Stable lattice Boltzmann scheme for a moving Burgers shock wave

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- An hyperbolic partial differential equation like the Burgers equation

$$(1) \qquad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big(f(u)\big) \,=\, 0\,, \quad f(u) \,\equiv\, \frac{u^2}{2}$$

exhibits shock waves [5], *id est* discontinuities propagating with finite velocity. In order to select the physically relevant weak solution, it is necessary to enforce the so-called entropy condition

$$(2) \qquad \frac{\partial}{\partial t}\big(\eta(u)\big) \,+\, \frac{\partial}{\partial x}\big(\zeta(u)\big) \,\leq\, 0$$

as suggested by Friedrichs and Lax [4]. In the relation (2), $\eta(\bullet)$ is a strictly convex function and $\zeta(\bullet)$ the associated entropy flux (see [3] or [5]). For the Burgers equation, we consider usually the quadratic entropy : $\eta(u) \equiv \frac{u^2}{2}$, $\zeta(u) \equiv \frac{u^3}{3}$.

- When we use a lattice Boltzmann scheme like the so-called D1Q3 scheme with velocities $v_j \in \{-\lambda\,,\,0\,,\,\lambda\}$, we replace the "macroscopic" hyperbolic equation (1) by a system of "microscopic" Boltzmann equations : $\frac{\partial f_j}{\partial t} + v_j \frac{\partial f_j}{\partial x} = Q_j(f)$, with $Q_j(f)$ related to an "equilibrium" value $f_j^{\rm eq}$ of BGK type : $Q_j(f) = \frac{1}{\tau}\big(f_j^{\rm eq} - f_j\big)$. The best situation occurs when dissipation can be established with a "microscopic entropy" $H(f) \equiv \sum_j h_j(f_j)$ and the so-called "H-Theorem" : $\frac{\partial}{\partial t}\big(\sum_j h_j(f_j)\big) + \frac{\partial}{\partial x}\big(\sum_j v_j\, h_j(f_j)\big) \leq 0$.

- With the approach proposed by Karlin [6], the choice $H(f) \equiv \sum_j h_j(f_j)$ allows the determination of the equilibrium functions $f_j^{\rm eq}$ according to a minimization process. A natural question concerns the link between the microscopic entropy $H(\bullet)$ and the macroscopic entropy $\eta(\bullet)$. This question has been studied by F. Bouchut [1] in the context of finite volumes methods. The result for the Burgers equation can be stated as follows with the choice of the quadratic entropy done in this contribution. Introduce first the so-called "entropy variables" $\varphi \equiv {\rm d}\eta(u)$ (see *e.g.* [3]), the Legendre-Fenchel-Moreau dual function $\eta^*(\varphi) \equiv \varphi\, u - \eta(u)$ and the "dual entropy flux" $\zeta^*(\varphi) \equiv \varphi\, f(u) - \zeta(u)$. Following [1], if there exists **convex** functions $h_j^*(\varphi)$ of the entropy variable φ such that

$$(3) \qquad \sum_j h_j^*(\varphi) \,\equiv\, \eta^*(\varphi) \,=\, \frac{\varphi^2}{2}\,, \quad \sum_j v_j\, h_j^*(\varphi) \,\equiv\, \zeta^*(\varphi) \,=\, \frac{\varphi^3}{6}\,,$$

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then the equilibrium $f_j^{\rm eq}(u) \equiv \dfrac{{\rm d}h_j^*}{{\rm d}\varphi}$ defines a **stable** approximation.

• In this contribution, we consider two examples of stable equilibria in the context of lattice Boltzmann scheme. More precisely, following the approach proposed by d'Humières [2], we introduce a matrix M that links particle densities f_j $(j = -1, 0, 1)$ and momenta m_k:

$$(4) \qquad m \equiv M \bullet f \,, \quad M = \begin{pmatrix} 1 & 1 & 1 \\ -\lambda & 0 & \lambda \\ \lambda^2 & 0 & \lambda^2 \end{pmatrix} , \quad u \equiv f_{-1} + f_0 + f_1 = m_1 \,, \quad \lambda = \frac{\Delta x}{\Delta t} \,.$$

We propose a first equilibrium distribution under the form : $m^{\rm eq,1} \equiv \big(u \,,\, \frac{u^2}{2} \,,\, \lambda \, {\rm sgn}(u) \, \frac{u^2}{2} \big)^{\rm t}$. Then it is possible to re-construct the dual-entropy η^* and a dual entropy flux ζ^* in order to satisfy (3). The corresponding h_j^* functions are convex if the Courant-Friedrichs-Lewy condition $\mid u \mid \leq \lambda$ is satisfied. When using the other algebraic form $m^{\rm eq,2} \equiv \big(u \,,\, \frac{u^2}{2} \,,\, \frac{\lambda^2}{2} \, u \big)^{\rm t}$, the stability constraint takes the form $\mid u \mid \leq \frac{\lambda}{2}$. Then the relaxation step is nonlinear and local in space : $m_1^* = m_1^{\rm eq} = u$, $m_k^* = m_k + s_k \, (m_k^{\rm eq} - m_k)$ for $k \geq 2$, with $s_2 = s_3 = 1.8$ in our simulations. The particle distribution f_j^* after relaxation is obtained by inversion of relation (4) : $f^* = M^{-1} \bullet \, m^*$. The time iteration of the scheme follows the characteristic directions of velocity v_j : $f_j(x,\, t+\Delta t) = f_j^*(x - v_j \, \Delta t,\, t)$.

• We have tested these two numerical schemes for the Burgers equation with the initial condition $u_0(x) = 1$ for $x \leq 0$, $u_0(x) = 1 - x$ for $0 \leq x \leq 1$, $u_0(x) = 0$ for $x \geq 1$, which exhibits a focalizing shock wave for time $t \geq 1$. The results will be presented at the Icmmes meeting.

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