;+\; \eta\, \mathrm{d}u(W)) \\
=\; & \mathrm{d}u(W) \big(\varphi \bullet W \;-\; \eta(W)\big) \;+\; \varphi \bullet \mathrm{d}j(W) \qquad\qquad \text{because } \ \mathrm{d}W \;=\; \mathrm{Id}\,(\mathbb{R}^{m,\,\mathrm{t}}) \\
=\; & \eta^\star(\varphi)\, \mathrm{d}u(W) \;+\; \varphi \bullet \mathrm{d}j(W)
\end{aligned}$$

and the desired result is established. □

Example. Gas dynamics (i).
For the Euler equations of gas dynamics, we have $m = 3$,

(2.10) $$\Omega \;=\; \Big\{ (\rho\,,\, q\,,\, \epsilon)^{\mathrm{t}} \in \mathbb{R}^{3,\,\mathrm{t}}\,, \quad \rho > 0\,, \quad \epsilon > \frac{q^2}{2\rho} \Big\}\,.$$

Classically, the velocity u and the internal energy e are introduced thanks to the relations

(2.11) $$q \;=\; \rho\, u\,, \quad \epsilon \;=\; \rho\, e \;+\; \frac{1}{2}\, \rho\, u^2$$

and because the pair (mathematical entropy , entropy flux) is of the form

(2.12) $$(\eta\,,\, \xi) \;=\; (\,-\rho\, s\,,\, -\rho\, s\, u\,)$$

where the entropy s is the specific thermostatic entropy (see e.g. [Du90]), the relation (2.7) is satisfied and the definition 3 is perfectly compatible with this classical physical model for gas dynamics. Moreover, the classical relation between the massic internal energy e, the massic entropy s and the specific volume $\tau \;\equiv\; \frac{1}{\rho}$:

(2.13) $\qquad \mathrm{d}e \;=\; T\,\mathrm{d}s \,-\, p\,\mathrm{d}\tau$

allows us to define the temperature T and the thermodynamic static pressure p (see e.g. Callen [Ca85]). Then the Gibbs-Duhem relation associated to extensive fields shows that the specific chemical potential μ can be defined as a function of temperature and of static pressure according to the relation

(2.14) $\qquad \mu \;=\; \mu(T,\,p) \;\equiv\; e \,-\, T\,s \,+\, p\,\tau\,.$

After some lines of easy algebra, it is possible to express the entropy variables for the system of Euler equations for gas dynamics :

(2.15) $\qquad \varphi \;=\; \dfrac{1}{T}\,\big(\,\mu \,-\, \dfrac{1}{2}\,u^2\,,u\,,\,-\,1\,\big)\,.$

It is also clear that the thermodynamic flux $j(\bullet)$ admits the following simple form :

(2.16) $\qquad j(W) \;=\; \big(\,0\;,\;p\;,\;p\,u\,\big)^{\mathrm{t}}\,.$

The explicit evaluation of matrix $Y(v)$ is simple ; we have

(2.17) $$Y(v) \;=\; \begin{pmatrix} 1 & 0 & 0 \\ -v & 1 & 0 \\ \frac{1}{2}\,v^2 & -v & 1 \end{pmatrix}$$

and the orbit of the particular state $W \;=\; \big(\,\rho,\,\rho\,u\,,\,\rho\,e \,+\, \dfrac{1}{2}\,\rho\,u^2\,\big)^{\mathrm{t}} \;\in \Omega$ is a parabola that is expressed by

(2.18) $$G\bullet W \;=\; \Big\{ \big(\,\rho,\,\rho\,(u-v)\,,\,\rho\,e \,+\, \frac{1}{2}\,\rho\,(u-v)^2\,\big)^{\mathrm{t}},\quad v \in \mathbb{R} \Big\}\,.$$

3. Invariance

Hypothesis 3.
The mathematical entropy is invariant under the Galileo group.

(3.1) $\qquad \forall\, g \in G,\quad \forall\, W \in \Omega,\qquad \eta(g\bullet W) \;=\; \eta(W)$

and this hypothesis implies in particular that

(3.2) $\qquad \forall\, v \in \mathbb{R},\quad \forall\, W \in \Omega,\qquad \eta(Y(v)\bullet W) \;=\; \eta(W)$

(3.3) $\qquad \forall\, W \in \Omega,\qquad \eta(R\bullet W) \;=\; \eta(W).$

Definition 5.
Systems of conservation laws invariant for the Galileo group.
The system of conservation laws (2.1) associated with a mathematical entropy η and an entropy flux satisfying hypothesis 2 (relation (2.7)) is said to be invariant

for the Galileo group G of transformations if for each **regular** solution $\mathbb{R}^2 \supset \Xi^{\rm t} \ni (t,\, x) \longmapsto W(t,x) \in \Omega$ of the conservation law (2.1) and each Galilean transformation $g \in G,$ the function $V(\theta,\, \xi)$ defined by the relation

$$(3.4) \qquad \mathbb{R}^2 \supset g(\Xi^{\rm t}) \ni (\theta,\, \xi) \longmapsto V(\theta,\, \xi) \,=\, (g \bullet W)(g^{-1}(\theta,\, \xi)) \in \Omega$$

is also a regular solution of the conservation law (2.1). It is the case in particular when $g \,=\, y_{\,v}$ is a special Galilean transform :

$$(3.5) \quad \left\{ \begin{array}{l} \dfrac{\partial}{\partial \theta} \big[Y(v) \bullet W(\theta,\, \xi + v\theta) \big] + \\ \quad + \dfrac{\partial}{\partial \xi} \Big[u(Y(v) \bullet W(\theta,\, \xi + v\theta) \, (Y(v) \bullet W(\theta,\, \xi + v\theta)) \\ \qquad + j(Y(v) \bullet W(\theta,\, \xi + v\theta)) \Big] \,=\, 0 \end{array} \right.$$

$$(3.6) \quad \left\{ \begin{array}{l} \dfrac{\partial}{\partial \theta} \eta \Big(Y(v) \bullet W(\theta,\, \xi + v\theta) \Big) + \\ \qquad + \dfrac{\partial}{\partial \xi} \Big[u(Y(v) \bullet W(\theta,\, \xi + v\theta)) \, \eta \big(Y(v) \bullet W(\theta,\, \xi + v\theta) \big) \Big] \,=\, 0 \end{array} \right.$$

and when $g \,=\, q$ is the reflection operator :

$$(3.7) \quad \left\{ \begin{array}{l} \dfrac{\partial}{\partial \theta} \big(R \bullet W(\theta,\, -\xi) \big) + \\ \quad + \dfrac{\partial}{\partial \xi} \Big[u(R \bullet W(\theta,\, -\xi)) \;\, R \bullet W(\theta,\, -\xi) \,+\, j \big(R \bullet W(\theta,\, -\xi) \big) \Big] \,=\, 0 \end{array} \right.$$

$$(3.8) \qquad \frac{\partial}{\partial \theta} \eta \Big(R \bullet W(\theta,\, -\xi) \Big) \,+\, \frac{\partial}{\partial \xi} \Big[u(R \bullet W(\theta,\, -\xi)) \;\, \eta \big(R \bullet W(\theta,\, -\xi) \big) \Big] \,=\, 0\,.$$

Proposition 3. Transformation of the velocity field.
We have the following properties for the velocity field associated with a special Galilean transformation and with the space reflection when the hypotheses 1, 2 and 3 are satisfied :

$$(3.9) \qquad \forall\, v \in \mathbb{R}, \quad \forall\, W \in \Omega, \qquad u\big(Y(v) \bullet W \big) \,=\, u(W) - v$$

$$(3.10) \qquad \forall\, W \in \Omega, \qquad u\big(R \bullet W \big) \,=\, -u(W)\,.$$

Proof of Proposition 3.
We first consider the elementary calculus that express the partial derivatives on each side of the Galilean transformation : $\frac{\partial}{\partial \theta} \,=\, \frac{\partial}{\partial t} \,+\, v \, \frac{\partial}{\partial x}$; $\frac{\partial}{\partial \xi} \,=\, \frac{\partial}{\partial x}$ and we remark that the relation (3.2) implies

$$(3.11) \qquad \eta \big(Y(v) \bullet W(\theta,\, \xi + v\theta) \big) \,=\, \eta \big(W(\theta,\, \xi + v\theta) \big)\,.$$

We develop the left hand side of relation (3.6). We get

$$\varphi\Big[\frac{\partial W}{\partial t} + v\,\frac{\partial W}{\partial x}\Big] + \frac{\partial}{\partial \xi}\Big[u(Y(v){\scriptstyle\bullet} W(\theta,\,\xi+v\theta))\ \eta\big(Y(v){\scriptstyle\bullet} W(\theta,\,\xi+v\theta)\big)\Big] =$$

$$= \quad -\frac{\partial}{\partial x}\big[u(W(\theta,\xi+v\theta))\ \eta(W(\theta,\xi+v\theta))\big]$$

$$= \quad +v\,\mathrm{d}\eta(W(\theta,\xi+v\theta)){\scriptstyle\bullet}\frac{\partial W}{\partial x}(\theta,\xi+v\theta) +$$

$$+\frac{\partial}{\partial \xi}\big[u(Y(v)\,{\scriptstyle\bullet} W(\theta,\xi+v\theta))\ \eta(W(\theta,\xi+v\theta))\big] \qquad \text{due to (3.11)}$$

$$= \quad \frac{\partial}{\partial x}\Big\{\Big[-u(W) + v + u(Y(v){\scriptstyle\bullet} W)\Big]\ \eta\big(W(\theta,\,\xi+v\theta)\big)\Big\} = 0\,.$$

This last expression is identically null for any regular solution $W(t,x)$. In consequence the coefficient in front of $\eta(W(\theta,\,\xi+v\theta))$ is null (see e.g. Serre [Se82]) and this fact is exactly expressed by the relation (3.9). In order to prove the relation (3.10), we develop the left hand side of the relation (3.8). We obtain

$$\mathrm{d}\eta(W(\theta,\,-\xi))\,\frac{\partial W}{\partial t}(\theta,\,-\xi) + \frac{\partial}{\partial \xi}\Big[u(R{\scriptstyle\bullet} W(\theta,\,-\xi))\ \ \eta(W(\theta,\,-\xi))\Big] =$$

$$= -\frac{\partial}{\partial x}\Big[u(W(\theta,\,-\xi))\ \eta(W(\theta,\,-\xi))\Big] + \frac{\partial}{\partial \xi}\Big[u(R{\scriptstyle\bullet} W(\theta,\,-\xi))\ \ \eta(W(\theta,\,-\xi))\Big]$$

$$= \frac{\partial}{\partial \xi}\Big\{\Big[u(W(\theta,\,-\xi)) + u(R{\scriptstyle\bullet} W(\theta,\,-\xi))\Big]\ \eta(W(\theta,\,-\xi))\Big\} = 0\,.$$

Then the bracket is null as above and the relation (3.10) is established. □

Proposition 4. Transformation of the thermodynamic flux.
Let (2.1) a system of conservation laws satisfying the hypotheses 1 to 3 and invariant for the Galileo group. Then for a special Galilean transformation y_v and the space reflection q we have

(3.12) $\qquad \forall\, v \in \mathbb{R},\quad \forall\, W\in\Omega,\qquad j\big(Y(v){\scriptstyle\bullet} W\big) = Y(v){\scriptstyle\bullet} j(W)$

(3.13) $\qquad \forall\, W\in\Omega,\qquad j\big(R{\scriptstyle\bullet} W\big) + R{\scriptstyle\bullet} j(W) = 0\,.$

Proof of Proposition 4.
For a special Galilean transformation, we have from the relation (3.5) :

$$Y(v){\scriptstyle\bullet}\Big[\frac{\partial W}{\partial t} + v\frac{\partial W}{\partial x}\Big] + \frac{\partial}{\partial \xi}\Big\{u(Y(v){\scriptstyle\bullet} W))\,(Y(v){\scriptstyle\bullet} W) + j(Y(v){\scriptstyle\bullet} W)\Big\} =$$

$$= -\frac{\partial}{\partial x}\Big\{Y(v){\scriptstyle\bullet}[u(W)\,W + j(W)]\Big\} + \frac{\partial}{\partial x}\Big(v\,Y(v){\scriptstyle\bullet} W\Big) +$$

$$+\frac{\partial}{\partial x}\Big\{u(Y(v){\scriptstyle\bullet} W))\,(Y(v){\scriptstyle\bullet} W) + j(Y(v){\scriptstyle\bullet} W)\Big\}$$

$$= \frac{\partial}{\partial x}\Big\{ \big[-u(W) + v + u(Y(v)\bullet W)\big]\,(Y(v)\bullet W) + $$
$$+ \big[-Y(v)\bullet j(W) + j(Y(v)\bullet W)\big] \Big\}$$
$$= \frac{\partial}{\partial x}\Big\{ \big[-Y(v)\bullet j(W) + j(Y(v)\bullet W)\big] \Big\} \qquad \text{due to (3.9)}$$
$$= 0 \qquad \text{according to the relation (3.5).}$$

Then the relation (3.12) is established ; the end of the proof is obtained by writing that the conservation law (3.7) is satisfied for $R\bullet W$:

$$\frac{\partial}{\partial \theta}\big(R\bullet W\big) + \frac{\partial}{\partial \xi}\big(f(R\bullet W)\big) =$$
$$= \frac{\partial}{\partial t}\big(R\bullet W\big) - \frac{\partial}{\partial x}\Big[u(R\bullet W)\,(R\bullet W) + j(R\bullet W)\Big]$$
$$= R\bullet\frac{\partial W}{\partial t} - \frac{\partial}{\partial x}\Big[-u(W)\;(R\bullet W) + j(R\bullet W)\Big]$$
$$= -R\bullet\frac{\partial}{\partial x}\Big[u(W)\,W + j(W)\Big] + \frac{\partial}{\partial x}\Big[u(W)\,R\bullet W - j(R\bullet W)\Big]$$
$$= -\frac{\partial}{\partial x}\Big[R\bullet j(W) + j(R\bullet W)\Big]$$
$$= 0\,.$$

Then the relation (3.13) is established and the proposition 4 is proven. □

4. Null-velocity manifold

Definition 6. Null-velocity manifold.
We denote by Ω_0 the null-velocity manifold, *i.e.* the set of all states $W \in \Omega$ whose associated velocity $u(W)$ is equal to zero :

(4.1) $\qquad W \in \Omega_0 \quad \text{if and only if} \quad u(W) = 0.$

We remark that the denonination of manifold is appropriate because the function $\Omega \ni W \longmapsto u(W) \in \mathbb{R}$ is regular.

Proposition 5. Decomposition of the space of states.
The open cone Ω is decomposed under the following form

(4.2) $$\Omega = \bigcup_{v \in \mathbb{R}} Y(v)\bullet\Omega_0$$

and the sets $Y(v)\bullet\Omega_0$ have no two by two intersection :

(4.3) $$\forall\, v,\, w \in \mathbb{R}\,, \qquad v \neq w \implies \big(Y(v)\bullet\Omega_0\big) \cap \big(Y(w)\bullet\Omega_0\big) = \emptyset\,.$$

In other words, the cone Ω is a boundle space with basis Ω_0, fiber $G \bullet \Omega_0$ over the manifold Ω_0 and projection Π defined by

$$(4.4) \qquad \Omega \ni W \longmapsto \Pi(W) \,=\, Y(u(W)) \bullet W \;\in \Omega_0 \,.$$

Proof of proposition 5.

The relation (4.2) is a consequence of the following remark :

$$(4.5) \qquad Y\big(u(W)\big) \bullet W \;\in\; \Omega_0$$

that takes into account the relation (3.9) and the definition 6 of Ω_0. In a similar way, $Y(v) \bullet \Omega_0$ is also the set of all the states having a velocity exactly equal to $-v$; then assertion (4.3) is clear. The end of the proposition consists simply in using the vocabulary of topologists. We refer the reader for example to the book of Godbillon [Go71]. $\square$

Proposition 6. The null velocity manifold is of co-dimension 1.

$$(4.6) \qquad \forall\, W \in \Omega_0 \,, \qquad \dim T_W \Omega_0 \,=\, m-1 \,.$$

Proof of Proposition 6.

We start from the relation (3.9) : $u\big(Y(v) \bullet W\big) \,=\, u(W) - v$ and we derive this expression relatively to the variable v. We obtain $\mathrm{d}u\big(Y(v) \bullet W\big) \bullet \big(\mathrm{d}Y(v) \bullet W\big) \,=\, -1\,,$ then we consider the particular value $v = 0$. It comes :

$$(4.7) \qquad \forall\, W \in \Omega \,, \qquad \mathrm{d}u(W) \bullet \big(\mathrm{d}Y(0) \bullet W\big) \,=\, -1 \,.$$

For $W \in \Omega$ and $r \in \mathbb{R}^{m,\,\mathrm{t}}$, we have

$$\mathrm{d}u(W) \bullet \big\{ r \,+\, (\mathrm{d}u(W) \bullet r)\; (\mathrm{d}Y(0) \bullet W) \big\} \,=$$
$$=\, (\mathrm{d}u(W) \bullet r) \,+\, (\mathrm{d}u(W) \bullet r)\,(-1) \,=\, 0$$

Then the vector $r \in \mathbb{R}^{m,\,\mathrm{t}}$ can be decomposed under the form $r \,=\, \rho + \theta\,\big(\mathrm{d}Y(0) \bullet W\big)$ with $\mathrm{d}u(W) \bullet \rho \,=\, 0$. If we suppose now that the state W belongs to the null-velocity manifold Ω_0, the condition $\mathrm{d}u(W) \bullet \rho \,=\, 0$ implies that $\rho \in T_W \Omega_0$. We deduce from this point the following decomposition of space $\mathbb{R}^{m,\,\mathrm{t}}$:

$$(4.8) \qquad \mathbb{R}^{m,\,\mathrm{t}} \,=\, T_W \Omega_0 \,+\, \mathbb{R}\,(\mathrm{d}Y(0) \bullet W) \,, \qquad W \in \Omega_0 \,.$$

The decomposition (4.8) is in fact a direct sum.

If $r \in \big(T_W \Omega_0\big) \cap \big(\mathbb{R}\,(\mathrm{d}Y(0) \bullet W)\big)$, there exists some scalar $\mu \in \mathbb{R}$ such that $r \,=\, \mu\,\mathrm{d}Y(0) \bullet W$. Then, the property that $r \in T_W \Omega_0$ implies $\mathrm{d}u(W) \bullet r \,=\, 0$. Due to the relation (4.7), we deduce that $\mu = 0$ and the vector r is null. Then the property (4.6) is established. $\square$

Hypothesis 4.
Null-velocity manifold is invariant by space reflection.
The null-velocity manifold Ω_0 is supposed to be invariant point by point by space reflection :

(4.9) $$\forall\, W \in \Omega_0 \,, \qquad R \bullet W \;=\; W \,.$$

Definition 7. Decomposition of space.
We introduce the two eigenspaces associated with the reflection operator R :

(4.10) $$\Lambda_1 \;=\; \big\{\, W \in {\rm I\!R}^{m,\,{\rm t}} \,,\; R \bullet W \;=\; W \,\big\}$$

(4.11) $$\Lambda_{-1} \;=\; \big\{\, W \in {\rm I\!R}^{m,\,{\rm t}} \,,\; R \bullet W \;=\; -W \,\big\}\,.$$

An immediate consequence of the property (1.13) is the decomposition

(4.12) $${\rm I\!R}^{m,\,{\rm t}} \;=\; \Lambda_1 \,\oplus\, \Lambda_{-1} \,.$$

Proposition 7. Constraint for the thermodynamic flux.

(4.13) $$\forall\, W \in \Omega_0 \,, \qquad j(W) \in \Lambda_{-1} \,.$$

Proof of Proposition 7.
It is an immediate consequence of the relation (3.13) : $j(R \bullet W) + j(W) \;=\; 0$ and of the hypothesis 4 : $R \bullet W \;=\; W$ when $W \in \Omega_0$. □

Remark 1. Linear geometry.
The hypothesis 4 can also be written as

(4.14) $$\Omega_0 \,\subset\, \Lambda_1$$

and the null-velocity manifold is flat.

Example. Gas dynamics (ii).
In the case of the Euler equations of gas dynamics, we have

(4.15) $$\Lambda_1 \;=\; \big\{\, (\rho \,,\, 0 \,,\, \epsilon)^{\rm t} \,,\, \rho \in {\rm I\!R} \,,\; \epsilon \in {\rm I\!R} \,\big\}$$

(4.16) $$\Lambda_{-1} \;=\; \big\{\, (0 \,,\, \sigma \,,\, 0)^{\rm t} \,,\, \sigma \in {\rm I\!R} \,\big\}\,.$$

We remark also that all the matrices $Y(v)$ have a common eigenvector that generates a linear space Γ_1 of dimension 1 included in Λ_1 :

(4.17) $$\Gamma_1 \;=\; \{\, (0 \,,\, 0 \,,\, \epsilon)^{\rm t} \,,\, \epsilon \in {\rm I\!R} \,\} \;\subset\; \Lambda_1.$$

Moreover, the half manifold $\Gamma_1^+ \;=\; \{\, (0 \,,\, 0 \,,\, \epsilon)^{\rm t} \,,\, \epsilon > 0 \,\}$ is a part of the boundary of Ω_0 which is composed by (unphysical ?) states without any matter $(\rho \;=\; 0)$, undefined velocity $\Big(u = \dfrac{\rho\, u}{\rho} = \dfrac{0}{0} \Big)$ and full of energy $(\epsilon > 0)$!

5. The case $m = 1$

Proposition 8.
It does not exist any hyperbolic equation ($m = 1$) invariant for the Galileo group.

Proof of Proposition 8.
The operator R is linear $\mathbb{R} \longrightarrow \mathbb{R}$ and satisfies $R^2 = \mathrm{Id}$. If $\dim \Lambda_1 = 1$, then $u(R\,W) + + R\,u(W) = 0$ implies that $u(W) \equiv 0$ for all $W \in \Omega$. Then $T_W \Omega_0 = \mathbb{R}$ and this property is in contradiction with the proposition 6. If $\dim \Lambda_{-1} = 1$, *i.e.* $R\,W = -W$, we have from (1.12) and the derivability of the function $\mathbb{R} \ni v \longmapsto Y(v) \,:\, Y(v) = a^v$ for some $a \geq 0$. On the other hand, from relation (1.14) : $Y(v)\,R\,Y(v) = R$ we deduce that $\forall\, v \in \mathbb{R},\ Y(v)^2 \equiv 1$. Then $a = 0$ and $Y(v) \equiv 1$. But this property is in contradiction with the property (3.9) : $u(Y(v)\,W) = u(W) - v$. The proposition is established. □

6. Galilean invariance for systems of two conservation laws

Theorem 1.
When $\Omega \subset \,]0, +\infty[\times^{\rm t} \mathbb{R} \subset \mathbb{R}^{2,\,\rm t}$, a system of two conservation laws invariant for the Galileo group is parameterized by the scalars $\alpha > 0$, $\beta > 0$ and by a derivable strictly convex function $]0, +\infty[\,\ni \xi \longmapsto \sigma(\xi) \in \mathbb{R}$. It takes one of the following forms :

(i) Hyperbolic Galileo.
We have for this first case

(6.1) $\qquad \sigma'(\xi) < 0$

and the space of states Ω is included in the following one :

$$(6.2) \qquad \Omega_+ \;=\; \left\{ W \;=\; \begin{pmatrix} \theta \\ \zeta \end{pmatrix} \in \mathbb{R}^{2,\,\rm t}, \quad \theta > 0\,, \quad |\,\zeta\,| < \sqrt{\frac{\alpha}{\beta}}\; |\,\theta\,| \right\}.$$

The system of conservation laws takes the algebraic form

$$(6.3) \qquad \frac{\partial}{\partial t} \begin{pmatrix} \theta \\ \zeta \end{pmatrix} + \frac{\partial}{\partial x} \left\{ u(W) \begin{pmatrix} \theta \\ \zeta \end{pmatrix} + \frac{\Pi\big(\sqrt{\theta^2 - \beta\,\zeta^2/\alpha}\,\big)}{\sqrt{\theta^2 - \beta\,\zeta^2/\alpha}} \begin{pmatrix} \beta\,\zeta/\alpha \\ \theta \end{pmatrix} \right\} \;=\; 0$$

with a velocity $u(\bullet)$ given by the relation

$$(6.4) \qquad u(W) \;=\; \frac{1}{\sqrt{\alpha\,\beta}}\, \mathrm{argth}\bigg(\sqrt{\frac{\beta}{\alpha}}\, \frac{\zeta}{\theta} \bigg)$$

and a function $\Pi(\bullet)$ named here the **mechanical pressure** and satisfying the relation

(6.5) $$\Pi(\xi) \;=\; -\frac{1}{\beta}\,\frac{\sigma^*(\sigma'(\xi))}{\sigma'(\xi)}$$

if we denote by $\sigma^*(\bullet)$ the dual function of $\sigma(\bullet)$. Moreover, the function $\eta(\bullet)$ defined by

(6.6) $$\eta(\theta,\,\zeta) \;=\; \sigma\bigg(\sqrt{\theta^2 - \frac{\beta\,\zeta^2}{\alpha}}\,\bigg)$$

is a mathematical entropy associated with the hyperbolic system (6.3).

(ii) Elliptic Galileo.

In this second case, we have

(6.7) $$\sigma'(\xi) > 0\,.$$

The elliptic Galileo system of conservation laws admits the expression

(6.8) $$\frac{\partial}{\partial t}\begin{pmatrix}\theta\\ \zeta\end{pmatrix} + \frac{\partial}{\partial x}\left\{ u(W)\,\begin{pmatrix}\theta\\ \zeta\end{pmatrix} + \frac{\Pi\big(\sqrt{\theta^2+\beta\,\zeta^2/\alpha}\,\big)}{\sqrt{\theta^2+\beta\,\zeta^2/\alpha}}\,\begin{pmatrix}-\beta\,\zeta/\alpha\\ \theta\end{pmatrix}\right\} \;=\; 0$$

with a velocity $u(\bullet)$, a mechanical pressure $\Pi(\bullet)$ and a mathematical entropy $\eta(\bullet)$ defined by the relations

(6.9) $$u(W) \;=\; \frac{1}{\sqrt{\alpha\,\beta}}\,\mathrm{arctg}\bigg(\sqrt{\frac{\beta}{\alpha}}\,\frac{\zeta}{\theta}\bigg)$$

(6.10) $$\Pi(\xi) \;=\; \frac{1}{\beta}\,\frac{\sigma^*(\sigma'(\xi))}{\sigma'(\xi)}$$

(6.11) $$\eta(\theta,\,\zeta) \;=\; \sigma\bigg(\sqrt{\theta^2 + \frac{\beta\,\zeta^2}{\alpha}}\,\bigg)\,.$$

Proof of Theorem 1.

• We have $R^2 \;=\; \mathrm{Id}$ in the linear space $\mathbb{R}^{2,\,\mathrm{t}}$. If $\dim\Lambda_1 \;=\; 2$, then $R \;=\; \mathrm{Id}$, $\Lambda_{-1} \;=\; \{0\}$ and due to the relation (4.13), $j(W_0)$ belongs to Λ_{-1} when W_0 lies in Ω_0. Then $j(W_0) \;=\; 0$ if $W_0 \in \Omega_0$. We deduce from (3.12) that $j\big(Y(v)\bullet W_0\big) \;=\; Y(v)\bullet j(W_0)$ for each $W_0 \in \Omega_0$. Then, according to (4.2) and the preceding point, we have $j(W) \;=\; 0$ for each $W \in \Omega$. We deduce from the proposition 2 and the property (2.9) that $\mathrm{d}u \;\equiv\; 0$ and this fact contradicts the relation (4.7). Then $\dim\Lambda_1 \leq 1$. Moreover the unidimensional (due to (4.6)) flat manifold $T_W\Omega_0$ is included in Λ_1. Then $\dim\Lambda_1 \geq 1$ and $\dim\Lambda_1 \;=\; \dim\Lambda_{-1} \;=\; 1$.

• We differentiate the relation (1.14) relatively to the variable v :

$\mathrm{d}Y(v)\bullet R\bullet Y(v) \,+\, Y(v)\bullet R\bullet \mathrm{d}Y(v) \;=\; 0$ and we take $v=0$:

(6.12) $\quad \mathrm{d}Y(0) \bullet R \,+\, R \bullet \mathrm{d}Y(0) \,=\, 0\,.$

For $r \in \Lambda_1$, we have $\mathrm{d}Y(0) \bullet r \,+\, R \bullet \mathrm{d}Y(0) \bullet r \,=\, 0$ and

(6.13) $\quad \forall\, r \in \Lambda_1\,, \qquad \mathrm{d}Y(0) \bullet r \in \Lambda_{-1}\,.$

In a similar way, for $r \in \Lambda_{-1}$, we have $-\mathrm{d}Y(0) \bullet r \,+\, R \bullet \mathrm{d}Y(0) \bullet r \,=\, 0$ and this implies

(6.14) $\quad \forall\, r \in \Lambda_{-1}\,, \qquad \mathrm{d}Y(0) \bullet r \in \Lambda_1\,.$

- Let $(r_+,\, r_-)$ be a basis of the linear space $\mathbb{R}^{2,\,\mathrm{t}}$ composed by a non null vector r_+ of Λ_1 and a non null vector r_- of Λ_{-1}. We define $\alpha \geq 0$ by the condition $\mathrm{d}Y(0) \bullet r_+ \,=\, -\alpha\, r_-$ after an eventual change of the sign of r_-. Due to (6.14), the vector $\mathrm{d}Y(0) \bullet r_-$ belongs to the linear space Λ_1 and can be written under the form : $\mathrm{d}Y(0) \bullet r_- \,=\, \beta\, r_+$. If the scalar α is null, we can express the matrix $Y(v)$ as

(6.15) $$Y(v) \,=\, \mathrm{e}^{v\, \mathrm{d}Y(0)} \,=\, \exp\left[v \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \right] \,=\, \begin{pmatrix} 1 & \beta\, v \\ 0 & 1 \end{pmatrix}\,;$$

and for $W_0 \,=\, (\theta_0,\, 0)^{\mathrm{t}} \,\in\, \Omega_0\,,$ we have $Y(v) \bullet W_0 \,=\, (\theta_0,\, 0)^{\mathrm{t}}$. Due to the relation (4.2), this property implies that Ω is included inside the subspace Λ_1, that contradicts the definition 2 that claims that Ω is an open set of $\mathbb{R}^{2,\,\mathrm{t}}$. Then $\alpha > 0\,.$

- If the scalar β is null, we can express the matrix $Y(v)$ as

(6.16) $$Y(v) \,=\, \mathrm{e}^{v\, \mathrm{d}Y(0)} \,=\, \exp\left[v \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix} \right] \,=\, \begin{pmatrix} 1 & 0 \\ -\alpha\, v & 1 \end{pmatrix}\,.$$

Then for $W_0 \,=\, (\theta_0,\, 0)^{\mathrm{t}} \,\in\, \Omega_0\,,$ we have $Y(v) \bullet W_0 \,=\, (\theta_0,\, -v\,\theta_0)^{\mathrm{t}}$ with $u(Y(v) \bullet W_0) \,=\, -v$ due to the relation (3.9). We deduce the expression $u(W) \,=\, \zeta/\theta$ for the velocity field. Moreover due to the invariance (3.2) of the mathematical entropy, we have the following calculus :

$$\eta(\theta_0,\, -v\,\theta_0) \,=\, \eta(W) \,=\, \eta\big(Y(v) \bullet W_0\big) \,=\, \eta(W_0) \,=\, \eta(\theta_0,\, 0)$$

and the mathematical entropy is function of the **unique** variable θ. In consequence the function $\eta(\bullet,\, \bullet)$ can not be a **strictly** convex function of the pair $(\theta,\, \zeta)$. Due to the general choices done in the section 2, this case must be excluded and the matrix of the operator $\mathrm{d}Y(0)$ has relatively to this basis one among the two following expressions :

(6.17) $$\mathrm{d}Y(0) \,=\, \begin{pmatrix} 0 & -\beta \\ -\alpha & 0 \end{pmatrix}\,, \qquad \alpha > 0\,, \quad \beta > 0\,,$$

(6.18) $$\mathrm{d}Y(0) \,=\, \begin{pmatrix} 0 & \beta \\ -\alpha & 0 \end{pmatrix}\,, \qquad \alpha > 0\,, \quad \beta > 0\,.$$

• **Case (i).** When the operator is defined in the basis $(r_+,\, r_-) \in \Lambda_1 \times \Lambda_{-1}$ with the matrix (6.17), the end of the construction of the system of conservation laws can be done as follows. We first remark that

$$(6.19) \qquad \big(\mathrm{d}Y(0)\big)^2 \;=\; \alpha\,\beta\;\,\mathrm{Id}\,;$$

we have the expansion

$$Y(v) \;=\; \mathrm{e}^{v\,\mathrm{d}Y(0)} \;=\; \mathrm{Id} + v\,\mathrm{d}Y(0) + \frac{v^2}{2!}\,\alpha\,\beta\,\mathrm{Id} + \frac{v^3}{3!}\,\alpha\,\beta\,\mathrm{d}Y(0) + \cdots$$

and the sum of the previous series is equal to

$$(6.20) \qquad Y(v) \;=\; \begin{pmatrix} \mathrm{ch}\big(v\,\sqrt{\alpha\,\beta}\,\big) & -\sqrt{\beta/\alpha}\;\mathrm{sh}\big(v\,\sqrt{\alpha\,\beta}\,\big) \\ -\sqrt{\alpha/\beta}\,\mathrm{sh}\big(v\,\sqrt{\alpha\,\beta}\,\big) & \mathrm{ch}\big(v\,\sqrt{\alpha\,\beta}\,\big) \end{pmatrix}.$$

For a state $W_0 \;=\; (\theta_0,\, 0)^{\mathrm{t}} \in \Omega_0\,,$ we have due to the expression (6.20),

$$W \;=\; Y(v) \bullet W_0 \;=\; \begin{pmatrix} \theta_0\,\mathrm{ch}\big(v\,\sqrt{\alpha\,\beta}\,\big) \\ -\sqrt{\alpha/\beta}\;\theta_0\;\,\mathrm{sh}\big(v\,\sqrt{\alpha\,\beta}\,\big) \end{pmatrix}$$

with $u(W) \;=\; -v$. We deduce the relation (6.4) with the adding condition

$$(6.21) \qquad \theta_0 \;=\; \frac{\theta}{\mathrm{ch}\,\big(\,u(W)\,\sqrt{\alpha\,\beta}\,\big)} \;=\; \sqrt{\theta^2 - \frac{\beta\,\zeta^2}{\alpha}} \qquad \text{if } \; W \;=\; (\theta,\, \zeta)^{\mathrm{t}} \in \Omega_+\,.$$

• We focus now on the mathematical entropy. We first note $\sigma(\bullet)$ the restriction of the mathematical entropy $\eta(\bullet)$ to the subset $\Omega_0 \;=\; \Omega \cap \big(\,]0,\, +\infty[\times^{\mathrm{t}}\{0\}\,\big)$. With the preceding notations, we have necessarily from the hypothesis (3.2) $\eta(W) \;=\; \eta(W_0) \;=\; \sigma(\theta_0)$ that establishes exactly the relation (6.6). A natural question is to verify that the mathematical entropy $\eta(\bullet)$ is a strictly convex function, when $\sigma(\bullet)$ satisfies the same property. We set

$$(6.22) \qquad \xi \;=\; \sqrt{\theta^2 - \frac{\beta\,\zeta^2}{\alpha}}$$

and we have from the relation (6.6) the following calculus :

$$\xi\,\mathrm{d}\xi \;=\; \theta\,\mathrm{d}\theta - \frac{\beta}{\alpha}\,\zeta\,\mathrm{d}\zeta\;, \qquad \frac{\partial\eta}{\partial\theta} \;=\; \frac{\theta}{\xi}\,\sigma'\,, \qquad \frac{\partial\eta}{\partial\zeta} \;=\; -\frac{\beta}{\alpha}\,\frac{\zeta}{\xi}\,\sigma'\,,$$

$$\frac{\partial^2\eta}{\partial\theta^2} \;=\; -\frac{\beta}{\alpha}\,\frac{\zeta^2}{\xi^3}\,\sigma' + \frac{\theta^2}{\xi^2}\,\sigma''\,, \qquad \frac{\partial^2\eta}{\partial\theta\,\partial\zeta} \;=\; \frac{\beta}{\alpha}\,\frac{\theta\,\zeta}{\xi^3}\,\sigma' - \frac{\beta}{\alpha}\,\frac{\theta\,\zeta}{\xi^2}\,\sigma''\,,$$

$$\frac{\partial^2\eta}{\partial\zeta^2} \;=\; -\frac{\beta}{\alpha}\,\frac{\theta^2}{\xi^3}\,\sigma' + \frac{\beta^2}{\alpha^2}\,\frac{\zeta^2}{\xi^2}\,\sigma''. \qquad \text{Then}$$

$$\det\big(\mathrm{d}^2\eta\big) \;=\; \frac{\partial^2\eta}{\partial\theta^2}\,\frac{\partial^2\eta}{\partial\zeta^2} - \Big(\frac{\partial^2\eta}{\partial\theta\,\partial\zeta}\Big)^2 \;=\;$$

$$= \Big(-\frac{\beta}{\alpha}\,\frac{\zeta^2}{\xi^3}\,\sigma' \,+\, \frac{\theta^2}{\xi^2}\,\sigma'' \Big)\,\Big(-\frac{\beta}{\alpha}\,\frac{\theta^2}{\xi^3}\,\sigma' \,+\, \frac{\beta^2}{\alpha^2}\,\frac{\zeta^2}{\xi^2}\,\sigma'' \Big) \,-\, \Big(\frac{\beta}{\alpha}\,\frac{\theta\,\zeta}{\xi^3}\,\sigma' \,-\, \frac{\beta}{\alpha}\,\frac{\theta\,\zeta}{\xi^2}\,\sigma'' \Big)^2$$

$$= \frac{\beta}{\alpha}\,\frac{\sigma'\,\sigma''}{\xi^5}\,\Big(-\theta^4 \,-\, \frac{\beta^2}{\alpha^2}\,\zeta^4 \,+\, 2\,\frac{\beta}{\alpha}\,\theta^2\,\zeta^2 \Big) \;=\; -\frac{\beta}{\alpha}\,\frac{\sigma'\,\sigma''}{\xi} \quad > \; 0$$

when the condition (6.1) is satisfied. Then, joined to the hypothesis $\sigma'' > 0$ the function $\eta(\bullet)$ is scritly convex.

• For $W_0 = (\theta_0,\, 0)^{\rm t} \in \Omega_0$, we have from (4.13) : $j(W_0) \in \Lambda_{-1}$, then we can write it under the form :

(6.23) $\quad \forall\, W_0 \,=\, (\theta_0,\, 0)^{\rm t} \,\in \Omega_0\,, \qquad j(W_0) \;=\; \big(\, 0 \,,\, \Pi(\theta_0) \,\big)^{\rm t} \,\in \Lambda_{-1}\,,$

where the function $]0,\, +\infty[\, \ni \theta_0 \longmapsto \Pi(\theta_0) \in {\rm I\!R}$ remains to be determined. In order to find the algebraic expression of the thermodynamic flux $j(\bullet)$, we derive now the relation (3.12) relatively to the variable v. It comes :

$${\rm d}j(Y(v) \bullet W) \bullet {\rm d}Y(v) \bullet W \;=\; {\rm d}Y(v) \bullet j(W)$$

and taking the particular value $v = 0$:

(6.24) $\quad \forall\, W \in \Omega\,, \qquad {\rm d}j(W) \bullet {\rm d}Y(0) \bullet W \;=\; {\rm d}Y(0) \bullet j(W)\,.$

We apply the compatibility condition (2.9) to the vector ${\rm d}Y(0) \bullet W_0$ with $W_0 \in \Omega_0$. Taking into account the relation (4.7), we get

$$\varphi \bullet {\rm d}Y(0) \bullet j(W_0) \;=\; \varphi \bullet {\rm d}j(W_0) \bullet {\rm d}Y(0) \bullet W_0 \;=\;$$

$$=\; -\eta^*(\varphi)\,{\rm d}u(W_0) \bullet {\rm d}Y(0) \bullet W_0 \;=\; \eta^*(\varphi) \qquad \textit{id est}$$

(6.25) $\quad \forall\, W_0 \in \Omega_0\,, \qquad \varphi \bullet {\rm d}Y(0) \bullet j(W_0) \;=\; \eta^*(\varphi) \qquad$ with $\varphi \,=\, {\rm d}\eta(W_0)\,.$

We have also from the relations (6.6) and (6.22) :

(6.26) $\quad \forall\, W \,=\, (\theta,\, \zeta)^{\rm t} \in \Omega\,, \qquad \varphi \;=\; \dfrac{\sigma'}{\xi}\,\Big(\theta\,,\, -\dfrac{\beta}{\alpha}\,\zeta \Big)$

and if $W \,=\, W_0 \,=\, (\theta_0,\, 0)^{\rm t} \,\in\, \Omega_0$, we have $\varphi \,=\, (\sigma',\, 0)$. We have also the following calculus :

$$\varphi \bullet {\rm d}Y(0) \bullet j(W_0) \;=\; \big(\, \sigma'\,,\, 0 \,\big) \begin{pmatrix} 0 & -\beta \\ -\alpha & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \Pi \end{pmatrix}$$

$$=\; \Big(0\,,\; -\beta\,\sigma' \Big)\, \begin{pmatrix} 0 \\ \Pi \end{pmatrix} \;=\; -\beta\,\sigma'(\theta_0)\,\Pi(\theta_0)\,.$$

We have also $\eta^*(\sigma'(\xi),\, 0) \,=\, \xi\,\sigma'(\xi) \,-\, \eta(\xi) \,=\, \xi\sigma'(\xi) \,-\, \sigma(\xi) \,=\, \sigma^*\big(\sigma'(\xi)\big)$ and due to the relation (6.25) and the preceding development, we have necessarily

(6.27) $\quad \Pi(\theta_0) \;=\; -\dfrac{\sigma^*\big(\sigma'(\theta_0)\big)}{\beta\,\sigma'(\theta_0)}$

in coherence with the relation (6.5). Then the thermodynamic flux function $j(W)$ can be easily deduced from the relation (3.12). We get, due to the condition $v = -u(W)$:

$$j\big(Y(v) \bullet W_0\big) = \begin{pmatrix} \mathrm{ch}\big(u\sqrt{\alpha\,\beta}\big) & \sqrt{\beta/\alpha}\ \mathrm{sh}\big(u\sqrt{\alpha\,\beta}\big) \\ \sqrt{\alpha/\beta}\ \mathrm{sh}\big(u\sqrt{\alpha\,\beta}\big) & \mathrm{ch}\big(u\sqrt{\alpha\,\beta}\big) \end{pmatrix} \begin{pmatrix} 0 \\ \Pi(\theta_0) \end{pmatrix} =$$

$$= \Pi(\theta_0)\ \mathrm{ch}\big(v\sqrt{\alpha\,\beta}\big) \begin{pmatrix} \sqrt{\beta/\alpha}\ \sqrt{\beta/\alpha}\ \zeta/\theta \\ 1 \end{pmatrix} \qquad \text{due to (6.4)}$$

$$= \Pi\Big(\sqrt{\theta^2 - \beta\,\zeta^2/\alpha}\Big)\ \frac{1}{\sqrt{\theta^2 - \frac{\beta\,\zeta^2}{\alpha}}} \begin{pmatrix} \beta\,\zeta/\alpha \\ \theta \end{pmatrix} \qquad \text{thanks to (6.21)}$$

and the expression (6.3) of the hyperbolic system is established in this first case.

• We still have to verify the global coherence of what have been done, *i.e.* that the function $\eta(\bullet)$ introduced at the relation (6.6) is really a mathematical entropy for the system (6.3). We first have $\mathrm{th}(u\sqrt{\alpha\,\beta}) = \sqrt{\beta/\alpha}\ \zeta/\theta$ then by derivation $\frac{1}{\mathrm{ch}^2(u\sqrt{\alpha\,\beta})}\sqrt{\alpha\,\beta}\,\mathrm{d}u = \sqrt{\frac{\beta}{\alpha}}\,\frac{1}{\theta^2}\big(\theta\,\mathrm{d}\zeta - \zeta\,\mathrm{d}\theta\big)$ and due to the relation (6.21) we have the following expression for the derivative of velocity :

$$\alpha\,\mathrm{d}u = \frac{1}{\theta^2 - \beta\,\zeta^2/\alpha}\,\big(\theta\,\mathrm{d}\zeta - \zeta\,\mathrm{d}\theta\big)\,. \tag{6.28}$$

Let $W(t,\,x) \equiv (\theta(t,\,x),\,\zeta(t,\,x))^{\mathrm{t}}$ be a regular solution of the system (6.3). We have, with the notation (6.22) :

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}(\eta\,u) = \Big(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\Big)\eta(W) + \eta\,\frac{\partial u}{\partial x}$$

$$= \frac{\mathrm{d}\sigma}{\mathrm{d}\xi}\Big(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\Big)\xi(W) + \eta\,\frac{\partial u}{\partial x}$$

$$= \sigma'\,\bigg(\frac{\theta}{\xi}\Big(\frac{\partial\theta}{\partial t} + u\,\frac{\partial\theta}{\partial x}\Big) - \frac{\beta}{\alpha}\,\frac{\zeta}{\xi}\Big(\frac{\partial\zeta}{\partial t} + u\,\frac{\partial\zeta}{\partial x}\Big)\bigg) + \eta\,\frac{\partial u}{\partial x}$$

$$= \frac{\sigma'}{\xi}\,\bigg(\theta\Big[-\theta\,\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\Big(\frac{\Pi}{\xi}\,\frac{\beta}{\alpha}\,\zeta\Big)\Big] + \frac{\beta}{\alpha}\,\zeta\,\Big[\zeta\,\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\Big(\frac{\Pi}{\xi}\,\theta\Big)\Big]\bigg) + \eta\,\frac{\partial u}{\partial x} \qquad \text{due to (6.3)}$$

$$= \frac{\sigma'}{\xi}\,\bigg(-\xi^2\,\frac{\partial u}{\partial x} + \frac{\beta}{\alpha}\,\frac{\Pi(\xi)}{\xi}\,\Big[\zeta\,\frac{\partial\theta}{\partial x} - \theta\,\frac{\partial\zeta}{\partial x}\Big]\bigg) + \eta\,\frac{\partial u}{\partial x} \qquad \text{due to (6.22)}$$

$$= \big(-\xi\,\sigma'(\xi) + \eta\big)\,\frac{\partial u}{\partial x} - \beta\,\sigma'(\xi)\ \Pi(\xi)\ \frac{\partial u}{\partial x} \qquad \text{due to (6.28)}$$

$$= \Big(-\xi\,\sigma'(\xi) + \sigma(\xi) + \sigma^*\big(\sigma'(\xi)\big)\Big)\,\frac{\partial u}{\partial x} \qquad \text{due to (6.5) and (6.6)}$$

$= 0$

by definition of the dual of a function in the sense of Moreau. Then any regular solution of the system (6.3) is also solution of the equation (2.2) of conservation of the entropy ; in other terms, the relation (2.9) is identically satisfied. This fact ends the first part of the theorem 1 where has been developed the case of a jacobian matrix $\mathrm{d}Y(0)$ given by the relation (6.17).

• **Case (ii).** When the matrix $\mathrm{d}Y(0)$ is given by the relation (6.18), we have

(6.29) $\quad \big(\mathrm{d}Y(0)\big)^2 \,=\, -\alpha\,\beta\ \mathrm{Id}\,.$

Then

$$Y(v) \,=\, \mathrm{e}^{v\,\mathrm{d}Y(0)}$$
$$= \mathrm{Id} + v\,\mathrm{d}Y(0) - \frac{v^2}{2!}\,\alpha\,\beta\,\mathrm{Id} - \frac{v^3}{3!}\,\alpha\,\beta\,\mathrm{d}Y(0) + \frac{v^4}{4!}\,(\alpha\,\beta)^2\ \mathrm{Id} + \cdots$$
$$= \cos\big(v\,\sqrt{\alpha\,\beta}\,\big)\ \mathrm{Id} + \frac{1}{\sqrt{\alpha\,\beta}}\ \sin\big(v\,\sqrt{\alpha\,\beta}\,\big)\ \mathrm{d}Y(0)\,;$$

(6.30) $$Y(v) \,=\, \begin{pmatrix} \cos\big(v\,\sqrt{\alpha\,\beta}\,\big) & \sqrt{\beta/\alpha}\ \sin\big(v\,\sqrt{\alpha\,\beta}\,\big) \\ -\sqrt{\alpha/\beta}\,\sin\big(v\,\sqrt{\alpha\,\beta}\,\big) & \cos\big(v\,\sqrt{\alpha\,\beta}\,\big) \end{pmatrix}.$$

For $W_0 \,=\, (\theta_0,\,0)^{\mathrm{t}} \in \Omega_0\,,$ we have

$$W \,=\, \begin{pmatrix} \theta \\ \zeta \end{pmatrix} \,=\, Y(v) \bullet W_0 \,=\, \begin{pmatrix} \theta_0\,\cos\big(v\,\sqrt{\alpha\,\beta}\,\big) \\ -\sqrt{\alpha/\beta}\ \theta_0\,\sin\big(v\,\sqrt{\alpha\,\beta}\,\big) \end{pmatrix} \qquad \text{with } u(W) \,=\, -v.$$

We deduce that necessarily the relation (6.9) is compatible with the previous expression and

(6.31) $$\theta_0 \,=\, \frac{\theta}{\cos\big(\,u(W)\,\sqrt{\alpha\,\beta}\,\big)} \,=\, \sqrt{\theta^2 + \frac{\beta\,\zeta^2}{\alpha}} \qquad \text{if } W \,=\, (\theta,\,\zeta)^{\mathrm{t}} \in \Omega\,.$$

• As in the hyperbolic case, the condition $u(W) \,=\, 0$ implies that the second component ζ of the state W is null. Then the necessary condition $\eta(W) \,=\, \sigma(\theta_0)$ conducts to fix the entropy on the null velocity manifold Ω_0 as a strictly convex function of some variable $\xi > 0\,;$ we set :

(6.32) $$\eta(\theta,\,\zeta) \,=\, \sigma(\xi)\ , \qquad \text{with} \quad \xi \,=\, \sqrt{\theta^2 + \frac{\beta\,\zeta^2}{\alpha}}\ .$$

We have again to determine the condition for the convex function $\sigma(\bullet)$ to construct a stricly convex entropy η from the relation (6.32). We have :

$$\xi\,\mathrm{d}\xi \,=\, \theta\,\mathrm{d}\theta + \frac{\beta}{\alpha}\,\zeta\,\mathrm{d}\zeta\ , \qquad \frac{\partial\eta}{\partial\theta} \,=\, \frac{\theta}{\xi}\,\sigma'\,, \qquad \frac{\partial\eta}{\partial\zeta} \,=\, \frac{\beta}{\alpha}\,\frac{\zeta}{\xi}\,\sigma'\,,$$

$$\frac{\partial^2\eta}{\partial\theta^2} = \frac{\beta}{\alpha}\,\frac{\zeta^2}{\xi^3}\,\sigma' + \frac{\theta^2}{\xi^2}\,\sigma'' , \qquad \frac{\partial^2\eta}{\partial\theta\,\partial\zeta} = -\frac{\beta}{\alpha}\,\frac{\theta\,\zeta}{\xi^3}\,\sigma' + \frac{\beta}{\alpha}\,\frac{\theta\,\zeta}{\xi^2}\,\sigma'' ,$$

$$\frac{\partial^2\eta}{\partial\zeta^2} = \frac{\beta}{\alpha}\,\frac{\theta^2}{\xi^3}\,\sigma' + \frac{\beta^2}{\alpha^2}\,\frac{\zeta^2}{\xi^2}\,\sigma'' . \qquad \text{Then}$$

$$\begin{aligned} \det\big(\mathrm{d}^2\eta\big) &= \frac{\partial^2\eta}{\partial\theta^2}\,\frac{\partial^2\eta}{\partial\zeta^2} - \Big(\frac{\partial^2\eta}{\partial\theta\,\partial\zeta}\Big)^2 \\ &= \Big(\frac{\beta}{\alpha}\,\frac{\zeta^2}{\xi^3}\,\sigma' + \frac{\theta^2}{\xi^2}\,\sigma''\Big)\Big(\frac{\beta}{\alpha}\,\frac{\theta^2}{\xi^3}\,\sigma' + \frac{\beta^2}{\alpha^2}\,\frac{\zeta^2}{\xi^2}\,\sigma''\Big) - \Big(-\frac{\beta}{\alpha}\,\frac{\theta\,\zeta}{\xi^3}\,\sigma' + \frac{\beta}{\alpha}\,\frac{\theta\,\zeta}{\xi^2}\,\sigma''\Big)^2 \\ &= \frac{\beta}{\alpha}\,\frac{\sigma'\,\sigma''}{\xi^5}\,\Big(\theta^4 + \frac{\beta^2}{\alpha^2}\,\zeta^4 + 2\,\frac{\beta}{\alpha}\,\theta^2\,\zeta^2\Big) = \frac{\beta}{\alpha}\,\frac{\sigma'\,\sigma''}{\xi} \quad > 0 \end{aligned}$$

when the condition $\sigma'(\xi) > 0$ of relation (6.7) is true. In these conditions, we have also $\frac{\partial^2\eta}{\partial\theta^2} > 0$, and we have established that the function $\eta(\bullet)$ is a scritly convex function of the pair $(\theta,\,\zeta)$.

• We search now the thermodynamic flux $j(W_0)$ on the form (6.23). The determination of the entropy variables is easy :

(6.33) $\quad \forall\, W = (\theta,\,\zeta)^{\rm t} \in \Omega\,, \qquad \varphi = \dfrac{\sigma'}{\xi}\,\Big(\theta\,,\,\dfrac{\beta}{\alpha}\,\zeta\Big)$

and $\varphi(W_0) = (\sigma',\,0)$ when $W_0 \in \Omega_0$. We have now

$$\varphi \bullet \mathrm{d}Y(0) \bullet j(W_0) = \big(\sigma'\,,\,0\big)\begin{pmatrix} 0 & \beta \\ -\alpha & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \Pi \end{pmatrix} = \big(0\,,\,-\beta\,\sigma'\big)\begin{pmatrix} 0 \\ \Pi \end{pmatrix}$$
$$= \beta\,\sigma'(\theta_0)\,\Pi(\theta_0)$$

and the relation (6.10) is established due to (6.25) which remains true for this second case. The end of the determination of the thermodynamic flux $j(W)$ for an arbitrary state W is easy, taking into account the relation (3.12) and the expression (6.30) :

$$\begin{aligned} j(W) &= Y(-u(W)) \bullet j\big(Y(u(W)) \bullet W\big) = Y(-u(W)) \bullet j(W_0) \\ &= \begin{pmatrix} \cos\big(u\,\sqrt{\alpha\,\beta}\,\big) & -\sqrt{\beta/\alpha}\,\sin\big(u\,\sqrt{\alpha\,\beta}\,\big) \\ \sqrt{\alpha/\beta}\,\sin\big(u\,\sqrt{\alpha\,\beta}\,\big) & \cos\big(u\,\sqrt{\alpha\,\beta}\,\big) \end{pmatrix}\begin{pmatrix} 0 \\ \Pi(\theta_0) \end{pmatrix} \\ &= \Pi(\theta_0)\,\cos\big(u\,\sqrt{\alpha\,\beta}\,\big)\begin{pmatrix} -\sqrt{\beta/\alpha}\,\sqrt{\beta/\alpha}\,\,\zeta\,/\theta \\ 1 \end{pmatrix} \\ &= \Pi\Big(\sqrt{\theta^2 + \beta\,\zeta^2/\alpha}\,\Big)\,\frac{1}{\sqrt{\theta^2 + \frac{\beta\,\zeta^2}{\alpha}}}\begin{pmatrix} -\beta\,\zeta/\alpha \\ \theta \end{pmatrix} \qquad \text{thanks to (6.31)} \end{aligned}$$

and the algebraic expression (6.8) of the hyperbolic system is established.

• As in the first case, we must verify that the candidate $\eta(\bullet)$ for a mathematical entropy is in fact a correct one, *i.e.* that the quantity $\eta(W)$ is advected with the velocity $u(W)$ for the regular solutions of the equation (6.8), with the entropy flux $u(W)\,\eta(W)$. Taking into account the relation (6.9), we have $\mathrm{tg}(u\sqrt{\alpha\,\beta}) = \sqrt{\beta/\alpha}\;\zeta/\theta$ then $\frac{1}{\cos^2(u\sqrt{\alpha\,\beta})}\sqrt{\alpha\,\beta}\,\mathrm{d}u = \sqrt{\frac{\beta}{\alpha}}\,\frac{1}{\theta^2}\,\big(\theta\,\mathrm{d}\zeta - \zeta\,\mathrm{d}\theta\big)$ and due to the relation (6.31) we have the following expression for the derivative of velocity :

$$\alpha\,\mathrm{d}u \;=\; \frac{1}{\theta^2 + \beta\,\zeta^2/\alpha}\,\big(\theta\,\mathrm{d}\zeta - \zeta\,\mathrm{d}\theta\big)\,. \tag{6.34}$$

Let $W(t,\,x) \equiv (\theta(t,\,x),\,\zeta(t,\,x))^{\mathrm{t}}$ be a regular solution of system (6.8). We have, with the notation (6.32) :

$$\begin{aligned}
\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}(\eta\,u) \;&=\; \Big(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\Big)\eta(W) + \eta\,\frac{\partial u}{\partial x}\\
&= \frac{\mathrm{d}\sigma}{\mathrm{d}\xi}\Big(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\Big)\xi(W) + \eta\,\frac{\partial u}{\partial x}\\
&= \sigma'\,\bigg(\frac{\theta}{\xi}\Big(\frac{\partial\theta}{\partial t} + u\,\frac{\partial\theta}{\partial x}\Big) + \frac{\beta}{\alpha}\,\frac{\zeta}{\xi}\Big(\frac{\partial\zeta}{\partial t} + u\,\frac{\partial\zeta}{\partial x}\Big)\bigg) + \eta\,\frac{\partial u}{\partial x}\\
&= \frac{\sigma'}{\xi}\,\bigg(\theta\Big[-\theta\,\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\Big(\frac{\Pi}{\xi}\,\frac{\beta}{\alpha}\,\zeta\Big)\Big] + \frac{\beta}{\alpha}\,\zeta\,\Big[-\zeta\,\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\Big(\frac{\Pi}{\xi}\,\theta\Big)\Big]\bigg) + \eta\,\frac{\partial u}{\partial x} && \text{due to (6.8)}\\
&= \frac{\sigma'}{\xi}\,\bigg(-\xi^2\,\frac{\partial u}{\partial x} + \frac{\beta}{\alpha}\,\frac{\Pi(\xi)}{\xi}\,\Big[-\zeta\,\frac{\partial\theta}{\partial x} + \theta\,\frac{\partial\zeta}{\partial x}\Big]\bigg) + \eta\,\frac{\partial u}{\partial x} && \text{due to (6.32)}\\
&= \big(-\xi\,\sigma'(\xi) + \eta\big)\,\frac{\partial u}{\partial x} + \beta\,\sigma'(\xi)\;\Pi(\xi)\;\frac{\partial u}{\partial x} && \text{due to (6.34)}\\
&= \Big(-\xi\,\sigma'(\xi) + \sigma(\xi) + \sigma^*\big(\sigma'(\xi)\big)\Big)\,\frac{\partial u}{\partial x} && \text{due to (6.10) and (6.11)}\\
&= 0\,.
\end{aligned}$$

The proof of Theorem 1 is completed. □

Remark 2. About the $p-$system.
Following a remark proposed by F. Coquel in march 2000, we can fix $\beta > 0$ and take the limit $\alpha \longrightarrow +\infty$ for the systems found at the theorem 1. Then the velocities $u(W)$ defined in (6.4) and (6.9) tend to zero and both systems of conservation laws (6.3) and (6.8) admit the following formal limit

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta\\ \zeta\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}0\\ \Pi(\theta)\end{pmatrix} \;=\; 0\,; \tag{6.35}$$

we observe here that this limit system is **not** the p-system that takes the classical form (see *e.g.* [GR96]) :

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta\\ \zeta\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}-\zeta\\ p(\theta)\end{pmatrix} = 0\,. \tag{6.36}$$

Remark 3. Preliminaries.
In all fairness, the elliptic Galileo version of two by two Galileo group preserving systems of conservation laws satisfies the relation (3.9) $u\big(Y(v)\bullet W\big) = = u(W) - v$ only for sufficiently small velocities v. It is clear from the relation (6.9) that the velocity $u(W)$ should be defined *modulo* some additive constant (equal to $\pi/\sqrt{\alpha\,\beta}$) because only the expression $\mathrm{tg}\big(\sqrt{\alpha\,\beta}\,u(W)\big)$ is well defined. Then the hypothesis 2 should be adapted to systems of conservation laws whose velocity belongs to some quotient group of the type $\mathbb{R}/(\mu\,\mathbb{Z})$. These developments have not been realized at this moment [november 2000], and this fact explains the word "preliminary" in the title of our contribution. This remark is also to be done for the three by three "elliptic Galileo" system developed in the next section.

7. Galilean invariance for systems of three conservation laws

Theorem 2.
When $\Omega \subset\,]0, +\infty[\, \times^{\mathrm{t}} \mathbb{R}^{2,\,\mathrm{t}} \subset \mathbb{R}^{3,\,\mathrm{t}}$, a system of three conservation laws invariant for the Galileo group has one of the three following types : hyperbolic Galileo, elliptic Galileo or nilpotent Galileo. The hyperbolic system is parameterized by the scalars $a>0$, $b>0$ and by a derivable strictly convex function $]0, +\infty[\, \times \mathbb{R} \ni (\alpha,\, \beta) \longmapsto \sigma(\alpha,\, \beta) \in \mathbb{R}$; we denote by $\sigma^*(\bullet)$ the dual function of $\sigma(\bullet)$.

(i) Hyperbolic Galileo.
We have for this first case

$$\forall\, \alpha > 0\,, \quad \forall\, \beta \in \mathbb{R}\,, \qquad \frac{\partial \sigma}{\partial \alpha}(\alpha,\, \beta) < 0 \tag{7.1}$$

and the space of states Ω is included in the cone Ω_+ defined by

$$\Omega_+ \;=\; \Big\{ W \;=\; \big(\theta,\, \zeta,\, \psi\big)^{\mathrm{t}} \in \mathbb{R}^{3,\,\mathrm{t}}, \quad \theta > 0\,, \quad |\zeta| < \sqrt{\frac{a}{b}}\; |\theta| \Big\}. \tag{7.2}$$

The system of conservation laws takes the algebraic form

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta\\ \zeta\\ \psi\end{pmatrix} + \frac{\partial}{\partial x}\left\{ u(W)\begin{pmatrix}\theta\\ \zeta\\ \psi\end{pmatrix} + \frac{\Pi\big(\sqrt{\theta^2 - b\,\zeta^2/a}\,,\, \psi\big)}{\sqrt{\theta^2 - b\,\zeta^2/a}}\begin{pmatrix} b\,\zeta/a\\ \theta\\ 0\end{pmatrix}\right\} = 0 \tag{7.3}$$

with a velocity $u(\bullet)$ given by the relation

$$(7.4) \qquad u(W) \,=\, \frac{1}{\sqrt{a\,b}} \, \text{argth}\Big(\sqrt{\frac{b}{a}} \, \frac{\zeta}{\theta} \Big)$$

and a so-called mechanical pressure function $\Pi(\bullet)$ satisfying

$$(7.5) \qquad \Pi(\xi,\, \beta) \,=\, -\frac{1}{b} \, \frac{\sigma^*(A,\, B)}{A} \,, \qquad \text{with } (A,\, B) \,=\, \mathrm{d}\sigma(\xi,\, \beta)\,.$$

Moreover, the function $\eta(\bullet)$ defined by

$$(7.6) \qquad \eta(\theta,\, \zeta,\, \psi) \,=\, \sigma\Big(\sqrt{\theta^2 - \frac{b\,\zeta^2}{a}} \,,\, \psi \Big)$$

is a mathematical entropy associated with the hyperbolic system (7.3).

(ii) Elliptic Galileo.

We have in this second case

$$(7.7) \qquad \forall\, \alpha > 0\,, \quad \forall\, \beta \in \mathbb{R}\,, \qquad \frac{\partial \sigma}{\partial \alpha}(\alpha,\, \beta) > 0\,.$$

The elliptic Galileo system of conservation laws admits the expression

$$(7.8) \quad \frac{\partial}{\partial t} \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} + \frac{\partial}{\partial x} \Bigg\{ u(W) \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} + \frac{\Pi\big(\sqrt{\theta^2 + b\,\zeta^2/a}\,,\, \psi \big)}{\sqrt{\theta^2 + b\,\zeta^2/a}} \begin{pmatrix} -b\,\zeta/a \\ \theta \\ 0 \end{pmatrix} \Bigg\} \,=\, 0$$

with a velocity $u(\bullet)$, a function $\Pi(\bullet)$ and a mathematical entropy $\eta(\bullet)$ defined by the relations

$$(7.9) \qquad u(W) \,=\, \frac{1}{\sqrt{a\,b}} \, \text{arctg}\Big(\sqrt{\frac{b}{a}} \, \frac{\zeta}{\theta} \Big)$$

$$(7.10) \qquad \Pi(\xi,\, \beta) \,=\, \frac{1}{b} \, \frac{\sigma^*(A,\, B)}{A} \,, \qquad \text{with } (A,\, B) \,=\, \mathrm{d}\sigma(\xi,\, \beta)$$

$$(7.11) \qquad \eta(\theta,\, \zeta,\, \psi) \,=\, \sigma\Big(\sqrt{\theta^2 + \frac{b\,\zeta^2}{a}} \,,\, \psi \Big)\,.$$

(iii) Nilpotent Galileo.

For this third case, we suppose

$$(7.12) \qquad \forall\, \alpha > 0\,, \quad \forall\, \beta \in \mathbb{R}\,, \qquad \frac{\partial \sigma}{\partial \beta}(\alpha,\, \beta) < 0\,.$$

The nilpotent Galileo system of conservation laws takes the algebraic form

$$(7.13) \qquad \frac{\partial}{\partial t} \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} + \frac{\partial}{\partial x} \Bigg\{ u(W) \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} + \Pi\Big(\theta\,,\, \psi - \frac{b}{2\,a} \, \frac{\zeta^2}{\theta} \Big) \begin{pmatrix} 0 \\ 1 \\ \frac{b}{a}\,\frac{\zeta}{\theta} \end{pmatrix} \Bigg\} \,=\, 0$$

with a velocity field $u(\bullet)$ given by the relation

$$(7.14) \qquad u(W) \,=\, \frac{\zeta}{a\,\theta}$$

and a mechanical pressure $\Pi(\bullet)$ satisfying

$$(7.15) \qquad \Pi(\alpha,\,\beta) \,=\, -\frac{1}{b}\,\frac{\sigma^*(A,\,B)}{B}\,, \qquad \text{with } (A,\,B) \,=\, \mathrm{d}\sigma(\alpha,\,\beta)$$

or in an equivalent way

$$(7.16) \qquad \Pi(\alpha,\,\beta) \,=\, -\frac{1}{b}\,\frac{\sigma^*\Big(\dfrac{\partial\sigma}{\partial\alpha}(\alpha,\,\beta)\,,\,\dfrac{\partial\sigma}{\partial\beta}(\alpha,\,\beta)\Big)}{\dfrac{\partial\sigma}{\partial\beta}(\alpha,\,\beta)}\,.$$

Moreover, the function $\eta(\bullet)$ defined by

$$(7.17) \qquad \eta(\theta,\,\zeta,\,\psi) \,=\, \sigma\Big(\theta\,,\,\psi - \frac{b}{2\,a}\,\frac{\zeta^2}{\theta}\Big)$$

is a mathematical entropy associated with the hyperbolic system (7.13).

Remark 4. Pressure and duality.
The case of gas dynamics corresponds to $\theta = \rho$, $\psi = \rho\,E$, $a = b = 1$ for the nilpotent Galileo system of conservation laws. The entropy $\sigma(\bullet)$ at null velocity is defined from the classical thermostatic entropy function $\Sigma(\bullet)$ which is concave and homogeneous of degree one relatively to the extensive physical variables of mass M, volume V and internal energy $\mathcal{E}$ [Du90]. Introducing the intensive thermostatic variables of temperature T, thermodynamic pressure p and massic chemical potential μ, we have the classical fundamental relation of thermostatics (see *e.g.* Callen [Ca85]),

$$(7.18) \qquad \mathrm{d}\mathcal{E} \,=\, T\,\mathrm{d}\Sigma(M,\,V,\,\mathcal{E}) \,-\, p\,\mathrm{d}V \,+\, \mu\mathrm{d}M$$

and taking into account the Euler relation for homogeneity of degree one

$$(7.19) \qquad \mathcal{E} \,\equiv\, T\,\Sigma \,-\, p\,V \,+\, \mu\,M\,,$$

we have necessarily :

$$(7.20) \qquad \eta\big(\rho\,,\,0\,,\,\psi\big) \,=\, -\Sigma\big(\rho\,,\,1\,,\,\psi\big) \,\equiv\, \sigma(\rho,\,\psi)\,.$$

We take $V = 1$, $M = \rho$ and $\mathcal{E} = \psi$; we deduce from the relations (7.19) and (7.20) the identity :

$$(7.21) \qquad \sigma \,=\, \frac{\mu}{T}\,\rho \,-\, \frac{1}{T}\,\psi \,-\, \frac{p}{T}$$

and by application of the relation (7.18) in the particular case $V \equiv 1$, we deduce :

(7.22) $\quad \mathrm{d}\sigma \;=\; \dfrac{\mu}{T}\,\mathrm{d}\rho - \dfrac{1}{T}\,\mathrm{d}\psi\,.$

Both identities (7.21) and (7.22) establish that we have

(7.23) $\quad \sigma^* \Big(\dfrac{\mu}{T} \,,\, -\dfrac{1}{T} \Big) \;=\; \dfrac{p}{T}\,.$

In this context, the relation (7.12) can be written as $\frac{1}{T} > 0$ and the following calculus, issued from the relation (7.16) :

$$\Pi(\rho,\,\psi) \;=\; -\frac{\sigma^*}{(-1/T)} \;=\; \frac{p/T}{(1/T)} \;=\; p\Big(\frac{\mu}{T} \,,\, -\frac{1}{T} \Big)$$

gives an **intrinsic** definition of the mechanical pressure $\Pi(\bullet)$ as identical to the thermodynamic pressure $p(\bullet)$ with the help of the relation (7.23) in terms of the **dual** of the thermostatic specific entropy. We remark also that, in some sense, the theorem 2 establishes theoretically the Euler equations of gas dynamics.

Proof of Theorem 2.

- We derive the relation (1.14) relatively to the velocity v :

$\mathrm{d}Y(v) \bullet R \bullet Y(v) \,+\, Y(v) \bullet R \bullet \mathrm{d}Y(v) \;=\; 0$

and we consider the particular case $v = 0$. It comes :

(7.24) $\quad \mathrm{d}Y(0) \bullet R \,+\, R \bullet \mathrm{d}Y(0) \;=\; 0\,.$

Then the range of the linear space Λ_1 by the operator $\mathrm{d}Y(0)$ is included inside the eigenspace Λ_{-1} and the range of the linear space Λ_{-1} by the same mapping is included inside the eigenspace Λ_{-1}. Moreover, the kernel of $\mathrm{d}Y(0)$ is stable under the action of the symmetry operator R :

(7.25) $\quad \mathrm{d}Y(0)\,(\Lambda_1) \;\subset\; \Lambda_{-1}$

(7.26) $\quad \mathrm{d}Y(0)\,(\Lambda_{-1}) \;\subset\; \Lambda_1$

(7.27) $\quad R\big(\ker \mathrm{d}Y(0)\big) \;\subset\; \ker \mathrm{d}Y(0)\,.$

We deduce from the relation (4.6) ($\dim T_W \Omega_0 = 2$ in our case) and from the relation (4.14) ($\Omega_0 \subset \Lambda_1$) that we have necessarily :

(7.28) $\quad \dim \Lambda_1 \;\geq\; 2\,, \qquad T_W \Omega_0 \,\subset\, \Lambda_1\,.$

- The case where $\dim \Lambda_1 \;=\; 3$ is not possible. Indeed we would have $\Lambda_{-1} \;=\; \{0\}$ and the relation (4.13) establishes that in this case the thermodynamic flux $j(\bullet)$ is identically null on the manifold Ω_0 then on the entire space Ω. Moreover, the relation (3.10) joined with the relation $R = \mathrm{Id}$ shows that $u(W) \;=\; 0$ for each $W \in \Omega$. This fact contradicts the relation (4.6) that claims that $\dim T_W \Omega_0 = 2$. Taking into account the relation (7.28), we have established that we have necessarily

(7.29) $\quad \dim \Lambda_1 \,=\, 2\,, \qquad \dim \Lambda_{-1} \,=\, 1\,.$

If $\dim \ker \mathrm{d}Y(0) \,=\, 3\,,$ then $Y(v) \equiv \mathrm{Id}$ for each real parameter v and the Galileo group does not operate anymore on the cone Ω. In particular, we have a contradiction with the property (3.9) : $u(Y(v)\bullet)\,W) = u(W) - v$.

• If $\dim \ker \mathrm{d}Y(0) \,=\, 2\,,$ due to the stability property (7.27), we are necessarily in one of the two following cases :

(7.30) $\quad \ker \mathrm{d}Y(0) \,=\, \Lambda_1$

(7.31) $\quad \exists\, r_0 \in \Lambda_1,\, r_0 \neq 0,\, \exists\, r_- \in \Lambda_1\,,\, r_- \neq 0,\, \ker \mathrm{d}Y(0) \,=\, \mathrm{span} \,<r_0\,,\, r_->\,.$

If the relation (7.30) holds, there exists a basis $(r_+\,,\, r_0)$ of the linear subspace Λ_1 with $\mathrm{d}Y(0)\bullet r_+ \,=\, \mathrm{d}Y(0)\bullet r_0 \,=\, 0\,,$ a non null vector $r_- \in \Lambda_{-1}$ and a scalar $a > 0$ such that inside the basis $(r_+\,,\, r_-\,,\, r_0)\,,$ the operator $\mathrm{d}Y(0)$ admits the following expression :

(7.32) $$\mathrm{d}Y(0) \,=\, \begin{pmatrix} 0 & -a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{in the basis } (r_+\,,\, r_-\,,\, r_0)\,.$$

Then $\quad Y(v) = \begin{pmatrix} 1 & -a\,v & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and for $W_0 = \begin{pmatrix} \theta_0 \\ 0 \\ \psi_0 \end{pmatrix} \in \Omega_0,$

we have $\; Y(v)\bullet W_0 = \begin{pmatrix} \theta_0 \\ 0 \\ \psi_0 \end{pmatrix}$

that belongs always in the null manifold Ω_0 and we have a contradiction exactly as in the preceding point. If the relation (7.31) is active, there exists a scalar $a > 0$ such that inside a basis $(r_+\,,\, r_-\,,\, r_0)$ consructed as previously, we have, taking into account the relation (7.25) :

(7.33) $$\mathrm{d}Y(0) \,=\, \begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{in the basis } (r_+\,,\, r_-\,,\, r_0)\,.$$

Then $\quad Y(v) \,=\, \begin{pmatrix} 1 & 0 & 0 \\ -a\,v & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and for $W_0 \,=\, \begin{pmatrix} \theta_0 \\ 0 \\ \psi_0 \end{pmatrix} \in \Omega_0\,,$

$$Y(v)\bullet W_0 \,=\, \begin{pmatrix} \theta_0 \\ -a\,v\,\theta_0 \\ \psi_0 \end{pmatrix} \in \Omega\,.$$

We deduce from the relation (3.9) that that $u(\theta,\, \zeta,\, \psi) \equiv \zeta/(a\,\theta)$ and for $W = (\theta,\, \zeta,\, \psi)^{\rm t} \in \Omega$, we have necessarily $\eta(\theta,\, \zeta,\, \psi) \equiv \sigma(\theta,\, \psi)$ where $\sigma(\bullet)$ is the restriction of the entropy $\eta(\bullet)$ to the null manifold Ω_0. Then the mathematical entropy $\eta(\bullet)$ cannot be a **strictly** convex function and this case has to be excluded. In consequence we have necessarily $\dim\,(\ker {\rm d}Y(0)) \leq 1$.

• We observe now that we have also necessarily $\dim \ker {\rm d}Y(0) \geq 1$ because taking into account the relation (7.25), the operator ${\rm d}Y(0)$ can be considered as a linear mapping between the linear space Λ_1 of dimension 2 and the linear space Λ_{-1} of dimension 1, therefore with a necessarily non degerated kernel. According to the previous point, we are inconsequence in the particular case :

(7.34) $\quad \dim\,\big(\ker\, {\rm d}Y(0)\big) \,=\, 1\,, \qquad \ker\, {\rm d}Y(0) \,\subset\, \Lambda_1\,.$

With analogous notations as in the previous subsection, we introduce a non null vector r_0 $(r_0 \in \Lambda_1)$ that generates the linear space $\ker\, {\rm d}Y(0)$. We complete the basis of Λ_1 by some non null vector r_+ and due to the inclusion (7.25), its image r_- by ${\rm d}Y(0)$ is necessarily a non null vector of the eigenspace Λ_{-1} :

(7.35) $\quad \exists\, a > 0\,, \qquad {\rm d}Y(0) \bullet r_+ \,=\, -a\, r_-\,.$

We write now the vector ${\rm d}Y(0) \bullet r_- \in \Lambda_1$ inside the basis $(r_+\,,\, r_0)$:

(7.36) $\quad {\rm d}Y(0) \bullet r_- \,=\, c\, r_+ \,-\, d\, r_0\,.$

If $c \neq 0$, we set $\widetilde{r}_+ \,=\, r_+ - d\, r_0/c$ and we have ${\rm d}Y(0) \bullet \widetilde{r}_+ \,=\, -a\, r_-$; ${\rm d}Y(0) \bullet r_- \,=\, c\, \widetilde{r}_+$. Inside the basis $(\widetilde{r}_+\,,\, r_-\,,\, r_0)$ the matrix of the operator ${\rm d}Y(0)$ takes the form :

(7.37) $$\quad {\rm d}Y(0) \,=\, \begin{pmatrix} 0 & c & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad a > 0\,, \quad c \neq 0\,.$$

We change our notations and replace the vector r_+ initially introduced by the new vector $\widetilde{r}_+$. When the scalar c introduced at the relation (7.36) is null, we have necessarily $d \neq 0$ and after an eventual change of the sign of the vector r_0, the matrix of the operator ${\rm d}Y(0)$ inside the basis $(r_+\,,\, r_-\,,\, r_0)$ can be expressed as

(7.38) $$\quad \exists\, a > 0\,,\; b > 0\,,\; {\rm d}Y(0) \,=\, \begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & 0 \\ 0 & -b & 0 \end{pmatrix} \text{ in the basis } (r_+\,,\, r_-\,,\, r_0)\,.$$

• **Case (i).** We develop now the two particular cases (7.37) and (7.38). If the relation (7.37) holds, we have in a first opportunity :

(7.39) $$\quad \exists\, a > 0\,,\; b > 0\,, \quad {\rm d}Y(0) \,=\, \begin{pmatrix} 0 & -b & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ in the basis } (r_+\,,\, r_-\,,\, r_0)$$

and we are exactly in the case (i) of an "hyperbolic Galileo" system of conservation laws. The proof follows what have been done at the theorem 1. By exponentiation of the relation (7.39), we have :

$$(7.40)\qquad Y(v) \;=\; \begin{pmatrix} \mathrm{ch}\big(v\sqrt{a\,b}\,\big) & -\sqrt{b/a}\;\mathrm{sh}\big(v\sqrt{a\,b}\big) & 0 \\ -\sqrt{a/b}\;\mathrm{sh}\big(v\sqrt{a\,b}\big) & \mathrm{ch}\big(v\sqrt{a\,b}\,\big) & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

For an arbitrary state $W \;=\; (\theta,\, \zeta,\, \psi)^{\mathrm{t}} \in \Omega$ we have :

$$Y(v) \bullet \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} \;=\; \begin{pmatrix} \theta\;\mathrm{ch}\big(v\sqrt{a\,b}\,\big) \,-\, \zeta\;\sqrt{b/a}\;\mathrm{sh}\big(v\sqrt{a\,b}\big) \\ -\sqrt{a/b}\;\theta\,\mathrm{sh}\big(v\sqrt{a\,b}\big) \,+\, \zeta\;\mathrm{ch}\big(v\sqrt{a\,b}\,\big) \\ \psi \end{pmatrix} .$$

Because $u\big(Y(u(W))\bullet W\big) \equiv 0$ and $\Omega_0 \,\subset\, \Lambda_1 \;=\; \mathbb{R} \times^{\mathrm{t}} \{0\} \times^{\mathrm{t}} \mathbb{R}\,,$ we deduce from the previous equality the expression (7.4) of the velocity field. We then have $\theta\,\mathrm{ch}\big(v\sqrt{a\,b}\,\big) \,-\, \zeta\;\sqrt{b/a}\;\mathrm{sh}\big(v\sqrt{a\,b}\big) \;=\; \sqrt{\theta^2 - b\,\zeta^2/a}$ and the necessary condition $\eta(W) \;=\; \eta\big(Y(u(W))\bullet W\big)$ shows that the relation (7.6) holds.

• We must look now to the precise conditions that makes the function $\eta(\bullet)$ defined in (7.6) a strictly convex function when the property is satisfied for the two-variables function $\sigma(\alpha,\,\beta)$. We set as in the proof of the theorem 1 :

$$(7.41)\qquad \xi \;=\; \sqrt{\theta^2 \,-\, \frac{b\,\zeta^2}{a}}\,, \qquad p \;=\; \frac{b}{a}$$

and we have :

$$\xi\,\mathrm{d}\xi \;=\; \theta\,\mathrm{d}\theta \,-\, \frac{b}{a}\,\zeta\,\mathrm{d}\zeta\,,\quad \frac{\partial\eta}{\partial\theta} \;=\; \frac{\theta}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\,,\qquad \frac{\partial\eta}{\partial\zeta} \;=\; -\frac{b}{a}\,\frac{\zeta}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\,,\quad \frac{\partial\eta}{\partial\psi} \;=\; \frac{\partial\sigma}{\partial\beta}\,,$$

$$\frac{\partial^2\eta}{\partial\theta^2} \;=\; -p\,\frac{\zeta^2}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} \,+\, \frac{\theta^2}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2}\,,\qquad \frac{\partial^2\eta}{\partial\theta\,\partial\zeta} \;=\; p\,\frac{\theta\,\zeta}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} \,-\, p\,\frac{\theta\,\zeta}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2}$$

$$\frac{\partial^2\eta}{\partial\theta\,\partial\psi} \;=\; \frac{\theta}{\xi}\,\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta}\,,\qquad \frac{\partial^2\eta}{\partial\zeta^2} \;=\; -p\,\frac{\theta^2}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} \,+\, p^2\,\frac{\zeta^2}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2}$$

$$\frac{\partial^2\eta}{\partial\zeta\,\partial\psi} \;=\; -p\,\frac{\zeta}{\xi}\,\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta}\,,\qquad \frac{\partial^2\eta}{\partial\psi^2} \;=\; \frac{\partial^2\sigma}{\partial\beta^2}\,.$$

Then

$$(7.42)\quad \mathrm{d}^2\eta \;=\; \begin{pmatrix} -p\,\dfrac{\zeta^2}{\xi^3}\,\dfrac{\partial\sigma}{\partial\alpha} \,+\, \dfrac{\theta^2}{\xi^2}\,\dfrac{\partial^2\sigma}{\partial\alpha^2} & p\,\dfrac{\theta\,\zeta}{\xi^3}\,\dfrac{\partial\sigma}{\partial\alpha} - p\,\dfrac{\theta\,\zeta}{\xi^2}\,\dfrac{\partial^2\sigma}{\partial\alpha^2} & \dfrac{\theta}{\xi}\,\dfrac{\partial^2\sigma}{\partial\alpha\,\partial\beta} \\ p\,\dfrac{\theta\,\zeta}{\xi^3}\,\dfrac{\partial\sigma}{\partial\alpha} - p\,\dfrac{\theta\,\zeta}{\xi^2}\,\dfrac{\partial^2\sigma}{\partial\alpha^2} & -p\,\dfrac{\theta^2}{\xi^3}\,\dfrac{\partial\sigma}{\partial\alpha} + p^2\,\dfrac{\zeta^2}{\xi^2}\,\dfrac{\partial^2\sigma}{\partial\alpha^2} & -p\,\dfrac{\zeta}{\xi}\,\dfrac{\partial^2\sigma}{\partial\alpha\,\partial\beta} \\ \dfrac{\theta}{\xi}\,\dfrac{\partial^2\sigma}{\partial\alpha\,\partial\beta} & -p\,\dfrac{\zeta}{\xi}\,\dfrac{\partial^2\sigma}{\partial\alpha\,\partial\beta} & \dfrac{\partial^2\sigma}{\partial\beta^2} \end{pmatrix}.$$

The two by two sub-matrix at the top and the left of the matrix of the expression (7.42) admits the following determinant :

$$\det \Big(\mathrm{d}^2\eta \Big)_{1\leq i \leq 2 \,,\, 1 \leq j \leq 2} \,=\, \frac{1}{\xi^5} \, \frac{\partial \sigma}{\partial \alpha} \, \frac{\partial^2 \sigma}{\partial \alpha^2} \Big(-p\,\theta^4 \,-\, p^3\,\zeta^4 \,+\, 2\,p^2\,\theta^2\,\zeta^2 \Big)$$

$$= \, -\frac{b}{a} \, \frac{1}{\xi} \, \frac{\partial \sigma}{\partial \alpha} \, \frac{\partial^2 \sigma}{\partial \alpha^2}$$

which is necessarily strictly positive in order to get the strict convexity of the function η. We know that the second derivative $\frac{\partial^2 \sigma}{\partial \alpha^2}$ is strictly positive and that a b and ξ have the same property. Then we have $-\frac{\partial \sigma}{\partial \alpha} > 0$ and the relation (7.1) is established. We develop now the determinant of $\mathrm{d}^2\eta$ by using the third column. It comes :

$$\det \Big(\mathrm{d}^2\eta \Big) \,=\, -\frac{p}{\xi} \, \frac{\partial \sigma}{\partial \alpha} \, \frac{\partial^2 \sigma}{\partial \alpha^2} \, \frac{\partial^2 \sigma}{\partial \beta^2} \,+$$

$$+ \, \frac{\theta}{\xi} \, \frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} \, \begin{vmatrix} p \, \frac{\theta\,\zeta}{\xi^3} \, \frac{\partial \sigma}{\partial \alpha} \,-\, p \, \frac{\theta\,\zeta}{\xi^2} \, \frac{\partial^2 \sigma}{\partial \alpha^2} & -p \, \frac{\theta^2}{\xi^3} \, \frac{\partial \sigma}{\partial \alpha} \,+\, p^2 \, \frac{\zeta^2}{\xi^2} \, \frac{\partial^2 \sigma}{\partial \alpha^2} \\ \frac{\theta}{\xi} \, \frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} & -p \, \frac{\zeta}{\xi} \, \frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} \end{vmatrix}$$

$$+ \, p \, \frac{\zeta}{\xi} \, \frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} \, \begin{vmatrix} -p \, \frac{\zeta^2}{\xi^3} \, \frac{\partial \sigma}{\partial \alpha} \,+\, \frac{\theta^2}{\xi^2} \, \frac{\partial^2 \sigma}{\partial \alpha^2} & p \, \frac{\theta\,\zeta}{\xi^3} \, \frac{\partial \sigma}{\partial \alpha} \,-\, p \, \frac{\theta\,\zeta}{\xi^2} \, \frac{\partial^2 \sigma}{\partial \alpha^2} \\ \frac{\theta}{\xi} \, \frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} & -p \, \frac{\zeta}{\xi} \, \frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} \end{vmatrix}$$

$$= \, -\frac{p}{\xi} \, \frac{\partial \sigma}{\partial \alpha} \, \frac{\partial^2 \sigma}{\partial \alpha^2} \, \frac{\partial^2 \sigma}{\partial \beta^2}$$

$$+ \, \frac{\theta}{\xi^4} \, \Big(\frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} \Big)^2 \, \begin{vmatrix} p \, \frac{\theta\,\zeta}{\xi} \, \frac{\partial \sigma}{\partial \alpha} \,-\, p\,\theta\,\zeta \, \frac{\partial^2 \sigma}{\partial \alpha^2} & -p \, \frac{\theta^2}{\xi} \, \frac{\partial \sigma}{\partial \alpha} \,+\, p^2\,\zeta^2 \, \frac{\partial^2 \sigma}{\partial \alpha^2} \\ \theta & -p\,\zeta \end{vmatrix}$$

$$+ \, p \, \frac{\zeta}{\xi^4} \, \Big(\frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} \Big)^2 \, \begin{vmatrix} -p \, \frac{\zeta^2}{\xi} \, \frac{\partial \sigma}{\partial \alpha} \,+\, \theta^2 \, \frac{\partial^2 \sigma}{\partial \alpha^2} & p \, \frac{\theta\,\zeta}{\xi} \, \frac{\partial \sigma}{\partial \alpha} \,-\, p\,\theta\,\zeta \, \frac{\partial^2 \sigma}{\partial \alpha^2} \\ \theta & -p\,\zeta \end{vmatrix}$$

$$= \, -\frac{p}{\xi} \, \frac{\partial \sigma}{\partial \alpha} \, \frac{\partial^2 \sigma}{\partial \alpha^2} \, \frac{\partial^2 \sigma}{\partial \beta^2}$$

$$+ \, \frac{1}{\xi^4} \, \Big(\frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} \Big)^2 \, \frac{\partial \sigma}{\partial \alpha} \Big[\frac{\theta^2}{\xi} \big(-p^2\,\zeta^2 \,+\, p\,\theta^2 \big) \,+\, \frac{p\,\zeta^2}{\xi} \, \big(p^2\,\zeta^2 \,-\, p\,\theta^2 \big) \Big]$$

$$= \, -\frac{p}{\xi} \, \frac{\partial \sigma}{\partial \alpha} \, \frac{\partial^2 \sigma}{\partial \alpha^2} \, \frac{\partial^2 \sigma}{\partial \beta^2} \,+\, \frac{p}{\xi^5} \, \Big(\frac{\partial^2 \sigma}{\partial \alpha \, \partial \beta} \Big)^2 \, \frac{\partial \sigma}{\partial \alpha} \, \big(\theta^2 \,-\, p\,\zeta^2 \big) \, \big(\theta^2 \,-\, p\,\zeta^2 \big)$$

$$= \; -\frac{p}{\xi}\,\frac{\partial \sigma}{\partial \alpha}\,\Big[\frac{\partial^2 \sigma}{\partial \alpha^2}\,\frac{\partial^2 \sigma}{\partial \beta^2} \,-\, \Big(\frac{\partial^2 \sigma}{\partial \alpha\,\partial \beta}\Big)^2\Big] \qquad \text{due to (7.41)}$$

and

$$(7.43) \qquad \det\,\big(\mathrm{d}^2 \eta\big) \;=\; -\frac{b}{a\,\sqrt{\theta^2 - b\,\zeta^2/a}}\,\frac{\partial \sigma}{\partial \alpha}\,\Big[\frac{\partial^2 \sigma}{\partial \alpha^2}\,\frac{\partial^2 \sigma}{\partial \beta^2} \,-\, \Big(\frac{\partial^2 \sigma}{\partial \alpha\,\partial \beta}\Big)^2\Big]\,.$$

The matrix $\mathrm{d}^2 \eta$ is definite positive if it is the case for the matrix $\mathrm{d}^2 \sigma$ and if $\frac{\partial \sigma}{\partial \alpha} < 0$, *i.e.* when the condition (7.1) is satisfied.

• In order to obtain the algebraic expression of the thermodynamic flux $j(\bullet)$, we first evaluate the entropy variables

$$(7.44) \qquad \varphi \;=\; \Big(\frac{\partial \eta}{\partial \theta}\,,\,\frac{\partial \eta}{\partial \zeta}\,,\,\frac{\partial \eta}{\partial \psi}\Big) \;=\; \Big(\frac{\theta}{\xi}\,\frac{\partial \sigma}{\partial \alpha}\,,\; -\frac{b\,\zeta}{a\,\xi}\,\frac{\partial \sigma}{\partial \alpha}\,,\;\frac{\partial \sigma}{\partial \beta}\Big)$$

on the null-velocity manifold Ω_0 that corresponds to $(\theta\,,\,\zeta\,,\,\psi) \;=\; (\theta_0\,,\,0\,,\,\psi_0)$. Then $\varphi_0 \;=\; \Big(\frac{\partial \sigma}{\partial \alpha}(\theta_0,\,\psi_0)\;,\;0\;,\;\frac{\partial \sigma}{\partial \beta}(\theta_0,\,\psi_0)\Big)$. Taking into account the relation (7.39), it comes :

$$(7.45) \qquad \varphi_0 \bullet \mathrm{d}Y(0) \;=\; \Big(0\;,\;-b\,\frac{\partial \sigma}{\partial \alpha}\;,\;0\Big)\,.$$

We introduce the mechanical pressure $\Pi(\bullet)$ as a notation :

$$(7.46) \qquad \forall\, W_0 \;=\; (\theta_0,\,0,\,\psi_0)^{\mathrm{t}} \,\in \Omega_0\,, \qquad j(W_0) \;=\; \big(0\,,\,\Pi(\theta_0,\,\psi_0)\,,\,0\big)^{\mathrm{t}}$$

and the relation $\varphi \bullet \mathrm{d}j \,+\, \eta^*\,\mathrm{d}u \;\equiv\; 0$ applied against the vector $\mathrm{d}Y(0) \bullet W_0$ gives

$$b\,\frac{\partial \sigma}{\partial \alpha}(\theta_0,\,\psi_0)\;\Pi(\theta_0,\,\psi_0) \;+\; \sigma^*(A,\,B) \;=\; 0$$

with $A = \frac{\partial \sigma}{\partial \alpha}(\theta_0,\,\psi_0)$, $B = \frac{\partial \sigma}{\partial \beta}(\theta_0,\,\psi_0)$, and this property is exactly the relation (7.5). The general expression of the thermodynamic flux $j(W)$ is obtained thanks to the relations (3.12) and (7.40) :

$$j(W) \;=\; Y(-u(W)) \bullet j\big(Y(u(W)) \bullet W\big)$$

$$= \begin{pmatrix} \mathrm{ch}\big(u\,\sqrt{a\,b}\,\big) & \sqrt{b/a}\;\mathrm{sh}\big(u\,\sqrt{a\,b}\big) & 0 \\ \sqrt{a/b}\;\mathrm{sh}\big(u\,\sqrt{a\,b}\big) & \mathrm{ch}\big(u\,\sqrt{a\,b}\,\big) & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ \Pi\big(\sqrt{\theta^2 - b\,\zeta^2/a}\,,\,\psi\big) \\ 0 \end{pmatrix}$$

$$(7.47) \qquad \forall\, W \;=\; \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} \in \Omega\,, \qquad j(W) \;=\; \frac{\Pi\big(\sqrt{\theta^2 - b\,\zeta^2/a}\,,\,\psi\big)}{\sqrt{\theta^2 - b\,\zeta^2/a}}\,\begin{pmatrix} b\,\zeta/a \\ \theta \\ 0 \end{pmatrix}\,.$$

Joined with the expression (7.4) of the velocity field, the relation (7.47) establishes the expression (7.3) of the hyperbolic Galileo system of conservation laws.

• We verify now that the function $\eta(\bullet)$ defined at the relation (7.6) is effectively a mathematical entropy associated with the flux $u(W)\,\eta(W)$; in other words we have the additional conservation law

$$\frac{\partial \eta(W)}{\partial t} + \frac{\partial \eta(W)}{\partial x}\big(u(W)\,\eta(W)\big) \;=\; 0 \tag{7.48}$$

if $W(x,\,t)$ is a regular solution of the conservation law (7.3). By differentiation of the relation (7.4), we have :

$$a\,\mathrm{d}u \;=\; \frac{1}{\theta^2 \,-\, b\,\zeta^2/a}\,\big(\theta\,\mathrm{d}\zeta \,-\, \zeta\,\mathrm{d}\theta\big)\,. \tag{7.49}$$

We have now do develop some algebraic calculus :

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta\,u) \;=\; \Big(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\Big)\eta(W) \,+\, \eta\,\frac{\partial u}{\partial x} \;=$$

$$= \frac{\partial \sigma}{\partial \alpha}\,\Big[\frac{\partial \xi}{\partial \theta}\,\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\theta + \frac{\partial \xi}{\partial \zeta}\,\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\zeta\Big] + \frac{\partial \sigma}{\partial \beta}\,\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\psi + \eta\,\frac{\partial u}{\partial x}$$

$$= \frac{\partial \sigma}{\partial \alpha}\,\Big[\frac{\theta}{\xi}\,\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\theta - \frac{b}{a}\,\frac{\zeta}{\xi}\,\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\zeta\Big] + \frac{\partial \sigma}{\partial \beta}\,\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\psi + \eta\,\frac{\partial u}{\partial x}$$

due to (7.44)

$$= \frac{\theta}{\xi}\,\frac{\partial \sigma}{\partial \alpha}\,\Big[-\theta\,\frac{\partial u}{\partial x} \,-\, \frac{\partial}{\partial x}\Big(\Pi(\xi,\,\psi)\,\frac{b\,\zeta}{a\,\xi}\Big)\Big]$$

$$\qquad - \frac{b}{a}\,\frac{\zeta}{\xi}\,\frac{\partial \sigma}{\partial \alpha}\,\Big[-\zeta\,\frac{\partial u}{\partial x} \,-\, \frac{\partial}{\partial x}\Big(\Pi(\xi,\,\psi)\,\frac{\theta}{\xi}\Big)\Big]$$

$$\qquad + \,\frac{\partial \sigma}{\partial \beta}\,\Big(-\psi\,\frac{\partial u}{\partial x}\Big) \,+\, \eta\,\frac{\partial u}{\partial x}$$

according to (7.3)

$$= \frac{\partial \sigma}{\partial \alpha}\,\frac{\partial u}{\partial x}\,\Big[-\frac{\theta^2}{\xi} \,+\, \frac{b}{a}\,\frac{\zeta^2}{\xi}\Big] \,-\, \frac{\partial \sigma}{\partial \beta}\,\frac{\partial u}{\partial x}\,\psi \,+\, \eta\,\frac{\partial u}{\partial x}$$

$$\qquad + \frac{1}{\xi}\,\frac{\partial \sigma}{\partial \alpha}\,\Pi(\xi,\,\psi)\,\Big[-\frac{\theta}{\xi}\,\frac{b}{a}\,\frac{\partial \zeta}{\partial x} \,+\, \frac{b}{a}\,\frac{\zeta}{\xi}\,\frac{\partial \theta}{\partial x}\Big]$$

$$= -\frac{\partial u}{\partial x}\,\Big[\,\xi\,\frac{\partial \sigma}{\partial \alpha}(\xi,\,\psi) \,+\, \psi\,\frac{\partial \sigma}{\partial \beta}(\xi,\,\psi) \,-\, \sigma(\xi,\,\psi)\Big]$$

$$\qquad - \frac{b}{a\,\xi^2}\,\frac{\partial \sigma}{\partial \alpha}\,\Pi(\xi,\,\psi)\,\Big[\theta\,\frac{\partial \zeta}{\partial x} \,-\, \zeta\,\frac{\partial \theta}{\partial x}\Big]$$

$$= -\frac{\partial u}{\partial x}\,\sigma^* \,-\, \frac{b}{a}\,\frac{\partial \sigma}{\partial \alpha}\,\Big(-\frac{1}{b}\,\frac{\sigma^*}{\partial \sigma\,/\,\partial \alpha}\Big)\,\Big(a\,\frac{\partial u}{\partial x}\Big)$$

due to (7.5) and (7.49)

$$= \;0$$

and the property (7.48) is established. The study of the first case is over.

• **Case (ii).** We consider again the expression (7.37) of the jacobian matrix $\mathrm{d}Y(0)$ now with $c > 0$ and the relation (7.37) can be rewritten as :

$$\exists\, a > 0,\ b > 0, \quad \mathrm{d}Y(0) \,=\, \begin{pmatrix} 0 & b & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{in the basis } (r_+\,,\, r_-\,,\, r_0)\,. \tag{7.50}$$

The relation (6.30) established during the proof of the theorem 1 can be reproduced without any modification and we have in consequence :

$$Y(v) \,=\, \begin{pmatrix} \cos\big(v\sqrt{a\,b}\,\big) & \sqrt{b/a}\,\sin\big(v\sqrt{a\,b}\big) & 0 \\ -\sqrt{a/b}\,\sin\big(v\sqrt{a\,b}\big) & \cos\big(v\sqrt{a\,b}\,\big) & 0 \\ 0 & 0 & 1 \end{pmatrix}\,; \tag{7.51}$$

then for $(\theta,\,\zeta,\,\psi)^{\mathrm{t}} \in \Omega\,,$ we have

$$Y(v)\bullet\begin{pmatrix}\theta\\ \zeta\\ \psi\end{pmatrix} \,=\, \begin{pmatrix} \theta\,\cos\big(v\sqrt{a\,b}\,\big) \,+\, \sqrt{b/a}\;\zeta\,\sin\big(v\sqrt{a\,b}\big) \\ -\sqrt{a/b}\;\theta\,\sin\big(v\sqrt{a\,b}\big) \,+\, \zeta\,\cos\big(v\sqrt{a\,b}\,\big) \\ \psi \end{pmatrix}.$$

We know that the reference basis $(r_+\,,\, r_-\,,\, r_0)$ belongs to the product of spaces $\Lambda_1 \times \Lambda_{-1} \times \Lambda_1$ and $\Omega_0 \subset \Lambda_1$. We deduce that the second component of the state $Y(v)\bullet W$ that belongs to the null velocity manifold is necessarily null and we have

$$\mathrm{tg}\,\big(u\sqrt{a\,b}\,\big) \,=\, \sqrt{\frac{b}{a}}\,\frac{\zeta}{\theta} \tag{7.52}$$

and the relation (7.9) is established. We deduce naturally

$$\cos(u\sqrt{a\,b}\,) \,=\, \frac{\theta}{\xi}\,, \qquad \sin(u\sqrt{a\,b}\,) \,=\, \sqrt{\frac{b}{a}}\,\frac{\zeta}{\xi}\,, \qquad \xi \,=\, \sqrt{\theta^2 \,+\, b\,\zeta^2/a} \tag{7.53}$$

$$Y(u(W))\bullet W \,=\, (\,\xi\,,\,0\,,\,\psi\,)^{\mathrm{t}}\,, \qquad \xi \,=\, \sqrt{\theta^2 \,+\, b\,\zeta^2/a} \tag{7.54}$$

and the relation (7.11) is a consequence of the invariance (3.2) of the mathematical entropy for the transformation $Y(v)$.

• We consider now as given the strictly convex function $\sigma(\bullet)$ which is the restriction of the mathematical entropy $\eta(\bullet)$ to the null manifold Ω_0. We must verify that the mathematical entropy is also a strictly convex function of the triplet $(\theta,\,\zeta,\,\psi)$. We set as above $p = b/a$ and we have

$$\varphi \,=\, \Big(\frac{\partial\eta}{\partial\theta}\,,\,\frac{\partial\eta}{\partial\zeta}\,,\,\frac{\partial\eta}{\partial\psi}\Big) \,=\, \Big(\frac{\theta}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\,,\,p\,\frac{\zeta}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\,,\,\frac{\partial\sigma}{\partial\beta}\Big)\,. \tag{7.55}$$

Then

$$\frac{\partial^2\eta}{\partial\theta^2} = p\,\frac{\zeta^2}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} \,+\, \frac{\theta^2}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2}\,, \qquad \frac{\partial^2\eta}{\partial\theta\,\partial\zeta} \,=\, -p\,\frac{\theta\,\zeta}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} \,+\, p\,\frac{\theta\,\zeta}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2}\,,$$

$$\frac{\partial^2\eta}{\partial\theta\,\partial\psi} = \frac{\theta}{\xi}\,\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta}\,, \qquad \frac{\partial^2\eta}{\partial\zeta^2} = p\,\frac{\theta^2}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} + p^2\,\frac{\zeta^2}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2}\,,$$

$$\frac{\partial^2\eta}{\partial\zeta\,\partial\psi} = p\,\frac{\zeta}{\xi}\,\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta}\,, \qquad \frac{\partial^2\eta}{\partial\psi^2} = \frac{\partial^2\sigma}{\partial\beta^2}$$

and

$$(7.56)\quad \mathrm{d}^2\eta = \begin{pmatrix} p\,\frac{\zeta^2}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} + \frac{\theta^2}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2} & -p\,\frac{\theta\,\zeta}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} + p\,\frac{\theta\,\zeta}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2} & \frac{\theta}{\xi}\,\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta} \\ -p\,\frac{\theta\,\zeta}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} + p\,\frac{\theta\,\zeta}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2} & p\,\frac{\theta^2}{\xi^3}\,\frac{\partial\sigma}{\partial\alpha} + p^2\,\frac{\zeta^2}{\xi^2}\,\frac{\partial^2\sigma}{\partial\alpha^2} & p\,\frac{\zeta}{\xi}\,\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta} \\ \frac{\theta}{\xi}\,\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta} & p\,\frac{\zeta}{\xi}\,\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta} & \frac{\partial^2\sigma}{\partial\beta^2} \end{pmatrix}.$$

This matrix is identical to the one presented at the relation (7.42), except that the variable p must be changed into $-p$. But this change of sign is exactly what is necessary to modify the definition of the variable ξ from (7.41) to (7.53). We observe that the two by two minor determinant composed by the two left lines and the two first columns of the right hand side of the relation (7.56) is equal to $\frac{p}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\,\frac{\partial^2\sigma}{\partial\alpha^2}$ and is strictly positive when $\frac{\partial\sigma}{\partial\alpha}$ is strictly positive. In consequence the hypothesis (7.7) is clearly established. The computation of $\det(\mathrm{d}^2\eta)$ done during the study of the hyperbolic case conducts to the relation (7.43) and the change of the variable p into $-p$ establishes that

$$(7.57)\qquad \det\,\big(\mathrm{d}^2\eta\big) = \frac{p}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\,\Big[\,\frac{\partial^2\sigma}{\partial\alpha^2}\,\frac{\partial^2\sigma}{\partial\beta^2} - \Big(\frac{\partial^2\sigma}{\partial\alpha\,\partial\beta}\Big)^2\,\Big]$$

which is strictly positive when $\sigma(\bullet)$ is strictly convex and when the relation (7.7) is satisfied.

• The constitution of the thermodynamic flux $j(\bullet)$ is obtained exactly as the hyperbolic case. Taking into account the relations (7.50) and (7.55), we have, for $W_0 = (\theta_0,\,0,\,\psi)^{\mathrm{t}} \in \Omega_0$:

$$\varphi(\theta_0,\,0,\,\psi)\bullet\mathrm{d}Y(0) = \Big(\frac{\partial\sigma}{\partial\alpha}\,,\,0\,,\,\frac{\partial\sigma}{\partial\beta}\Big)\begin{pmatrix} 0 & b & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \Big(0\,,\,b\,\frac{\partial\sigma}{\partial\beta}\,,\,0\Big)$$

and if we introduce the mechanical pressure $\Pi(\bullet)$ with the relation (7.46), we have from the relation (2.9) : $\varphi_0\bullet\mathrm{d}Y(0)\bullet j(W_0) + \sigma^*(-1) = 0\,,$ and this calculus proves exactly the relation (7.10). The general expression of the thermodynamic flux is given by somes lines of algebra that are consequence of the relations (3.12) and (7.51) :

$$j(W) = Y(-u(W))\bullet j\big(Y(u(W))\bullet W\big)$$

$$= \begin{pmatrix} \cos\big(u\sqrt{a\,b}\,\big) & -\sqrt{b/a}\,\sin\big(u\sqrt{a\,b}\big) & 0 \\ \sqrt{a/b}\,\sin\big(u\sqrt{a\,b}\big) & \cos\big(u\sqrt{a\,b}\,\big) & 0 \\ 0 & 0 & 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ \Pi\big(\sqrt{\theta^2 + b\,\zeta^2/a}\,,\,\psi\big) \\ 0 \end{pmatrix}$$

$$= \Pi(\xi,\,\psi)\,\begin{pmatrix} -\sqrt{\dfrac{b}{a}}\sqrt{\dfrac{b}{a}}\,\dfrac{\zeta}{\xi} \\ \theta/\xi \\ 0 \end{pmatrix} \qquad \text{taking into account the relation (7.53)}$$

$$= \frac{1}{\xi}\,\Pi(\xi,\,\psi)\,\begin{pmatrix} -\dfrac{b\,\zeta}{a} \\ \theta \\ 0 \end{pmatrix}.$$

Then the algebraic expression (7.8) of the elliptic Galileo system of conservation laws is established.

• We still have to show that any regular solution of the system (7.8) satisfy the conservation (7.48) of the mathematical entropy, with a velocity field given by the relation (7.9) that satisfies in consequence :

$$a\,\mathrm{d}u \;=\; \frac{1}{\theta^2 + b\,\zeta/a}\,(\theta\,\mathrm{d}\zeta - \zeta\,\mathrm{d}\theta)\,. \qquad (7.58)$$

We have very simply

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}(\eta\,u) \;=\; \Big(\frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x}\Big)\eta(W) \,+\, \eta\,\frac{\partial u}{\partial x} \;=$$

$$= \frac{\partial\sigma}{\partial\alpha}\,\Big[\frac{\partial\xi}{\partial\theta}\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\theta + \frac{\partial\xi}{\partial\zeta}\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\zeta\Big] + \frac{\partial\sigma}{\partial\beta}\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\psi + \eta\,\frac{\partial u}{\partial x}$$

$$= \frac{\partial\sigma}{\partial\alpha}\,\Big[\frac{\theta}{\xi}\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\theta + \frac{b}{a}\,\frac{\zeta}{\xi}\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\zeta\Big] + \frac{\partial\sigma}{\partial\beta}\Big(\frac{\partial}{\partial t}+u\,\frac{\partial}{\partial x}\Big)\psi + \eta\,\frac{\partial u}{\partial x} \qquad \text{due to (7.55)}$$

$$= \frac{\theta}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\Big[-\theta\,\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}\Big(\Pi(\xi,\,\psi)\,\frac{b\,\zeta}{a\,\xi}\Big)\Big]$$

$$+ \frac{b}{a}\,\frac{\zeta}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\Big[-\zeta\,\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\Big(\Pi(\xi,\,\psi)\,\frac{\theta}{\xi}\Big)\Big]$$

$$+ \frac{\partial\sigma}{\partial\beta}\Big(-\psi\,\frac{\partial u}{\partial x}\Big) + \eta\,\frac{\partial u}{\partial x} \qquad \text{applying the equation (7.8)}$$

$$= -\frac{\partial u}{\partial x}\Big[\frac{\theta^2}{\xi}\,\frac{\partial\sigma}{\partial\alpha} + \frac{b}{a}\,\frac{\zeta^2}{\xi}\,\frac{\partial\sigma}{\partial\alpha} + \psi\,\frac{\partial\sigma}{\partial\beta} - \sigma\Big] + \frac{\partial}{\partial x}\Big(\frac{\Pi}{\xi}\Big)\Big[\frac{\theta}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\,\frac{b\,\zeta}{a} - \frac{b}{a}\,\frac{\zeta}{\xi}\,\frac{\partial\sigma}{\partial\alpha}\,\theta\Big]$$

$$+ \frac{b}{a}\,\frac{\partial\sigma}{\partial\alpha}\,\frac{\Pi}{\xi^2}\,\Big(\theta\,\frac{\partial\zeta}{\partial x} - \zeta\,\frac{\partial\theta}{\partial x}\Big)$$

$$= \; -\sigma^* \Big(\frac{\partial \sigma}{\partial \alpha}(\xi,\, \psi),\, \frac{\partial \sigma}{\partial \beta}(\xi,\, \psi) \Big) \, \frac{\partial u}{\partial x} \, + \, b \, \frac{\partial \sigma}{\partial \alpha}(\xi,\, \psi) \; \Pi(\xi,\, \psi) \, \frac{\partial u}{\partial x} \,,$$

$$= \; 0 \qquad\qquad \text{taking into account the relation (7.15).}$$

This property establishes the structure of the elliptic Galileo system.

• **Case (iii).** We detail now the third case, when the matrix $\mathrm{d}Y(0)$ is given by the relation (7.38) with $a > 0$ and $b > 0$. We first remark that the matrix $\mathrm{d}Y(0)$ is nilpotent because

$$\text{(7.59)} \qquad \mathrm{d}Y(0)^2 \; = \; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a\, b & 0 & 0 \end{pmatrix} \,, \qquad \mathrm{d}Y(0)^3 \; = \; 0 \,.$$

This remark justifies the name "nilpotent" given for this third Galileo group preserving system of conservation laws. The exponentiation of the matrix $Y(v)$ is easy :

$$\text{(7.60)} \qquad Y(v) \; = \; \begin{pmatrix} 1 & 0 & 0 \\ -a\, v & 1 & 0 \\ a\, b\, \frac{v^2}{2} & -bv & 1 \end{pmatrix} \,, \qquad v \in \mathbb{R} \,, \qquad a > 0 \,, \qquad b > 0 \,.$$

The determiation of the velocity field $u(\bullet)$ is a direct consequence of the evaluation of the product $Y(v) \bullet W$ and of the remark that $u\big(Y(u(W)) \bullet W\big) \; = \; 0 \,.$ We have

$$\forall\, W \; = \; \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} \in \Omega \,, \qquad Y(v) \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} \; = \; \begin{pmatrix} \theta \\ -a\, v\, \theta \, + \, \zeta \\ a\, b\, \frac{v^2}{2}\, \theta \, - \, b\, v\, \zeta \, + \, \psi \end{pmatrix}$$

and we deduce that we have necessarily $u(W) \; = \; \frac{\zeta}{a\, \theta}$. This expression is exactly the relation (7.14). We deduce :

$$\text{(7.61)} \qquad \forall\, W \; = \; (\theta,\, \zeta,\, \psi)^{\mathrm{t}} \, \in \Omega \,, \qquad Y\big(u(W) \bullet W\big) \; = \; \Big(\theta \,,\, 0 \,,\, \psi \, - \, \frac{b}{2\, a} \, \frac{\zeta^2}{\theta} \Big)^{\mathrm{t}} \,.$$

• The computation of the entropy variables is a simple consequence of the relations (7.17) and (7.61). It comes

$$\text{(7.62)} \qquad \varphi \; = \; \Big(\frac{\partial \sigma}{\partial \alpha} \, + \, \frac{b}{2\, a} \, \frac{\zeta^2}{\theta^2} \, \frac{\partial \sigma}{\partial \beta} \,,\, -\frac{b}{a} \, \frac{\zeta}{\theta} \, \frac{\partial \sigma}{\partial \beta} \,,\, \frac{\partial \sigma}{\partial \beta} \Big) \,.$$

With the notation $p \; = \; b/a \,,$ the exact expression of the Hessian of the mathematical entropy only needs some care :

$$\frac{\partial^2 \eta}{\partial \theta^2} \; = \; \frac{\partial^2 \sigma}{\partial \alpha^2} - p \, \frac{\zeta^2}{\theta^3} \, \frac{\partial \sigma}{\partial \beta} \, + \, \frac{1}{4} \, p^2 \, \frac{\zeta^4}{\theta^4} \, \frac{\partial^2 \sigma}{\partial \beta^2} \,, \qquad \frac{\partial^2 \eta}{\partial \theta \, \partial \zeta} \; = \; p \, \frac{\zeta}{\theta^2} \, \frac{\partial \sigma}{\partial \beta} \, - \, \frac{p^2}{2} \, \frac{\zeta^3}{\theta^3} \, \frac{\partial^2 \sigma}{\partial \beta^2} \,,$$

$$\frac{\partial^2 \eta}{\partial \theta \, \partial \psi} \; = \; \frac{p}{2} \, \frac{\zeta^2}{\theta^2} \, \frac{\partial^2 \sigma}{\partial \beta^2} \,, \qquad \frac{\partial^2 \eta}{\partial \zeta^2} \; = \; -\frac{p}{\theta} \, \frac{\partial \sigma}{\partial \beta} \, + \, p^2 \, \frac{\zeta^2}{\theta^2} \, \frac{\partial^2 \sigma}{\partial \beta^2} \,,$$

$$\frac{\partial^2 \eta}{\partial \zeta \, \partial \psi} \,=\, -p \, \frac{\zeta}{\theta} \, \frac{\partial^2 \sigma}{\partial \beta^2} \,, \qquad \frac{\partial^2 \eta}{\partial \psi^2} \,=\, \frac{\partial^2 \sigma}{\partial \beta^2} \,,$$

and with the complementary notation $q \equiv \zeta / \theta$ we get

$$(7.63) \quad \left\{ \begin{array}{l} \mathrm{d}^2 \eta \,= \\ \begin{pmatrix} \frac{\partial^2 \sigma}{\partial \alpha^2} - p \, \frac{q^2}{\theta} \, \frac{\partial \sigma}{\partial \beta} + \frac{1}{4} p^2 \, q^4 \, \frac{\partial^2 \sigma}{\partial \beta^2} & \frac{p \, q}{\theta} \, \frac{\partial \sigma}{\partial \beta} - \frac{1}{2} p^2 q^3 \frac{\partial^2 \sigma}{\partial \beta^2} & \frac{1}{2} \, pq^2 \, \frac{\partial^2 \sigma}{\partial \beta^2} \\ \frac{p \, q}{\theta} \, \frac{\partial \sigma}{\partial \beta} - \frac{1}{2} p^2 q^3 \frac{\partial^2 \sigma}{\partial \beta^2} & -\frac{p}{\theta} \, \frac{\partial \sigma}{\partial \beta} + p^2 \, q^2 \, \frac{\partial^2 \sigma}{\partial \beta^2} & -p \, q \, \frac{\partial^2 \sigma}{\partial \beta^2} \\ \frac{1}{2} \, pq^2 \frac{\partial^2 \sigma}{\partial \beta^2} & -p \, q \, \frac{\partial^2 \sigma}{\partial \beta^2} & \frac{\partial^2 \sigma}{\partial \beta^2} \end{pmatrix} . \end{array} \right.$$

The two by two minor determinant at the bottom and the right of $\mathrm{d}^2 \eta$ is equal to $-\frac{b}{a \, \theta} \, \frac{\partial \sigma}{\partial \beta} \, \frac{\partial^2 \sigma}{\partial \beta^2}$, which establishes the necessary condition (7.12) when we have supposed that the domain Ω is composed by states $W \,=\, (\theta, \, \zeta, \, \psi)^{\mathrm{t}}$ that satisfy the condition

$$(7.64) \qquad \theta > 0 \,.$$

The evaluation of the determinant of the relation (7.63) is easy. We multiply the last line by $p \, q$ and we add it to the second line. We obtain :

$$\det (\mathrm{d}^2 \eta) \,= \\ = \begin{vmatrix} \frac{\partial^2 \sigma}{\partial \alpha^2} - p \, \frac{q^2}{\theta} \, \frac{\partial \sigma}{\partial \beta} + \frac{1}{4} p^2 \, q^4 \, \frac{\partial^2 \sigma}{\partial \beta^2} & \frac{p \, q}{\theta} \, \frac{\partial \sigma}{\partial \beta} - \frac{1}{2} p^2 \, q^3 \, \frac{\partial^2 \sigma}{\partial \beta^2} & \frac{1}{2} \, p \, q^2 \, \frac{\partial^2 \sigma}{\partial \beta^2} \\ \frac{p \, q}{\theta} \, \frac{\partial \sigma}{\partial \beta} & -\frac{p}{\theta} \, \frac{\partial \sigma}{\partial \beta} & 0 \\ \frac{1}{2} \, p \, q^2 \, \frac{\partial^2 \sigma}{\partial \beta^2} & -p \, q \, \frac{\partial^2 \sigma}{\partial \beta^2} & \frac{\partial^2 \sigma}{\partial \beta^2} \end{vmatrix} ;$$

then we multiply the third column of the previous expression by $\frac{1}{2} \, p \, q^2$ and we substract it from the first column ; in an analogous way, we multiply the last column by $p \, q$ and we add it to the second column of the determinant. We find :

$$\det (\mathrm{d}^2 \eta) \,= \begin{vmatrix} \frac{\partial^2 \sigma}{\partial \alpha^2} - p \, \frac{q^2}{\theta} \, \frac{\partial \sigma}{\partial \beta} & \frac{p \, q}{\theta} \, \frac{\partial \sigma}{\partial \beta} & \frac{1}{2} \, p \, q^2 \, \frac{\partial^2 \sigma}{\partial \beta^2} \\ \frac{p \, q}{\theta} \, \frac{\partial \sigma}{\partial \beta} & -\frac{p}{\theta} \, \frac{\partial \sigma}{\partial \beta} & 0 \\ 0 & 0 & \frac{\partial^2 \sigma}{\partial \beta^2} \end{vmatrix} \,=\, \frac{\partial^2 \sigma}{\partial \beta^2} \Big(-\frac{p}{\theta} \Big) \, \frac{\partial \sigma}{\partial \beta} \, \frac{\partial^2 \sigma}{\partial \alpha^2} \,.$$

The sign of $\det (\mathrm{d}^2 \eta)$ is the one of $-\frac{\partial \sigma}{\partial \beta}$ when the condition (7.64) is satisfied then is coherent with the inequality (7.12). Then the function $\eta(\bullet)$ defined with the relation (7.17) is strictly convex.

• The thermodynamic flux $j(\bullet)$ is constructed exactly as in the two previous cases. We use the definition (7.46) of the function $\Pi(\bullet)$; then the expression (7.62) of the entropy variables shows :

$$\varphi \bullet {\rm d}Y(0) \bullet j(W_0) \,=\, \Big(\frac{\partial \sigma}{\partial \alpha} \,,\, 0 \,,\, \frac{\partial \sigma}{\partial \beta} \Big) \begin{pmatrix} 0 & 0 & 0 \\ -a & 0 & 0 \\ 0 & -b & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \Pi \\ 0 \end{pmatrix} \,=\, -b \, \Pi \, \frac{\partial \sigma}{\partial \beta} \,.$$

The relation (7.16) is then a consequence of the remark that ${\rm d}j(W) \bullet {\rm d}Y(0) \bullet W = {\rm d}Y(0) \bullet j(W)$ joined with the relation (4.7). The general expression for the function $j(W)$ is a consequence of the relation (3.12) and the expresison (7.61) of a state inside the null velocity manifold :

$$\begin{aligned} j(W) \,&=\, Y\big(-u(W)\big) \bullet j\Big(\theta \,,\, 0 \,,\, \psi - \frac{b}{2\,a} \, \frac{\zeta^2}{\theta} \Big) \\ &=\, \begin{pmatrix} 1 & 0 & 0 \\ a\,u & 1 & 0 \\ a\,b\,\frac{u^2}{2} & b\,u & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \Pi\Big(\theta \,,\, \psi - \frac{b}{2\,a} \, \frac{\zeta^2}{\theta} \Big) \\ 0 \end{pmatrix} \\ &=\, \Pi\Big(\theta \,,\, \psi - \frac{b}{2\,a} \, \frac{\zeta^2}{\theta} \Big) \begin{pmatrix} 0 \\ 1 \\ \frac{b}{a} \, \frac{\zeta}{\theta} \end{pmatrix} \qquad \text{due to the relation (7.14).} \end{aligned}$$

The algebraic expression (7.13) of the nilpotent Galileo system of conservation laws is an immediate consequence of what have been done at the previous line.

• As in the two preceding cases, we verify that the candidate (7.17) for beeing a mathematical entropy satisfies the relation (7.48) for regular solutions of the conservation law (7.13). We have

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta\, u) \,&=\, \Big(\frac{\partial}{\partial t} + u \, \frac{\partial}{\partial x} \Big) \eta(W) \,+\, \eta \, \frac{\partial u}{\partial x} \\ &= \Big(\frac{\partial \sigma}{\partial \alpha} + \frac{a\,b}{2} \, u^2 \, \frac{\partial \sigma}{\partial \beta} \Big) \Big(\frac{\partial}{\partial t} + u \, \frac{\partial}{\partial x} \Big) \theta \,-\, b\,u \, \frac{\partial \sigma}{\partial \beta} \, \Big(\frac{\partial}{\partial t} + u \, \frac{\partial}{\partial x} \Big) \zeta \\ &\qquad + \frac{\partial \sigma}{\partial \beta} \Big(\frac{\partial}{\partial t} + u \, \frac{\partial}{\partial x} \Big) \psi \,+\, \eta \, \frac{\partial u}{\partial x} \qquad \text{due to the expression (7.62)} \\ &= \Big(\frac{\partial \sigma}{\partial \alpha} + \frac{a\,b}{2} \, u^2 \, \frac{\partial \sigma}{\partial \beta} \Big) \Big(-\theta \, \frac{\partial u}{\partial x} \Big) \,-\, b\,u \, \frac{\partial \sigma}{\partial \beta} \, \Big(-\zeta \, \frac{\partial u}{\partial x} \,-\, \frac{\partial \Pi}{\partial x} \Big) \\ &\qquad + \frac{\partial \sigma}{\partial \beta} \, \Big(-\psi \, \frac{\partial u}{\partial x} \,-\, \frac{\partial}{\partial x} \, \big(\Pi\, u\, b \big) \Big) \,+\, \sigma \, \frac{\partial u}{\partial x} \\ &\qquad\qquad \text{taking into account the equation (7.13)} \end{aligned}$$

$$= \; -\frac{\partial u}{\partial x} \Big[\theta \, \frac{\partial \sigma}{\partial \alpha} \, + \, \Big(\psi - \frac{b}{2\,a} \, \frac{\zeta^2}{\theta} \Big) \, \frac{\partial \sigma}{\partial \beta} \, - \, \sigma \Big] \, - \, b \, \Pi \, \frac{\partial \sigma}{\partial \beta} \, \frac{\partial u}{\partial x} \qquad \text{due to (7.14)}$$

$$= \; -\frac{\partial u}{\partial x} \, \Big[\theta \, \frac{\partial \sigma}{\partial \alpha} \Big(\theta ,\, \psi - \frac{b}{2\,a} \, \frac{\zeta^2}{\theta} \Big) \, + \, \Big(\psi - \frac{b}{2\,a} \, \frac{\zeta^2}{\theta} \Big) \, \frac{\partial \sigma}{\partial \beta} \Big(\theta ,\, \psi - \frac{b}{2\,a} \, \frac{\zeta^2}{\theta} \Big)$$

$$- \, \sigma \Big(\theta ,\, \psi - \frac{b}{2\,a} \, \frac{\zeta^2}{\theta} \Big) \, - \, \sigma^* \Big(\frac{\partial \sigma}{\partial \alpha} ,\, \frac{\partial \sigma}{\partial \beta} \Big) \Big]$$

$= \; 0$ and the theorem 2 is proven. □

8. The Cemracs System

This section describes the funny hyperbolic system of conservation laws that we have derived during the Cemracs 99. It corresponds to the elliptic Galileo system of conservation laws with $a = b = 1$ and the thermodynamics of the polytropic perfect gas.

• Let $\gamma > 1$ be a fixed real number, *e.g.* $\gamma = 7/5$ for the air (di-atomic gas) at usual conditions of tempeature and pressure, see [Ca85]. We set

$$\sigma(\theta,\, \psi) \; = \; -\theta \, \log \Big[\frac{(\gamma - 1)\, \psi}{\theta^\gamma} \Big] , \qquad \theta > 0 \,, \quad \psi > 0 \tag{8.1}$$

and as considered in the remark 4, we can introduce the density ρ and the internal energy e according to

$$\theta \; = \; \rho \,, \qquad \psi \; = \; \rho \, e \,. \tag{8.2}$$

We have $\quad \mathrm{d}\sigma \; = \; Q(\theta,\, \psi) \, \mathrm{d}\theta \, + \, \chi(\theta,\, \psi) \, \mathrm{d}\psi \quad$ with

$$Q(\theta,\, \psi) \; = \; \gamma \, + \, \frac{\sigma(\theta,\, \psi)}{\theta} \tag{8.3}$$

$$\chi(\theta,\, \psi) \; = \; -\frac{\theta}{\psi} \quad > 0 \,. \tag{8.4}$$

The determination of the dual function of the entropy is easy. We have to solve, for a given pair $(A,\, B)$, the system

$$\begin{cases} \gamma \, + \, \dfrac{\sigma(\theta,\, \psi)}{\theta} \; = \; A \\ \qquad -\dfrac{\theta}{\psi} \; = \; B \,. \end{cases} \tag{8.5}$$

Then, following the relation (7.10), we evaluate the "mechanical pressure" $\Pi(\bullet)$:

$$\Pi(\theta,\, \psi) \; = \; \frac{\sigma^*(A,\, B)}{A} \; = \; \frac{1}{A} \big(\theta \, A \, + \, \psi \, B \, - \, \sigma(\theta,\, \psi) \big)$$

$$= \theta + \frac{1}{\gamma + \sigma/\theta} (-\theta - \sigma) = \frac{(\gamma - 1)\,\theta}{\gamma + \sigma/\theta} \,,$$

$$(8.6) \qquad \Pi(\theta, \psi) = \frac{(\gamma - 1)\,\theta}{\gamma - \log\Big[\dfrac{(\gamma - 1)\,\psi}{\theta^\gamma}\Big]} \,.$$

Taking into account the relation (7.9), we set

$$(8.7) \qquad u(W) = \mathrm{arctg}\, \frac{\zeta}{\theta} \,, \qquad W = (\theta, \zeta, \psi)^{\rm t} \in \Omega$$

and the Cemracs system is a simple re-writing of the relation (7.8) :

$$(8.8) \qquad \left\{ \begin{array}{l} \dfrac{\partial}{\partial t} \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} + \dfrac{\partial}{\partial x} \left\{ u(W) \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} + \dfrac{\Pi\big(\sqrt{\theta^2 + \zeta^2}\,, \psi\big)}{\sqrt{\theta^2 + \zeta^2}} \begin{pmatrix} -\zeta \\ \theta \\ 0 \end{pmatrix} \right\} = 0 \,, \\ \qquad\qquad\qquad\qquad W = \begin{pmatrix} \theta \\ \zeta \\ \psi \end{pmatrix} . \end{array} \right.$$

Proposition 9. The Cemracs system is hyperbolic.
Under the hypotheses

$$(8.9) \qquad \theta > 0 \,, \qquad \psi > 0 \,, \qquad \Pi\big(\sqrt{\theta^2 + \zeta^2}\,, \psi\big) > 0 \,,$$

the system (8.6) (8.7) (8.8) of conservation laws is hyperbolic.

Proof of proposition 9.

• It is some kind of exercice for the studients. We do a direct proof of this proposition to improve the preceding assertions. The simplest way to do it is to consider the following new set of variables :

$$(8.10) \qquad V \equiv \Big(\theta \,, \xi \equiv \frac{\zeta}{\theta} \,, \varphi \equiv \frac{1}{\sqrt{1 + \xi^2}} \, \Pi(\theta \sqrt{1 + \xi^2}, \psi) \Big)^{\rm t} .$$

We have

$$(8.11) \qquad (1 + \xi^2)\, {\rm d}u = {\rm d}\xi = {\rm d}\Big(\frac{\zeta}{\theta}\Big) = -\frac{\zeta}{\theta^2}\, {\rm d}\theta + \frac{1}{\theta}\, {\rm d}\zeta$$

and

$$(8.12) \qquad \frac{\Pi(\theta \sqrt{1 + \xi^2}, \psi)}{\sqrt{1 + \xi^2}} = \frac{(\gamma - 1)\,\theta}{\gamma - \log\Big[\dfrac{(\gamma - 1)\,\psi}{\theta^\gamma\, (1 + \xi^2)^{\gamma/2}}\Big]} \equiv \varphi \,.$$

Then $\quad \dfrac{{\rm d}\varphi}{\varphi} = \dfrac{{\rm d}\theta}{\theta} - \dfrac{\varphi}{(\gamma - 1)\,\theta} \Big[-\dfrac{{\rm d}\psi}{\psi} + \gamma\, \dfrac{{\rm d}\theta}{\theta} + \dfrac{\gamma}{1 + \xi^2}\, \xi\, {\rm d}\xi \Big]$

$$(8.13)\qquad {\rm d}\varphi \,=\, \Big(1 - \frac{\gamma}{\gamma-1}\,\frac{\varphi}{\theta}\Big)\,\frac{\varphi}{\theta}\,{\rm d}\theta \,-\, \frac{\gamma}{\gamma-1}\,\frac{\varphi^2}{\theta}\,\frac{\xi}{1+\xi^2}\,{\rm d}\xi \,+\, \frac{\varphi^2}{(\gamma-1)\,\theta}\,\frac{{\rm d}\psi}{\psi}\,.$$

- We write again the two first equations of the system (8.8) :

$$(8.14)\qquad \frac{\partial\theta}{\partial t} + u\,\frac{\partial\theta}{\partial x} + \theta\,\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(\varphi\,\xi) \,=\, 0$$

$$(8.15)\qquad \frac{\partial\zeta}{\partial t} + u\,\frac{\partial\zeta}{\partial x} + \zeta\,\frac{\partial u}{\partial x} + \frac{\partial\varphi}{\partial x} \,=\, 0\,.$$

We multiply the equation (8.14) by $-\zeta/\theta^2 = -\xi/\theta$ and the equation (8.15) by $1/\theta$. Thanks to the relation (8.11), we have the following evolution in time of the variable ξ :

$$(8.16)\qquad \frac{\partial\xi}{\partial t} + u\,\frac{\partial\xi}{\partial x} + \frac{\varphi\,\xi}{\theta}\,\frac{\partial\xi}{\partial x} + \frac{1+\xi^2}{\theta}\,\frac{\partial\varphi}{\partial x} \,=\, 0\,.$$

The equation for the variable φ is composed from the relations (8.14), (8.16), the third equation of the system (8.8) that is :

$$(8.17)\qquad \frac{\partial\psi}{\partial t} + u\,\frac{\partial\psi}{\partial x} + \psi\,\frac{\partial u}{\partial x} \,=\, 0$$

and from the relation (8.13). We set

$$(8.18)\qquad y \,\equiv\, \frac{\varphi}{\theta}\,;$$

we multiply the equation (8.14) by the coefficient $\big(1-\frac{\gamma}{\gamma-1}\,\frac{\varphi}{\theta}\big)\,\frac{\varphi}{\theta}$, the equation (8.16) by $-\frac{\gamma}{\gamma-1}\,\frac{\varphi^2}{\theta}\,\frac{\xi}{(1+\xi^2)}$, the equation (8.17) by $\frac{\varphi^2}{(\gamma-1)\,\theta\,\psi}$ and we add these three equations. The $\frac{\partial\theta}{\partial x}$ term is absent, the term associated to $\frac{\partial\xi}{\partial x}$ is equal to

$$\theta\,\Big[\,\Big(\frac{1}{1+\xi^2} - y\Big)\,\Big(1 - \frac{\gamma}{\gamma-1}\,y\Big)\,y \,-\, \frac{\gamma}{\gamma-1}\,y^3\,\frac{\xi^2}{1+\xi^2} \,+\, \frac{1}{\gamma-1}\,\frac{y^2}{1+\xi^2}\,\Big]$$
$$=\, \frac{\theta}{1+\xi^2}\,\Big(y \,-\, (2+\xi^2)\,y^2 \,+\, \frac{\gamma}{\gamma-1}\,y^3\Big)$$

and the coefficient of the term relatively to $\frac{\partial\varphi}{\partial x}$ is :

$$-\xi\,y\,\Big(1 - \frac{\gamma}{\gamma-1}\,y\Big) \,-\, \frac{\gamma}{\gamma-1}\xi\,y^2 \,=\, -\xi\,y\,.$$

We deduce from this calculus that with the variables V introduced in (8.10), the system (8.8) takes the form

$$(8.19)\qquad \frac{\partial V}{\partial t} + u\,\frac{\partial V}{\partial x} + B(V)\,\frac{\partial V}{\partial x} \;=\; 0$$

with

$$(8.20)\qquad B(V) \;=\; \begin{pmatrix} 0 & \dfrac{\theta}{1+\xi^2} - \varphi & -\xi \\ 0 & \xi\, y & \dfrac{1+\xi^2}{\theta} \\ 0 & \dfrac{\theta}{1+\xi^2}\,\Big(y \,-\, (2+\xi^2)\, y^2 \,+\, \dfrac{\gamma}{\gamma-1}\, y^3 \Big) & -\xi\, y \end{pmatrix}.$$

• We see in an clear way that the real number zero is an eigenvalue of the matrix $B(V)$ of the relation (8.20). The two other eigenvalues λ satisfy the equation :

$$(8.21)\qquad \lambda^2 + \Big[-y \,+\, 2\,y^2 \,-\, \frac{\gamma}{\gamma-1}\, y^3 \Big] \;=\; 0\,.$$

The equation (8.21) has two opposite real eigenvalues when $y > 0$ as proposed in our study [Du99]. We have effectively in this case :

$$-y \,+\, 2\,y^2 \,-\, \frac{\gamma}{\gamma-1}\, y^3 \;=\; -y\,(1-y)^2 \,-\, \frac{1}{\gamma-1}\, y^3 \;<\; 0$$

and the proposition 9 is established. □

Remark 5. Curious Physics.

The condition $y > 0$ *i.e.* $\varphi/\theta > 0$ or in an equivalent way $\Pi > 0$ because $\theta > 0$ by convention. It corresponds to

$$(8.22)\qquad \Pi \;\equiv\; \frac{p}{\mu} \;>\; 0$$

because as claimed in the relation (7.22) : $\mathrm{d}\sigma \;=\; \frac{\mu}{T}\,\mathrm{d}\theta \,-\, \frac{1}{T}\,\mathrm{d}\psi$. Then $\Pi \;=\; \frac{1}{\mu/T}\,\frac{p}{T} \;=\; \frac{p}{\mu}$ and the mechanical pressure $\Pi(\bullet)$ is the quotient of the physical pressure $p(\bullet)$ divided by the chemical potential μ which is **not** natural. It is easier to interpret the Cemracs system (8.8) as an elliptic Galieo system of conservation laws whose associated thermostatics is obtained by **exchanging** the roles of the variables θ and $\psi \equiv \rho\, e$ inside the relation (8.2). This corresponds to :

$$(8.23)\qquad \theta \;=\; \rho\, e\,, \qquad \psi \;=\; \rho\,.$$

Then the relation $\mathrm{d}\sigma \;=\; -\dfrac{1}{T}\,\theta + \dfrac{\mu}{T}\,\mathrm{d}\psi$ and the condition (7.7) can be written

$$(8.24)\qquad \frac{\partial \sigma}{\partial \theta} \;=\; -\frac{1}{T} \;>\; 0$$

corresponding to a thermostatics system with a negative temperature and associated with a thermostatic pressure p equal to the mechanical pressure Π and given by the formula

$$\Pi\,(\rho\, e,\, \rho) \;=\; \frac{(\gamma-1)\,\rho\, e}{\gamma \,-\, \log\Big[\dfrac{(\gamma-1)\,\rho}{(\rho\, e)^{\gamma}}\Big]}\;. \tag{8.25}$$

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