A Non-Parameterized Entropy Correction For Roe's Approximate Riemann Solver

François DUBOIS * and Guillaume MEHLMAN * †

* Aerospatiale, Division Systèmes Stratégiques et Spatiaux
Département Mathématiques Appliquées
78133 Les Mureaux Cedex, France

† Ecole Polytechnique
Centre de Mathématiques Appliquées
91128 Palaiseau Cedex, France

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Abstract

The main drawback with Roe's approximate Riemann solver is that non-physical expansion shocks can occur in the vicinity of sonic points. Previous work aimed at enforcing the entropy condition is based on the representation of sonic rarefaction waves. We propose a new non-parameterized approach which is based on a nonlinear Hermite interpolation of an approximate flux function and the exact resolution of non convex scalar Riemann problems. Convergence and consistency with the entropy condition are proved for scalar convex conservation laws with arbitrarily large initial data. When considering strictly hyperbolic systems of conservation laws, consistency of the resulting scheme with the entropy condition is also proved for initial data sufficiently close to a constant. Numerical results on a one-dimensional shock-tube and a two-dimensional supersonic forward facing step confirm our theoretical results.

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1 Introduction

In this paper, we consider numerical solutions of the initial-value problem for hyperbolic systems of conservation laws

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 . \tag{1.1}$$

Here, $U(x,t)$ is a column vector of m unknowns and $F(U)$, the flux, is a vector-valued function of m components. To allow for discontinuous solutions we admit weak solutions that satisfy (1.1) in the sense of distributions, i.e.,

$$\int_0^\infty \int_{-\infty}^\infty \left[\frac{\partial \phi}{\partial t} U + \frac{\partial \phi}{\partial x} F(U)\right] dxdt + \int_{-\infty}^\infty \phi(x,0) U_0(x) dx = 0 ,$$

for all C^∞ test functions $\phi(x,t)$ that vanish for $|x|+t$ large.

Since weak solutions of (1.1) are not uniquely determined by their initial data, we select physically relevant solutions, defined as those solutions that are limits as $\varepsilon \to 0$ of solutions $U(\varepsilon)$ of the viscous equations

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = \varepsilon \frac{\partial^2 U}{\partial x^2} , \quad \varepsilon > 0 . \tag{1.2}$$

In the sequel, we consider systems of conservation laws that possess an *entropy function* $\eta(U)$ (see Friedrichs and Lax [6]), defined as follows :

- η is a convex function of U, i.e., $d^2\eta(U) > 0$.
- η satisfies
$$d\eta(U) \cdot dF(U) = dq(U) ,$$

where q is a function called *entropy flux*. It follows that every smooth solution of (1.1) also satisfies

$$\frac{\partial \eta(U)}{\partial t} + \frac{\partial q(U)}{\partial x} = 0 .$$

It is well known that limit solutions of (1.2) satisfy, in the weak sense, the following inequality :

$$\frac{\partial \eta(U)}{\partial t} + \frac{\partial q(U)}{\partial x} \le 0 ; \tag{1.3}$$

i.e., for all nonnegative smooth test functions $\phi(x,t)$ of compact support

$$-\int_0^\infty \int_{-\infty}^\infty [\phi_t \eta + \phi_x q]\, dxdt - \int_{-\infty}^\infty \phi(x,0)\eta(U_0(x)) dx \le 0 .$$

In the following, we shall describe numerical approximations to weak solutions of (1.1) that are obtained by solving the following system of ordinary differential equations (method of lines) :

$$\frac{dU_j^h}{dt} + \frac{1}{h}\left(\Phi(U_j^h, U_{j+1}^h) - \Phi(U_{j-1}^h, U_j^h)\right) = 0 , \tag{1.4}$$

with the initial data

$$U_j^h(0) = \frac{1}{h}\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} U^0(x) dx , \tag{1.5}$$

where h is the mesh step.

The Godunov scheme [7] for the general hyperbolic system of conservation laws (1.1) consists in replacing the flux Φ by the exact flux obtained by solving the Riemann problem at each cell interface and at each time step. It is evident that, due to averaging, all of the information contained in the exact solution of each problem is not retained. Therefore, the exact solution of local Riemann problems is replaced by an approximation (see Harten, Lax and Van Leer [11]).

Roe [23] initially proposed a purely algebraic approximate Riemann solver for the Euler system of equations governing the flow of a thermally perfect inviscid fluid. A similar course of reasoning may be followed for any system of conservation laws. The main drawback with this method is that nonphysical expansion shocks can occur in the vicinity of sonic points. Nevertheless, Roe's flux is very popular (see Yee [31], Hollanders and Marmignon [12],and Chakravarthy [2] amongst others). In the following we restrict ourselves to this particular choice for the numerical flux Φ of (1.4).

In the next section, we briefly recall Roe's method in general and introduce some notations which will be useful. In Section 3, we will review two entropy corrections due to Harten [9] and Roe [24] which have been used in order to cope with non-physical solutions. Both approaches consist in modeling the approximate solution in situations where the linearized Roe flux violates the entropy condition. These corrections appear more as "spreading devices" which act upon the approximate solution rather than a remedy to the fact that the approximate flux is linear in situations where a linear description is not relevant. Our approach will be described in Section 4. It is based on a nonlinear modification of the flux in the vicinity of sonic points. The exact solution of the Riemann problem associated with this modified flux function is determined algebraically. In Sections 5 and 6, we will prove that, in the scalar case, this modified Roe scheme converges to the unique entropy solution of (1.1) and that for general hyperbolic systems, limit solutions of (1.4) (1.5) satisfy the entropy inequality (1.3). In the last section, numerical results are presented.

2 Roe's Method

Roe's method consists in replacing the solution of (1.1) at each cell interface by the exact solution of the following linear problem :

$$\begin{cases} \dfrac{\partial U}{\partial t} + \dfrac{\partial F^R_{j+\frac{1}{2}}(U)}{\partial x} = 0 \\ U(x,0) = \begin{cases} U_j\,, & x < x_{j+\frac{1}{2}}\,, \\ U_{j+1}\,, & x > x_{j+\frac{1}{2}}\,, \end{cases} \end{cases} \tag{2.6}$$

where $x_{j+\frac{1}{2}} = (j+\frac{1}{2})h$ is the point at the cell interface. The linearized flux function $F^R_{j+\frac{1}{2}}$ is defined by

$$F^R_{j+\frac{1}{2}}(U) = F(U_j) + A(U_j, U_{j+1}) \cdot (U - U_j) \tag{2.7}$$

where $A(U_j, U_{j+1})$ is an approximation of the jacobian $A(U) = dF(U)$.

The matrix $A(U_l, U_r)$ is called a *Roe-type linearization* and is required to have the following properties [23]:

$$F(U_r) - F(U_l) = A(U_l, U_r) \cdot (U_r - U_l) \tag{2.8}$$

$$A(U,U) = dF(U) \tag{2.9}$$

$$A(U_l, U_r) \quad \textit{has real eigenvalues and a complete set of eigenvectors.} \tag{2.10}$$

Let $r_j(U)$ and $\lambda_j(U)$ denote the eigenvectors and associated eigenvalues of the jacobian $dF(U)$. Similarly, let $R_j(U_l, U_r)$ and $\lambda_j(U_l, U_r)$ denote the eigenvectors and associated eigenvalues of the matrix $A(U_l, U_r)$. Then, if

$$U_r - U_l \quad = \quad \sum_{j=1}^{m} \alpha_j \, R_j(U_l, U_r) \, , \tag{2.11}$$

it is easy to verify that Roe's numerical flux has the following expression :

$$\begin{aligned} \Phi^R(U_l, U_r) &= F(U_l) + \sum_{j=1}^{m} \lambda_j(U_l, U_r)^- \, \alpha_j \, R_j(U_l, U_r) \\ &= F(U_r) - \sum_{j=1}^{m} \lambda_j(U_l, U_r)^+ \, \alpha_j \, R_j(U_l, U_r) \end{aligned}$$

where

$$\lambda^\pm = \pm \, max(\, 0 \, ; \, \pm \, \lambda \,) \, .$$

With obvious notations, we have

$$\Phi^R(U_l, U_r) = F(U_l) \, + \, A^-(U_l, U_r) \cdot (U_r - U_l) = F(U_r) \, - \, A^+(U_l, U_r) \cdot (U_r - U_l) \, .$$

For general hyperbolic systems, the existence of a mathematical entropy ensures the existence of a Roe-type linearization (see Harten, Lax, Van Leer [11]). It is a well known fact that such a linearization is not unique and that Roe's scheme does not satisfy the entropy inequality. This is one of the serious drawbacks of Roe's method.

Indeed, let us consider a scalar conservation law. We assume that the flux function F is strictly convex. If we consider two states U_l and U_r such that

$$\begin{aligned} U_l &\le U_r \, , \\ F(U_l) &= F(U_r) \, , \end{aligned}$$

then, it is obvious that

$$\Phi^R(U_l, U_r) \; = \; F(U_l) \; = \; F(U_r) \, ,$$

and that the jump between U_l and U_r will not spread into a rarefaction fan as it should.

3 Coping with Non-Physical Solutions

We review in this section two entropy corrections for Roe's method. The first one has been proposed by Harten and Hyman [10] and the other by Roe himself [24].

Harten's Entropy Correction

By rewriting the Roe flux in a centered manner, the numerical viscosity associated with this scheme appears

$$\Phi^R(U_l, U_r) = \frac{1}{2} \left(F(U_l) + F(U_r) - \sum_{j=1}^{m} Q^R(\lambda_j(U_l, U_r)) \, \alpha_j \, R_j(U_l, U_r) \right)$$

where

$$Q^R(\lambda) = |\lambda| \, .$$

The previous analysis shows that Roe's scheme is not viscous enough in the vicinity of sonic points. A very popular correction due to Harten [9] consists in modifying the numerical viscosity by replacing, for $|\lambda| \leq \delta$, $Q^R(\lambda)$ by

$$Q^H(\lambda) = \frac{1}{2}\left(\frac{\lambda^2+\delta^2}{\delta}\right) . \tag{3.12}$$

The parameter δ measures the amount of artificial viscosity which is added. The actual tuning of δ crucially affects convergence towards the correct solution and depends very much on the problem considered (see the numerical experience of Yee [31] or Hollanders and Marmignon [12]). Moreover, when considering the Euler equations, extensions to viscous flows are quite delicate.

In fact, a more rigorous form of Harten's correction (see [10]) gives clear indications on how to modify Q^R. Denote by $W = (w_j)_{j=1,\ldots,m}$ the characteristic variables of the increment state $U - U_l$:

$$U - U_l = \sum_{j=1}^{m} w_j R_j(U_l, U_r) . \tag{3.13}$$

The linearized Riemann problem (2.6) decouples into m scalar Riemann problems for the characteristic variables

$$\begin{cases} \dfrac{\partial w_j}{\partial t} + \lambda_j(U_l,U_r)\cdot\dfrac{\partial w_j}{\partial x} = 0 \\ w_j(x,0) = \begin{cases} 0\,, & x<0\,, \\ \alpha_j\,, & x>0\,, \end{cases} \end{cases}$$

the solution to which is

$$w_j(x,t) = \begin{cases} 0\,, & \dfrac{x}{t} < \lambda_j(U_l,U_r)\,, \\ \alpha_j\,, & \dfrac{x}{t} > \lambda_j(U_l,U_r)\,. \end{cases} \tag{3.14}$$

Thus, each j-wave in the Riemann problem is approximated by a jump discontinuity that propagates with a speed $\lambda_j(U_l, U_r)$. This is a reasonable approximation when the j-wave is a shock or a contact discontinuity. When the j-wave is a rarefaction, Harten modifies approximation (3.14) by introducing an intermediate state α_j^* :

$$w_j(x,t) = \begin{cases} 0\,, & \dfrac{x}{t} < \lambda_j^l\,, \\ \alpha_j^*\,, & \lambda_j^l < \dfrac{x}{t} < \lambda_j^r\,, \\ \alpha_j\,, & \lambda_j^r < \dfrac{x}{t}\,, \end{cases}$$

where

$$\lambda_j^l = \lambda_j(U_l,U_r) - \delta_j, \quad \lambda_j^r = \lambda_j(U_l,U_r) + \delta_j, \quad \text{and} \quad \delta_j \geq 0. \tag{3.15}$$

Writing conservation out in full for this approximate Riemann solver yields the expression of the numerical flux :

$$\Phi^{HH}(U_l,U_r) = \frac{1}{2}\left(F(U_l) + F(U_r) - \sum_{j=1}^{m} Q_j^{HH}(\lambda_j(U_l,U_r))\,\alpha_j\,R_j(U_l,U_r)\right) ,$$

with the numerical viscosity :

$$Q_j^{HH}(\lambda) = \begin{cases} |\lambda|\,, & |\lambda| > \delta_j, \\ \delta_j\,, & |\lambda| \leq \delta_j. \end{cases}$$

In the scalar case, the theoretical choice

$$\delta = \max_{0\leq\theta\leq 1} \left(0\,;\,\lambda(U_l,U_r)-\lambda(U_l,U_l+\theta(U_r-U_l))\,;\,\lambda(U_l+\theta(U_r-U_l),U_r)-\lambda(U_l,U_r)\right)$$

ensures entropy consistency of the approximate Riemann solver (see [10]). Consistency of the modified Roe-scheme follows from the Harten-Lax theorem [8]. This choice of δ gives left and right wave speeds in (3.15) that are exactly those given by Oleinik's entropy condition [19]. Note that δ_j is to vanish if and only if the j-wave is a shock. The modification of the flux is only active in rarefaction waves.

This scheme, which reduces to Harten's "one state solver" [11] when the wave considered is a rarefaction, appears to be quite viscous. According to Roe [24], Harten has abandoned this more rigorous approach to entropy enforcement in favor of the empirical device mentioned previously which gives better resolution for the Euler equations.

Roe's Entropy Correction

In [24], Roe proposes a more selective correction which is only active when a sonic rarefaction is detected. Consider a scalar conservation law and assume that the flux function F is strictly convex. Then, Roe's scheme reduces to

$$\Phi^R(U_l,U_r) = \frac{1}{2}\left(F(U_l)+F(U_r)-|\lambda|\,(U_r-U_l)\right)$$

where

$$\lambda = \frac{F(U_r)-F(U_l)}{U_r-U_l}\,.$$

Sonic rarefactions are characterized by

$$F'(U_l) \quad < \quad 0 \quad < \quad F'(U_r)\,.$$

Roe's correction is based on a modelization of sonic rarefaction waves : these waves are assumed to be not centered but **distributed**. A modified flux based on a model of sonic expansion waves has also been recently proposed by Van Leer, Lee and Powell [26]. Both the solution and the characteristic field are assumed to vary linearly through the rarefaction. By introducing the concept - which is not very familiar to the authors of the present paper - of fluctuation functions (see [24]), Roe finds the following modified flux function :

$$\begin{aligned}\Phi^M(U_l,U_r) &= \frac{1}{2}(F(U_l)+F(U_r)) - \frac{1}{4}\left(F'(U_r)-F'(U_l)\right)\,(U_r-U_l)\\ &= \frac{1}{2}(F(U_l)+F(U_r)) - \frac{1}{4}\,\delta\,\frac{\phi}{\lambda}\end{aligned} \tag{3.16}$$

where

$$\delta = F'(U_r)-F'(U_l)\,,$$

and

$$\phi = F(U_r)-F(U_l)\,.$$

As suggested by Roe [24], this numerical flux can be determined through purely geometrical arguments (see figure 1). In fact, the coefficient $\frac{1}{4}$ in the right hand side of (3.16) is sufficient to ensure that, when considering convex flux functions,

$$\operatorname{sign}(U_r-U_l)\,\left(\Phi^M(U_l,U_r)-f(U)\right) \leq 0\,,$$

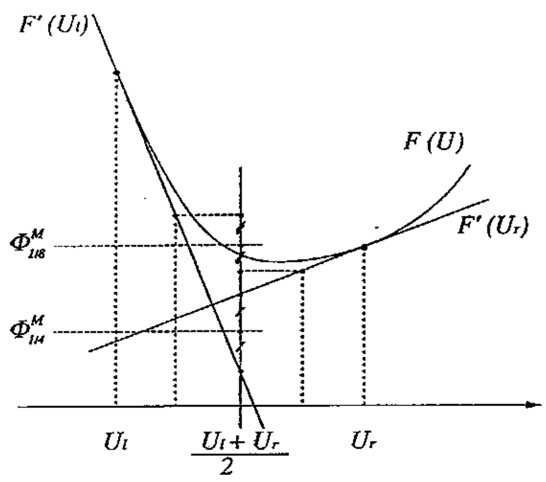

Figure 1: Geometric interpretation of the modified fluxes (3.16) and (3.17).

for all U between U_l and U_r. Roe thus proves that the resulting scheme is an E-scheme (see Osher [20]). In the litterature and in practice, it appears that this procedure gives satisfactory results for one-dimensional numerical experiments (see Chakravarthy [2]).

If we actually explicit the assumptions of a linearly distributed sonic rarefaction wave in the following manner :

$$\frac{\partial U}{\partial x} = \frac{\Delta U}{\Delta x} = \frac{\phi}{\lambda \Delta x}$$
$$F(\xi) = F(U_l) + \phi\xi + \xi(1-\xi)A ,$$

where

$$\xi = \frac{x - x_l}{\Delta x},$$

we find that

$$\frac{dF}{d\xi} = \frac{dF}{dU}\frac{dU}{d\xi} = \frac{\partial F}{\partial U}\frac{\phi}{\lambda} .$$

This leads to

$$\frac{dF}{d\xi}(0) = \phi + A = F'(U_l)\,\frac{\phi}{\lambda} ,$$
$$\frac{dF}{d\xi}(1) = \phi - A = F'(U_r)\,\frac{\phi}{\lambda} ,$$

and

$$-2A = \delta\frac{\phi}{\lambda} .$$

Finally, we obtain a **different** modified numerical flux

$$\Phi(U_l, U_r) = \frac{1}{2}(F(U_l) + F(U_r)) - \frac{1}{8}\delta\frac{\phi}{\lambda} \tag{3.17}$$

which does **not** define an E-scheme for a general convex flux function (see figure1).

The main drawback with Roe's approach is that the "switching" from Roe's flux to the modified Roe flux is not regular. The numerical flux is **not** a continuous function of its arguments in the vicinity of sonic points. From a theoretical point of view, it is often essential for the numerical flux to be at least Lipschitz-continuous. In particular, the scheme (1.4) (1.5) which defines the method of lines is ill-posed if the numerical flux is not continuous. Nevertheless, there seems to be no evidence to support the fact that this loss of continuity may hinder numerical performance.

4 A New Modified Roe-Scheme

There are several objections to the "spreading devices" mentionned in the previous section. In both examples, the underlying idea is to give an a priori representation of the solution. We present below a new approach [4] based on a nonlinear modification of the flux function. Since problems occur at sonic points, we decide to modify F^R of (2.7) in problem (2.6) only at sonic points. We recall that the linearized flux function has the expression :

$$F^R(U_l, U_r, U) = F(U_l) + A(U_l, U_r) \cdot (U - U_l) \, .$$

Definition 4.1 (Modified Flux)

We define m intermediate states :

$$\begin{aligned} U_0 &= U_l \\ &\vdots \\ U_j &= U_{j-1} + \alpha_j R_j(U_l, U_r) \\ &\vdots \\ U_m &= U_r \, . \end{aligned}$$

Let S be the set of sonic indices

$$S = \{i \, , \; \lambda_i(U_{i-1}) < 0 < \lambda_i(U_i)\} \, .$$

We introduce the following modified flux function parameterized by U_l and U_r :

$$F^{DM}(U_l, U_r, U) = F(U_l) + \sum_{j=1}^{m} g_j(w_j) \, R_j(U_l, U_r) \, ,$$

where the w_js are the characteristic variables defined in (3.13). The g_js are parameterized by the states $(U_j)_{j=1,\dots,m}$ and are defined according to :

if $j \notin S$,

$$\forall w, \quad g_j(w) = \lambda_j(U_l, U_r) \cdot w \, ,$$

if $j \in S$,

$$g_j(w) = \begin{cases} p_j(w) \quad , & \forall w \in (0, \alpha_j) \, , \\ \lambda_j(U_l, U_r) \cdot w \quad , & \forall w \notin (0, \alpha_j), \end{cases}$$

where p_j is the unique Hermite polynomial of degree 3 defined by the conditions :

$$p_j(0) = 0 \, , \quad p_j(\alpha_j) = \lambda_j(U_l, U_r)\alpha_j \, , \quad p_j'(0) = \lambda_j(U_{j-1}) \, , \quad p_j'(\alpha_j) = \lambda_j(U_j) \, .$$

Note that $\lambda_j(U_{j-1})$ and $\lambda_j(U_j)$ are the true eigenvalues of the physical flux at the intermediate states U_j given by the Roe-matrix $A(U_l, U_r)$. Away from sonic points, F^{DM} coincides with the linearized Roe flux F^R. If the initial flux F in (1.1) is at least of class C^1, and if the matrix $A(U_l, U_r)$ is continuous with respect to U_l and U_r, then the modified flux F^{DM} is a continuous function of all three arguments.

Proposition 4.2 (Solution of the Modified Riemann Problem)

The Riemann problem

$$\left\{ \begin{array}{l} \dfrac{\partial V}{\partial t} + \dfrac{\partial F^{DM}(U_l, U_r, V)}{\partial x} = 0 \\ V(x,0) = \left\{ \begin{array}{ll} V_l\,, & x < 0\,, \\ V_r\,, & x > 0\,, \end{array} \right. \end{array} \right. \tag{4.18}$$

has a unique entropy solution.

Proof : By considering the family of mathematical entropies

$$\eta = \sum_{j=1}^{m} \eta_j(w_j)$$

where η_j is any scalar convex function, it is clear that the solution of (4.18) is the superposition

$$V(x,t) = \sum_{j=1}^{m} Z_j(\frac{x}{t}) \cdot R_j(U_l, U_r)$$

where $Z_j(\frac{x}{t})$ is the unique entropy solution (see Oleinik [19], Ballou [1], and Leroux [15]) of the scalar Riemann problem

$$\left\{ \begin{array}{l} \dfrac{\partial Z}{\partial t} + \dfrac{\partial g_j(Z)}{\partial x} = 0 \\ Z(x,0) = \left\{ \begin{array}{ll} w_j^l\,, & x < 0\,, \\ w_j^r\,, & x > 0\,. \end{array} \right. \end{array} \right. \tag{4.19}$$

□

Following [11], a Godunov-type scheme associated with this modified flux can naturally be defined :

Definition 4.3 (Modified Numerical Flux)

Let $V_{l,r}$ be the exact solution of (4.18) associated with the initial data

$$\left\{ \begin{array}{l} V_l = 0\,, \\ V_r = U_r - U_l\,, \end{array} \right.$$

we define the numerical flux (which is classically well-defined independently of t)

$$\Phi^{DM}(U_l, U_r) = F^{DM}(U_l, U_r, V_{l,r}(0,t))$$

We now show that Φ^{DM} can be derived in a straightforward manner.

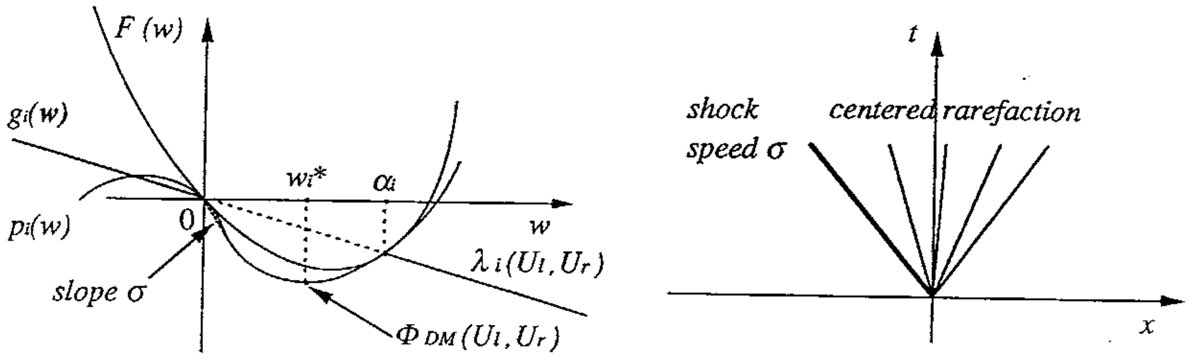

Figure 2: Exact solution of the scalar Riemann problem (4.19).

5

Proposition 4.4 (Expression of the Numerical Flux)

If $i \in S$, the Hermite polynomial $p_i(w)$ in (4.1) is defined by

$$p_i(w) = aw^3 + bw^2 + cw \ ,$$

with

$$\begin{cases} a = \dfrac{\lambda_i(U_i) + \lambda_i(U_{i-1}) - 2\lambda_i(U_l, U_r)}{\alpha_i^2} \ , \\ b = \dfrac{3\lambda_i(U_l, U_r) - 2\lambda_i(U_{i-1}) - \lambda_i(U_i)}{\alpha_i} \ , \\ c = \lambda_i(U_{i-1}) \ . \end{cases}$$

The modified numerical flux has the expression

$$\Phi^{DM}(U_l, U_r) = F(U_l) + \sum_{i \notin S,\ \lambda_i(U_l,U_r)<0} \lambda_i(U_l, U_r)\alpha_i \, R_i(U_l, U_r) + \sum_{i \in S} g_i(w_i^*) \, R_i(U_l, U_r)$$

where

$$w_i^* = \frac{-\lambda_i(U_{i-1}) \cdot \alpha_i}{3\lambda_i(U_l, U_r) - 2\lambda_i(U_{i-1}) - \lambda_i(U_i) + \sqrt{(3\lambda_i(U_l, U_r) - \lambda_i(U_i) - \lambda_i(U_{i-1}))^2 - \lambda_i(U_{i-1})\lambda_i(U_i)}}$$

is the argument of the unique extremum of g_i between 0 and α_i (see figure 2).

Proof : The conditions $g_i'(0) < 0$, $g_i'(\alpha_i) > 0$, and the fact that g_i is a third order polynomial ensure that there is a unique extremum between 0 and α_i. The exact solution of Riemann problem (4.19) is obtained by considering the lower convex hull of g_i on $[0, \alpha_i]$ if $\alpha_i > 0$, or the upper convex hull on $[\alpha_i, 0]$ if $\alpha_i < 0$ (see [19], [1], and [15]). The solution of scalar problem (4.19) contains a unique centered rarefaction wave and at most one shock (see figure 2). The point "$\frac{x}{t} = 0$" lies inside the rarefaction wave for which the integral equation is :

$$g_i'\left(Z_i(\frac{x}{t})\right) = \frac{x}{t} \ .$$

The exact expressions of the polynomial p_i and of w_i^* are obtained through elementary algebra.

□

Remark 4.5

Since $g_i(w_i^*)$ is the unique extremum of the polynomial p_i on the interval $(0, \alpha_i)$, we have

$$\begin{cases} \dfrac{g_i(w_i^*)}{\alpha_i} \leq 0 \,, \\ \dfrac{g_i(w_i^*)}{\alpha_i} \leq \lambda_i(U_l, U_r) \,. \end{cases}$$

It is easy to see that our numerical flux can be written in the centered form :

$$\begin{aligned} \Phi^{DM}(U_l, U_r) &= \Phi^R(U_l, U_r) + \sum_{i \in S} \sup\left(\frac{g_i(w_i^*)}{\alpha_i}; \frac{g_i(w_i^*)}{\alpha_i} - \lambda_i(U_l, U_r)\right) \alpha_i R_i(U_l, U_r) \\ &= \Phi^R(U_l, U_r) + \sum_{i \in S} (g_i(w_i^*) - \lambda_i(U_l, U_r)^- \alpha_i) R_i(U_l, U_r) \,, \end{aligned}$$

which makes the added numerical viscosity explicit.

5 Study of the Scalar Case

For convex scalar conservation laws, we show that the method of lines associated with our modification of the Roe scheme converges to the unique entropy solution (see Kruzkov [13]).

Theorem 5.1 (Convergence)

Let F be a convex scalar flux and U^0 initial data in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$. The semi-discrete numerical scheme (1.4) (1.5) converges to the unique entropy solution of (1.1) with initial data U^0.

Proof :

The proof follows the ideas developed by Osher [22]. We first establish the TVD property which garanties convergence to a weak solution by compactness, then the following weak entropy inequality

$$\int_{U_j}^{U_{j+1}} \eta''(w) \left(\Phi^{DM}(U_j, U_{j+1}) - F(w)\right) dw \leq 0 \,. \tag{5.20}$$

for the single entropy $\eta(U) = \frac{1}{2}U^2$. This ensures that the limit solution satisfies the weak entropy inequality (1.3) for any strictly convex entropy η (see Di Perna [3]). In order to simplify the notations, we assume $F'(0) = 0$.

TVD property : If we write the scheme under the form

$$\begin{aligned} h\frac{dU_j}{dt} &= \frac{F(U_j) - \Phi^{DM}(U_j, U_{j+1})}{U_{j+1} - U_j}(U_{j+1} - U_j) - \frac{F(U_j) - \Phi^{DM}(U_{j-1}, U_j)}{U_j - U_{j-1}}(U_j - U_{j-1}) \\ &= C_{j+\frac{1}{2}}(U_{j+1} - U_j) - D_{j-\frac{1}{2}}(U_j - U_{j-1}) \,, \end{aligned}$$

it is sufficient (see Sanders [25]) to prove

$$C_{j+\frac{1}{2}} \geq 0, \qquad D_{j+\frac{1}{2}} \geq 0 \,.$$

Since Roe's scheme satisfies this property, only the case

$$U_j < 0 < U_{j+1}$$

must be examined. By proposition (4.4),

$$\Phi^{DM}(U_j, U_{j+1}) = F(U_j) + g(w^*)(U_{j+1} - U_j)$$

which implies

$$C_{j+\frac{1}{2}} = -g(w^*)\ , \qquad D_{j+\frac{1}{2}} = \lambda\ (U_{j+1} - U_j) - g(w^*)$$

From Remark 4.5, and the fact that $\alpha = U_{j+1} - U_j$, we deduce $C_{j+\frac{1}{2}} \geq 0$ and $D_{j+\frac{1}{2}} \geq 0$.

Entropy inequality : Away from the sonic point, the scheme reduces to the standard upwind scheme. Once again, the only case to be taken under consideration is the "sonic" case :

$$U_j < 0 < U_{j+1}\ .$$

In fact, we will prove inequality (5.20) by establishing (see figure 3)

$$\Phi^{DM}(U_j, U_{j+1}) \leq \frac{1}{U_{j+1} - U_j} \int_{U_j}^{U_{j+1}} P(F, w)dw \leq \frac{1}{U_{j+1} - U_j} \int_{U_j}^{U_{j+1}} F(w)dw\ , \qquad (5.21)$$

where P is the piecewise linear function defined by

$$P(F, w) = \sup \left\{ F(U_j) + F'(U_j)(w - U_j); F(U_{j+1}) + F'(U_{j+1})(w - U_{j+1}) \right\}\ .$$

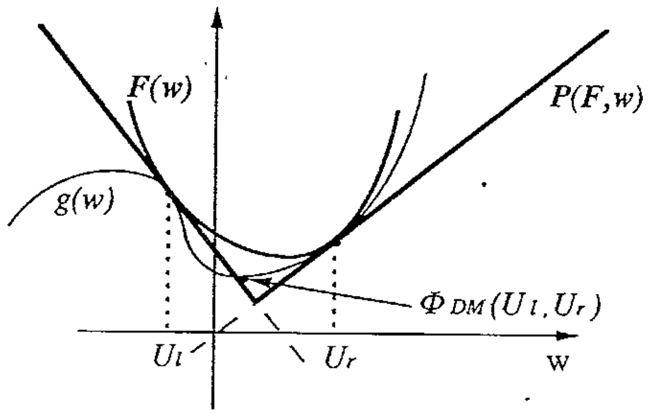

Figure 3: Geometric interpretation of condition (5.21)

The inequality on the right hand side of (5.21) is obvious since F is convex. Since $\Phi^{DM}(U_j, U_{j+1})$ is the minimum value of the Hermite polynomial, it is sufficient to show that for any polynomial ϕ of degree less than 3, and for any two points a and b such that

$$a < 0 < b \quad \text{and} \quad \phi'(a) < 0 < \phi'(b)\ ,$$

we have

$$\min_{a \leq \xi \leq b} (\phi(\xi)) \leq \frac{1}{b - a} \int_a^b P(\phi, \xi)d\xi\ .$$

By change of variables, we can assume that

$$\phi(0) = \phi'(0) = 0 \quad \text{and} \quad \phi''(0) \geq 0\ .$$

Hence, we will consider ϕ of the form

$$\phi(\xi) = \alpha\xi^3 + \beta\xi^2 ,$$

where

$$\beta \geq 0 .$$

We now prove that

$$\frac{1}{b-a}\int_a^b P(\phi,\xi)d\xi \geq 0.$$

Since $P(\phi,\xi)$ is piecewise linear, one easily obtains after multiplication by the positive quantity $\phi'(b) - \phi'(a)$:

$$\begin{aligned}
&\frac{\phi'(b)-\phi'(a)}{b-a}\int_a^b P(\phi,\xi)d\xi \\
&= \phi(a)\phi'(b) - \phi(b)\phi'(a) + \frac{b-a}{2}\left(\left(\frac{\phi(b)-\phi(a)}{b-a}\right)^2 + \phi'(a)\phi'(b)\right) \\
&= \left(3\alpha^2a^2b^2 + 2\alpha\beta ab(a+b) + \left(\alpha(a^2+ab+b^2) + \beta(a+b)\right)^2\right)\left(\frac{b-a}{2}\right) \\
&= \left(3\alpha^2a^2b^2 + \alpha^2(a^2+ab+b^2)^2 + 2\alpha\beta(a+b)^3 + \beta^2(a+b)^2\right)\left(\frac{b-a}{2}\right) \\
&\geq \left(3\alpha^2a^2b^2 + \alpha^2(a^2+ab+b^2)^2 - \alpha^2(a+b)^4\right)\left(\frac{b-a}{2}\right) \\
&= (-ab)\alpha^2(a^2+b^2)(b-a) \\
&\geq 0 .
\end{aligned}$$

This last inequality ends our proof of the entropy inequality (5.20).

□

Remark 5.2

The above theorem is not trivial. In fact our modified flux does not define an E-scheme. We think that we have added a "minimal numerical viscosity". Enlightened by the previous analysis, one could define an even less viscous flux by choosing :

$$\Phi(U_j, U_{j+1}) = \frac{1}{U_{j+1} - U_j}\int_{U_j}^{U_{j+1}} P(F,w)dw .$$

This flux also satisfies the entropy inequality (5.20). But then, the sufficient conditions of Sanders for the TVD property are no longer true. Moreover, we prefer the Godunov-type scheme approach.

Remark 5.3

The idea of solving a Riemann problem associated with a modified flux function has been independently proposed by LeVeque [16]. The modified flux is chosen to be the function $P(F,.)$ which is not of class C^1. The solution of the corresponding Riemann problem is a version of Harten's one-intermediate-state approximate Riemann solver (see [11]) for a particular and non-parameterized choice of the wave speeds. Since the numerical flux is in this case equal to $inf_{w\in[U_j,U_{j+1}]}(P(F,w))$, it is clear from figure 3 that this scheme is an E-scheme as observed by LeVeque. Nevertheless, this approximate Riemann solver is quite viscous and our approach guarantees, without any additional information on the physical flux and at a comparable computational cost, convergence to the entropy solution through weaker entropy requirements.

6 An Entropy Inequality for Hyperbolic Systems

In the sequel, we show that our modified Roe approximate Riemann solver satisfies an entropy inequality. The first order Godunov scheme satisfies such an inequality in the discrete case (see, e.g. [11]). In the semi-discrete case (1.4)(1.5), Osher and Solomon [21] have proved that the O-version of the Osher scheme satisfies a semi-discrete entropy inequality. We now prove an analogous result for our modified Roe scheme.

Theorem 6.1 (Entropy Inequality)

Let us assume that (1.1) is a strictly hyperbolic system of conservation laws which admits a mathematical entropy η. We assume that all the fields are genuinely non-linear. We consider two sates U_l and U_r that are sufficiently close. Then we have the following inequality :

$$\int_{U_l}^{U_r} d^2\eta(U)(dU, \Phi^{DM}(U_l, U_r) - F(U)) \leq 0 . \tag{6.22}$$

Corollary 6.2

We consider the method of lines (1.4)(1.5) associated with Φ^{DM}. We suppose that the sequence $(U_j^h(t))_{j\in Z,\, t\geq 0}$ has sufficiently small oscillation in the (x,t) space and that it converges boundedly almost everywhere to $u(x,t)$ as $h \to 0$. Then, the limit solution satifies the weak entropy inequality (1.3).

This corollary is a direct consequence of the previous theorem and the general results of Osher [20].

Proof of Theorem 6.1 :

We assume that the states U_l and U_r are in the neighborhood $\mathcal{V}$ of some given state U_0. Either all of the eigenvalues $\lambda_j(U_0)$ are non null, and in this case we assume that the neighborhood $\mathcal{V}$ of U_0 is sufficiently small in order that

$$\forall\, j \in \{1, \ldots, m\}, \quad \forall U \in \mathcal{V}, \quad |\lambda_j(U)| \geq \beta > 0 \,,$$

or, by strict hyperbolicity, there exists a single index i such that $\lambda_i(U_0) = 0$. In this case,

$$\forall\, j{\neq}i, \quad \forall U \in \mathcal{V}, \quad |\lambda_j(U)| \geq \beta > 0 \,,$$

and $\lambda_i(U)$ is arbitrarily small. We divide the proof into four parts. First we decompose the left hand side of (6.22) along the characteristic directions associated with the Roe-matrix. Secondly, we consider the eigenvalues bounded away from 0. Then we focus on the eigenvalue λ_i close to 0. Finally, we establish the estimate (6.22).

• The left hand side of (6.22) does not depend on the path between U_l and U_r. Therefore we choose the following path Γ :

$$U(\theta) = U_{j-1} + \theta(U_j - U_{j-1}), \quad 0 \leq \theta \leq 1, \quad j = 1, \ldots, m \tag{6.23}$$

which in fact consists in m different segments joining $U_0 \equiv U_l$ to $U_m \equiv U_r$. We recall that

$$U_j - U_{j-1} = \alpha_j R_j(U_l, U_r) \,.$$

The eigenvectors $R_j(U_l, U_r)$ of the Roe-matrix are continuously oriented in order to maintain the classical normalization :

$$\text{grad}\lambda_j(U) \cdot r_j(U) \equiv 1 \,,$$

where j is a genuinely nonlinear field of $dF(U)$. We introduce the dual basis associated with the $R_j(U_l;U_r)$:

$$L_j(U_l;U_r)\cdot R_k(U_l,U_r)=\delta_{jk}\ ,\quad \forall j,k\in\{1,\dots,m\}\,.$$

By definition of η, we have under intrinsic form :

$$\forall U,\ \forall\xi,\quad dq(U)\cdot\xi=d\eta(U)\cdot(dF(U)\cdot\xi)\ .$$

By differentiating along the direction ζ, we have :

$$\forall U,\ \forall\xi,\ \forall\zeta,\quad d^2q(U)(\xi,\zeta)=d\eta(U)\cdot\left(d^2F(U)(\xi,\zeta)\right)+d^2\eta(U)\left(dF(U)\cdot\xi,\zeta\right)\ .$$

Taking $\xi=r_k(U)$ and $\zeta=r_l(U)$ with $l\neq k$, we obtain by strict hyperbolicity the classical result :

$$d^2\eta(r_k(U),r_l(U))=0\,,\quad k\neq l\,.$$

Let η_j'' denote the strictly positive constant defined by :

$$\eta_j''=d^2\eta(r_j(U_0),r_j(U_0))\,.$$

By continuity of the eigenvectors of the Roe-matrix with respect to the left and right arguments, we have :

$$d^2\eta\,(U(\theta))\,(R_j(U_l,U_r),\ \cdot\ )=(1+C_{jj}(\theta))\,\eta_j''\,L_j(U_l,U_r)\ +\sum_{k\neq j}C_{kj}(\theta)\,L_k(U_l,U_r)$$

$$\exists\, C_1>0\,,\quad \forall\theta\in[0;1]\,,\qquad |C_{kl}(\theta)|\leq C_1|\alpha|\ ,$$

where the constant C_1 is independent of k, l, U_l, U_r, and θ and where $|\alpha|$ denotes $\sum_{l=1}^{m}|\alpha_l|$. We deduce :

$$\begin{aligned}&\int_\Gamma d^2\eta(U)(dU,\Phi^{DM}(U_l,U_r)-F(U))\\ &\leq\sum_{j=1}^{m}\eta_j''\alpha_j\int_0^1(1+C_{jj}(\theta))L_j(U_l,U_r)(\Phi^{DM}(U_l,U_r)-F(U(\theta))d\theta\\ &\quad+\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j|\int_0^1|C_{kj}(\theta)L_k(U_l,U_r)(\Phi^{DM}(U_l,U_r)-F(U(\theta))|d\theta\end{aligned}$$

Since we have :

$$\Phi^{DM}(U_l,U_r)=F(U_l)+A^-(U_l,U_r)(U_r-U_l)+\sum_{i\in S}(g_i(w_i^*)-\lambda_i(U_l,U_r)^-\alpha_i)R_i(U_l,U_r)\ ,\tag{6.24}$$

and

$$|F(U(\theta))-F(U_l)-A(U_l,U_r)(U(\theta)-U_l)|\leq C_2|\alpha|^2\ ,\tag{6.25}$$

we deduce

$$|L_k(U_l,U_r)(\Phi^{DM}(U_l,U_r)-F(U(\theta)))|\leq C_3|\alpha_k|+C_2|\alpha|^2\ .$$

Therefore, we have

$$\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j|\int_0^1 |C_{kj}(\theta)L_k(U_l,U_r)(\Phi^{DM}(U_l,U_r)-F(U(\theta))|d\theta$$

$$\leq \sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j|\,C_1|\alpha|\,(C_3|\alpha_k|+C_2|\alpha|^2)$$

$$\leq C_1C_3\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k||\alpha| + mC_1C_2\sum_{j=1}^{m}|\alpha_j||\alpha|^3$$

$$\leq C_1C_3\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k||\alpha| + mC_1C_2\sum_{j=1}^{m}(|\alpha_j|+\sum_{k\neq j}|\alpha_k|)|\alpha_j||\alpha|^2$$

$$\leq (C_1C_3+mC_1C_2|\alpha|)\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k||\alpha| + mC_1C_2\sum_{j=1}^{m}\alpha_j^2|\alpha|^2$$

$$\leq (C_1C_3+mC_1C_2|\alpha|)\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k||\alpha| + mC_1C_2\sum_{j=1}^{m}\alpha_j^2|\alpha|(|\alpha_j|+\sum_{k\neq j}|\alpha_k|)$$

$$\leq (C_1C_3+2mC_1C_2|\alpha|)\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k||\alpha| + mC_1C_2\sum_{j=1}^{m}|\alpha||\alpha_j|^3 \,.$$

Finally, by combining the previous inequalities, we obtain the following upper bound for the integral in the left hand side of (6.22) :

$$\begin{aligned}&\int_\Gamma d^2\eta(U)(dU,\Phi^{DM}(U_l,U_r)-F(U))\\ &\leq \sum_{j=1}^{m}\eta_j''\alpha_j\int_0^1(1+C_{jj}(\theta))L_j(U_l,U_r)(\Phi^{DM}(U_l,U_r)-F(U(\theta))d\theta \\ &\quad +C_4\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k||\alpha| + C_5\sum_{j=1}^{m}|\alpha||\alpha_j|^3\end{aligned}\tag{6.26}$$

where the constants C_4 and C_5 can be chosen equal to :

$$\begin{cases} C_4 = C_1(2mC_2+C_3)\\ C_5 = C_1C_2m\end{cases}$$

when the α_l satisfy $|\alpha_l|\leq 1, \quad l=1,\dots,m$.

• We consider an index j which is associated with an eigenvalue of $dF(U_0)$ bounded away from 0. By continuity of the Roe-matrix $A(U_l,U_r)$ and Property 1.5, the neighborhood $\mathcal{V}$ can be chosen so that $\lambda_j(U_l,U_r)$ also satisfies :

$$\forall\, U_l\,,\ U_r\ \in\mathcal{V},\quad |\lambda_j(U_l,U_r)|\ \geq\ \beta\ >\ 0\,.$$

Let I_j denote the quantity :

$$I_j = \alpha_j\int_0^1(1+C_{jj}(\theta))L_j(U_l,U_r)(\Phi^{DM}(U_l,U_r)-F(U(\theta))d\theta \tag{6.27}$$

which appears in the right hand side of inequality (6.26). By writing

$$L_j(U_l,U_r)\Phi^{DM}(U_l,U_r) = \frac{1}{2}L_j(U_l,U_r)\left[F(U_l)+F(U_r)-|A(U_l,U_r)|(U_r-U_l)\right]\ ,$$

one easily obtains

$$\begin{aligned} I_j &= -\alpha_j \int_0^1 (1+C_{jj}(\theta))\alpha_j \frac{|\lambda_j(U_l,U_r)|}{2} d\theta \\ &+ \alpha_j \int_0^1 (1+C_{jj}(\theta)) L_j(U_l,U_r) \left(\frac{F(U_l)+F(U_r)}{2} - F(U(\theta)) \right) d\theta . \end{aligned}$$

The inequality

$$\begin{aligned} &\left| \int_0^1 (1+C_{jj}(\theta)) L_j(U_l,U_r) \left[F(U(\theta)) - \frac{F(U_l)+F(U_r)}{2} \right] d\theta \right| \\ &= \left| \int_0^1 (1+C_{jj}(\theta)) L_j(U_l,U_r) \left[F(U(\theta)) - F(U_l) + \frac{F(U_l)-F(U_r)}{2} \right] d\theta \right| \\ &\leq \left| \int_0^1 (1+C_{jj}(\theta))\, L_j(U_l,U_r) \left[F(U(\theta)) - F(U_l) - A(U_l,U_r)(U(\theta)-U_l) \right] d\theta \right| \\ &\qquad + \left| \alpha_j \int_0^1 (1+C_{jj}(\theta))(\theta - \frac{1}{2}) d\theta \right| \\ &\leq \int_0^1 |1+C_{jj}(\theta)|\; |L_j(U_l,U_r) \left[F(U(\theta)) - F(U_l) - A(U_l,U_r)(U(\theta)-U_l) \right]| \, d\theta \\ &\qquad + |\alpha_j| \int_0^1 |C_{jj}(\theta)||\theta - \frac{1}{2}| d\theta \\ &\qquad \leq (1+C_1|\alpha|)C_2|\alpha|^2 \; + \; \frac{1}{4}|\alpha_j| C_1 |\alpha| , \end{aligned}$$

implies

$$\left| I_j + \frac{|\lambda_j(U_l,U_r)|}{2} \alpha_j^2 \right| \leq C_1|\alpha|\alpha_j^2 \frac{|\lambda_j(U_l,U_r)|}{2} \; + \; |\alpha_j|(1+C_1|\alpha|)C_2|\alpha|^2 + \frac{1}{4}\alpha_j^2 C_1 |\alpha| .$$

We finally obtain for $|\alpha|$ small enough

$$I_j \quad \leq \quad -\frac{\beta}{4}\alpha_j^2 , \tag{6.28}$$

which is valid for all indices j associated with eigenvalues bounded away from 0. This estimate remains valid even if the j-field is linearly degenerated, as noticed by Osher [20].

• We assume now that $\lambda_i(U_0) = 0$ (i is the only index which satisfies this property by strict hyperbolicity). We emphasize on the fact that only two cases or possible :

$$S = \{i\} \qquad \text{or} \qquad S = \emptyset .$$

In order to simplify the notations, we define the quantity ζ_i :

$$\bullet \text{ if } S = \{i\} , \qquad \zeta_i = \frac{g_i(w_i^*)}{\alpha_i} - \lambda_i(U_l,U_r)^- ,$$

$$\bullet \text{ if } S = \emptyset , \qquad \zeta_i = 0 .$$

From Remark 4.5, ζ_i **is always negative** .

We now estimate the quantity I_i defined in (6.23). For $U(\theta)$ defined in (6.23) by

$$U(\theta) = U_{i-1} + \theta(U_i - U_{i-1}) = U_l + \sum_{j=1}^{i-1} \alpha_j R_j(U_l,U_r) + \theta \alpha_i R_i(U_l,U_r) , \quad 0 \leq \theta \leq 1 ,$$

we estimate

$$\begin{aligned} F(U(\theta)) = \; & F(U_l) + A(U_l,U_r)\cdot(U(\theta)-U_l) \\ & + [dF(U_l) - A(U_l,U_r)]\cdot(U(\theta)-U_l) \\ & + \frac{1}{2}d^2F(U_l)(U(\theta)-U_l, U(\theta)-U_l) + O(|\alpha|^3)\,. \end{aligned} \tag{6.29}$$

We now estimate the third term in the right hand side of (6.29). From property (1.4), which is one of the generic properties of Roe matrices, we have after two differentiations with respect to U_l :

$$dF(U_l) = A(U_l,U_r) - \frac{\partial A(U,V)}{\partial U}(U_l,U_r)\cdot(U_r-U_l) \tag{6.30}$$

$$d^2F(U_l) = 2\frac{\partial A(U,V)}{\partial U}(U_l,U_r) - \frac{\partial^2 A(U,V)}{\partial U^2}(U_l,U_r)(U_r-U_l,\cdot) \tag{6.31}$$

which imply :

$$\begin{aligned} & dF(U_l) - A(U_l,U_r) \\ = \; & -\frac{1}{2}d^2F(U_l)(U_r-U_l,\cdot) \; - \; \frac{1}{2}\frac{\partial^2 A(U,V)}{\partial U^2}(U_l,U_r)(U_r-U_l,U_r-U_l)\,. \end{aligned} \tag{6.32}$$

By inserting (6.32) in (6.29), we have :

$$F(U(\theta)) = F(U_l) + A(U_l,U_r)\cdot(U(\theta)-U_l) + \frac{1}{2}d^2F(U_l)(U(\theta)-U_l,U(\theta)-U_r) + O(|\alpha|^3)\,.$$

From (6.24) and the previous expansion we have

$$\begin{aligned} L_i(U_l,U_r)(\Phi^{DM}(U_l,U_r) - F(U(\theta))) &= \lambda_i(U_l,U_r)^-\alpha_i + \zeta_i\alpha_i - \lambda_i(U_l,U_r)\theta\alpha_i \\ & -\frac{L_i(U_l,U_r)}{2}d^2F(U_l)(U(\theta)-U_l,U(\theta)-U_r) + O(|\alpha|^3)\,. \end{aligned}$$

Since $L_i(U_l,U_r) = l_i(U_l) + O(|\alpha|)$ where $(l_j)_{j=1,\dots,m}$ is the dual basis associated with the $(r_j)_{j=1,\dots,m}$, we have :

$$\begin{aligned} L_i(U_l,U_r)(\Phi^{DM}(U_l,U_r) - F(U(\theta))) \; &= \lambda_i(U_l,U_r)^-\alpha_i + \zeta_i\alpha_i - \lambda_i(U_l,U_r)\theta\alpha_i \\ & +\frac{1}{2}\alpha_i^2\theta(1-\theta) \\ & +\frac{1}{2}l_i(U_l)\cdot d^2F(U_l)(U_r-U_i,U_{i-1}-U_l) \\ & +\frac{1}{2}l_i(U_l)\cdot d^2F(U_l)(U_r-U_i,U(\theta)-U_{i-1}) \\ & +\frac{1}{2}l_i(U_l)\cdot d^2F(U_l)(U_i-U(\theta),U_{i-1}-U_l) \\ & +O(|\alpha|^3)\,. \end{aligned}$$

After integration along the segment $[U_{i-1},U_i]$, we have :

$$I_i = \alpha_i^2\int_0^1 (1+C_{ii}(\theta))(\lambda_i(U_l,U_r)^- - \lambda_i(U_l,U_r)\theta + \zeta_i + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 \tag{6.33}$$

where

$$|R_i^1| \le C_6|\alpha_i|\sum_{j\neq k}|\alpha_j||\alpha_k| + C_7|\alpha_i||\alpha|^3\,.$$

The constant C_6 is chosen so that

$$\forall U \in \mathcal{V}\,, \quad \forall \xi\,,\ \eta\,, \qquad l_i(U)\cdot d^2F(U)(\xi,\eta) \le C_6\ |\xi||\eta|$$

and C_7 is associated with the $O(|\alpha|^3)$.

We now estimate the eigenvalue $\lambda_i(U_l, U_r)$ by differentiating the relation

$$A(U_l,U_r)\,R_i(U_l,U_r) = \lambda_i(U_l,U_r)\,R_i(U_l,U_r)$$

with respect to U_r and by taking $U_r = U_l$ in relation (6.31). We obtain :

$$\forall \xi\,, \quad \frac{\partial \lambda_i(U,V)}{\partial U}(U_l,U_l)\cdot\xi = \frac{1}{2}\ l_i(U_l)\cdot d^2F(U_l)(\xi, r_i(U_l))$$

from which we deduce the following estimate :

$$\lambda_i(U_l,U_r) = \lambda_i(U_l) + \frac{1}{2} l_i(U_l)\cdot d^2F(U_l)(U_r - U_l, r_i(U_l)) + O(|\alpha|^2)\ .$$

Also, by using the fact that

$$\forall U\,, \quad d\lambda_i(U)\cdot\xi = l_i(U)\cdot d^2F(U)(\xi, r_i(U))\ ,$$

we can derive the following Taylor expansions :

$$\lambda_i(U_{i-1}) = \lambda_i(U_l) + l_i(U_l)\cdot d^2F(U_l)(U_{i-1} - U_l, r_i(U_l)) + O(|\alpha|^2)\ ,$$

and

$$\lambda_i(U_i) = \lambda_i(U_l) + l_i(U_l)\cdot d^2F(U_l)(U_i - U_l, r_i(U_l)) + O(|\alpha|^2)\ .$$

From the estimates of $\lambda_i(U_l,U_r)$ and $\lambda_i(U_{i-1})$, we deduce :

$$\begin{aligned}
\lambda_i(U_l,U_r) - \lambda_i(U_{i-1}) &= \frac{1}{2}\ l_i(U_l)\cdot d^2F(U_l)(U_r - U_l - 2(U_{i-1} - U_l), r_i(U_l)) + O(|\alpha|^2) \\
&= \frac{1}{2}\ l_i(U_l)\cdot d^2F(U_l)((U_r - U_{i-1}) + (U_l - U_{i-1}), r_i(U_l)) + O(|\alpha|^2) \\
&= \frac{\alpha_i}{2} + R_i^2\ ,
\end{aligned} \tag{6.34}$$

where

$$|R_i^2| \le \frac{C_6}{2}\sum_{j\neq i}|\alpha_j| + C_8|\alpha|^2\ .$$

With similar arguments we obtain

$$\lambda_i(U_i) - \lambda_i(U_l,U_r) = \frac{\alpha_i}{2} + R_i^3\ , \tag{6.35}$$

where $|R_i^3|$ is bounded above by the same quantity as $|R_i^2|$.

From now on, we assume that $|\alpha| \le \dfrac{1}{2C_1}$ which implies

$$\forall\,\theta \in [0,1]\,, \qquad \frac{1}{2} \le 1 + C_{ii}(\theta) \le \frac{3}{2}\ .$$

The integral I_i can be split into

$$\begin{aligned}
I_i = \quad &\alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))((1-\theta)\lambda_i(U_l,U_r)^- - \theta\lambda_i(U_l,U_r)^+ + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta \\
&+\alpha_i^2\int_0^1 (1 + C_{ii}(\theta))\zeta_i d\theta + R_i^1\ .
\end{aligned}$$

Three cases must now be ditinguished in order to estimate I_i :

Admissible shocks in the sense of Lax : $\alpha_i \leq 0$

Since ζ_i is negative, we have the following inequalities :

$$\begin{aligned} I_i &\leq \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))((1-\theta)\lambda_i(U_l, U_r)^- - \theta\lambda_i(U_l, U_r)^+ + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 \\ &\leq \frac{\alpha_i^3}{2} \int_0^1 (1 + C_{ii}(\theta))\, \theta(1-\theta)d\theta + R_i^1 \\ &\leq \frac{\alpha_i^3}{4} \int_0^1 \theta(1-\theta)d\theta + R_i^1 \\ &\leq -\frac{|\alpha_i|^3}{24} + R_i^1 \,. \end{aligned}$$

Non-centered rarefactions : $\alpha_i \geq 0$ and either $\lambda_i(U_{i-1}) \geq 0$ or $\lambda_i(U_i) \leq 0$

In either of these two cases $\zeta_i = 0$. Therefore,

$$I_i = \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))((1-\theta)\lambda_i(U_l, U_r)^- - \theta\lambda_i(U_l, U_r)^+ + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 \,.$$

Let us assume that $\alpha_i \geq 0$ and $\lambda_i(U_{i-1}) \geq 0$. Then, by introducing estimate (6.34) in the previous relation for $|\alpha|$ small enough, we obtain

$$\begin{aligned} I_i &= \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))(\lambda_i(U_l, U_r)^- - \theta\lambda_i(U_l, U_r) + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 \\ &\leq \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))(-\theta\lambda_i(U_l, U_r) + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 \\ &\leq \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))(-\theta(\lambda_i(U_{i-1}) + \frac{\alpha_i}{2}) + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 + \frac{3}{2}\alpha_i^2|R_i^2| \\ &\leq \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))(-\theta\frac{1}{2}\alpha_i + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 + \frac{3}{2}\alpha_i^2|R_i^2| \\ &\leq -\frac{|\alpha_i|^3}{2} \int_0^1 (1 + C_{ii}(\theta))\theta^2 d\theta + R_i^1 + \frac{3}{2}\alpha_i^2|R_i^2| \,. \end{aligned}$$

Through similar arguments, we obtain the same estimate if we assume that $\alpha_i \geq 0$ and $\lambda_i(U_i) \leq 0$. Indeed, by using estimate (6.35) we obtain

$$\begin{aligned} I_i &= \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))(-\lambda_i(U_l, U_r)^+ + (1-\theta)\lambda_i(U_l, U_r) + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 \\ &\leq \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))((1-\theta)\lambda_i(U_l, U_r) + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 \\ &\leq \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))((1-\theta)(\lambda_i(U_i) - \frac{\alpha_i}{2}) + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 + \frac{3}{2}\alpha_i^2|R_i^3| \\ &\leq \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))(-\frac{1}{2}\alpha_i(1-\theta) + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 + \frac{3}{2}\alpha_i^2|R_i^3| \\ &\leq -\frac{|\alpha_i|^3}{2} \int_0^1 (1 + C_{ii}(\theta))(1-\theta)^2 d\theta + R_i^1 + \frac{3}{2}\alpha_i^2|R_i^3| \,. \end{aligned}$$

Finally, in both cases, for $|\alpha|$ small enough we have :

$$I_i \leq -\frac{|\alpha_i|^3}{12} + R_i^4 \,,$$

where R_i^4 is estimated as follows :

$$
\begin{aligned}
R_i^4 &\leq |R_i^1| + \frac{3}{2}\alpha_i^2(\frac{C_6}{2}\sum_{j\neq i}|\alpha_j| + C_8|\alpha|^2) \\
&\leq C_6|\alpha_i|\sum_{j\neq k}|\alpha_j||\alpha_k| + C_7|\alpha_i||\alpha|^3 + \frac{3}{4}C_6\alpha_i^2\sum_{j\neq i}|\alpha_j| + \frac{3}{2}C_8\alpha_i^2|\alpha|^2 \\
&\leq \frac{7}{4}C_6|\alpha|\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k| + C_7|\alpha|^2|\alpha_i|\sum_{j\neq i}|\alpha_j| + (C_7 + \frac{3}{2}C_8)|\alpha_i|^2|\alpha|^2 \\
&\leq (\frac{7}{4}C_6 + (2C_7 + \frac{3}{2}C_8)|\alpha|)|\alpha|\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k| + (C_7 + \frac{3}{2}C_8)|\alpha||\alpha_i|^3 \\
&\leq (\frac{7}{4}C_6 + m(2C_7 + \frac{3}{2}C_8))|\alpha|\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k| + (C_7 + \frac{3}{2}C_8)|\alpha||\alpha_i|^3
\end{aligned}
$$

for $|\alpha_i| \leq 1, \quad i = 1, \ldots, m$.

Sonic points : $\lambda_i(U_{i-1}) \leq 0 \leq \lambda_i(U_i)$

In this case, $\zeta_i = \dfrac{g_i(w_i^*)}{\alpha_i} - \lambda_i(U_l, U_r)^-$ and

$$I_i = \alpha_i \int_0^1 (1 + C_{ii}(\theta))(g_i(w_i^*) - \lambda_i(U_l, U_r)\theta\alpha_i + \frac{1}{2}\alpha_i^2\theta(1-\theta))d\theta + R_i^1 .$$

Moreover, we can assume that α_i is positive since the case $\alpha_i \leq 0$ has already been studied. We now estimate the quantity $g_i(w_i^*)$ by using the classical Hermite interpolation polynomials :

$$g_i(w) = \frac{\lambda_i(U_{i-1})w(\alpha_i - w)^2 - \lambda_i(U_i)w^2(\alpha_i - w) + \lambda_i(U_l, U_r)w^2(3\alpha_i - 2w)}{\alpha_i^2} .$$

By substituting the expansions (6.34) and (6.35) in the previous formula, one finds :

$$\forall t \in [0,1], \quad \frac{g_i(t\alpha_i)}{\alpha_i} = \lambda_i(U_{i-1})t + \frac{\alpha_i}{2}t^2 + t^2(2-t)R_i^2 - t^2(1-t)R_i^3 .$$

Since w_i^* is the argument of the unique minimum of g_i on the interval $[0, \alpha_i]$, we have :

$\forall t \in [0,1]$,

$$
\begin{aligned}
I_i &\leq \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))(\frac{g_i(t\alpha_i)}{\alpha_i} - (\lambda_i(U_{i-1}) + \frac{\alpha_i}{2} + R_i^2)\theta + \frac{1}{2}\alpha_i\theta(1-\theta))d\theta + R_i^1 \\
&\leq \alpha_i^2 \int_0^1 (1 + C_{ii}(\theta))(\lambda_i(U_{i-1})t + \frac{\alpha_i}{2}t^2 - \lambda_i(U_{i-1})\theta - \frac{\alpha_i}{2}\theta^2)d\theta + R_i^5 ,
\end{aligned}
$$

where R_i^5 is bounded above as follows (see previous estimate of R_i^4) :

$$
\begin{aligned}
R_i^5 &\leq |R_i^1| + (\frac{3}{4} + \frac{3}{2}t^2(2-t))\alpha_i^2|R_i^2| + \frac{3}{2}t^2(1-t)\alpha_i^2|R_i^3| \\
&\leq |R_i^1| + 3\alpha_i^2(\frac{C_6}{2}\sum_{j\neq i}|\alpha_j| + C_8|\alpha|^2) \\
&\leq (\frac{5}{2}C_6 + m(2C_7 + 3C_8))|\alpha|\sum_{j=1}^{m}\sum_{k\neq j}|\alpha_j||\alpha_k| + (C_7 + 3C_8)|\alpha||\alpha_i|^3
\end{aligned}
$$

for $|\alpha_i| \leq 1, \quad i = 1, \ldots, m$.

Consider a small positive parameter ε $(0 < \varepsilon < 1)$ and assume that $|\alpha| \leq \dfrac{\varepsilon}{C_1}$. By applying the following uniform bounds

$$\forall\, \theta \in [0,1]\,, \qquad 1-\varepsilon \leq 1 + C_{ii}(\theta) \leq 1+\varepsilon\,.$$

to the integral in I_i and by using the fact that $\lambda_i(U_{i-1}) \leq 0$ and $\alpha_i \geq 0$, we obtain the inequality :

$$\forall t \in [0,1], \qquad I_i \leq \alpha_i^2 \left((1-\varepsilon)t - \frac{(1+\varepsilon)}{2}\right) \lambda_i(U_{i-1}) + \frac{\alpha_i^3}{2}\left((1+\varepsilon)t^2 - \frac{(1-\varepsilon)}{3}\right) + R_i^5$$

Finally, by choosing $\varepsilon = \dfrac{1}{50}$ and $t = \dfrac{13}{24}$ we obtain the upper bound

$$I_i \leq -\frac{1}{80}\alpha_i^3 + R_i^5\,.$$

- From the previous points, we have for $|\alpha|$ small enough the following upper bound for the entropy production, i.e. the left hand side of (6.22) :

$$\begin{aligned}\int_{U_l}^{U_r} d^2\eta(U)(dU, \Phi^{DM}(U_l,U_r) - F(U)) \leq\ & -\frac{\beta}{4} \sum_{j\,,\,\lambda_j(U_0)\neq 0} \alpha_j^2 \\ & -\frac{1}{80} \sum_{j\,,\,\lambda_j(U_0)=0} |\alpha_j|^3 + \sum_{j\,,\,\lambda_j(U_0)=0} |R_j^5| \\ & + C_4 \sum_{j=1}^{m}\sum_{k\neq j} |\alpha_j||\alpha_k||\alpha| + C_5 \sum_{j=1}^{m} |\alpha||\alpha_j|^3\,,\end{aligned}$$

which yields, after elementary algebra :

$$\begin{aligned}\int_{U_l}^{U_r} d^2\eta(U)(dU, \Phi^{DM}(U_l,U_r) - F(U)) \leq\ & -\frac{\beta}{4} \sum_{j\,,\,\lambda_j(U_0)\neq 0} \alpha_j^2 \\ & -\frac{1}{80} \sum_{j\,,\,\lambda_j(U_0)=0} |\alpha_j|^3 \\ & + C_9 \sum_{j=1}^{m}\sum_{k\neq j} |\alpha_j||\alpha_k||\alpha| + C_{10} \sum_{j=1}^{m} |\alpha||\alpha_j|^3\,.\end{aligned}$$

where the constants C_9 and C_{10} can be chosen equal to :

$$\begin{cases} C_9 = C_4 + \dfrac{5}{2}C_6 + m(2C_7 + 3C_8)\,, \\ C_{10} = C_5 + C_7 + 3C_8\,. \end{cases}$$

By writing

$$C_{10} \sum_{j=1}^{m} |\alpha||\alpha_j|^3 \quad \leq \quad C_{10}\,|\alpha|^2 \sum_{j\,,\,\lambda_j(U_0)\neq 0} \alpha_j^2 + C_{10}\,|\alpha| \sum_{j\,,\,\lambda_j(U_0)=0} |\alpha_j|^3\,,$$

and by choosing $|\alpha|$ small enough, we finally obtain

$$\begin{aligned} &\int_{U_l}^{U_r} d^2\eta(U)(dU, \Phi^{DM}(U_l, U_r) - F(U)) \\ \leq \quad & -\frac{\beta}{8} \sum_{j\,,\,\lambda_j(U_0)\neq 0} \alpha_j^2 - \frac{1}{160} \sum_{j\,,\,\lambda_j(U_0)=0} |\alpha_j|^3 + C_9 \sum_{j=1}^{m} \sum_{k\neq j} |\alpha_j||\alpha_k||\alpha| \,. \end{aligned} \tag{6.36}$$

Define x and y by

$$x = \sum_{j\,,\,\lambda_j(U_0)=0} |\alpha_j| \,,$$

$$y = \sum_{j\,,\,\lambda_j(U_0)\neq 0} |\alpha_j| \,.$$

Since there exists, by strict hyperbolicity, at most one index j such that $\lambda_j(U_0) = 0$, we have

$$x^3 = \sum_{j\,,\,\lambda_j(U_0)=0} |\alpha_j|^3 \,.$$

By splitting into two parts the C_9 term in the right hand side of (6.36) according to

$$C_9 \sum_{j=1}^{m} \sum_{k\neq j} |\alpha_j||\alpha_k||\alpha| \leq C_9|\alpha|(xy + (x+y)y) \leq 2C_9|\alpha|xy + C_9|\alpha|y^2 \,,$$

we obtain the following upper bound for the entropy production :

$$\int_{U_l}^{U_r} d^2\eta(U)(dU, \Phi^{DM}(U_l, U_r) - F(U)) \leq \quad -\frac{1}{160} f(x,y) \,.$$

with

$$f(x,y) = -x^3 - ay^2 + b(x^2y + xy^2) \,,$$

and where a and b can be chosen as two positive constants for $|\alpha|$ small enough.

We now show that, for x and y positive and small enough, $f \leq 0$. We can rewrite f in the form :

$$f(x,y) = x(-x^2 + bxy + by^2) - ay^2 \,.$$

The polynomial

$$P(x) = -x^2 + bxy + by^2$$

reaches its maximum value at $\frac{by}{2}$:

$$P(\frac{by}{2}) = (\frac{b^2}{4} + b)\, y^2 \,.$$

Hence,

$$\forall\, x, y > 0 \,, \quad f(x,y) \,\leq\, ((\frac{b^2}{4} + b)\, x - a)\, y^2 \,.$$

Therefore, for all positive x and y in the ball of center $(0,0)$ and of radius $\frac{a}{\frac{b^2}{4} + b}$, $f \leq 0$.

This last inequality ends our proof of theorem 6.1.

□

7 Numerical Results

In this section, numerical results obtained with our entropy correction are presented for two classical gas dynamics problems. The first problem is a 1D shock-tube problem with a strong sonic expansion and the second one is the classical 2D supersonic forward facing step popularized by Woodward and Colella [30].

The Euler equations governing the flow of an inviscid compressible thermally perfect gas with a constant specific heat ratio of 1.4 are solved by a finite volume MUSCL appproach [27]. Finite volume methods are based on the resolution of an approximate Riemann problem at each cell interface [7] [11]. For the shock-tube problem, Roe's original linearization [23] has been used. A Roe-type approximate Riemann solver in the sense of properties (2.8)-(2.10) which is based on a shock curve decomposition [17] has been used for the second problem. In both test cases, computations have been performed with and without the entropy correction. Second order accuracy is achieved spatially through a multidimensionnal MUSCL interpolation of the density, momentum and pressure (see details in [5]). The limitation procedure is a 2D version of the minmod limiter [18]. The time discretization is a second order Runge-Kutta scheme (Heun's scheme).

We have selected the following shock-tube problem

$$\frac{\rho_r}{\rho_l} = 0.03\,, \qquad \frac{p_r}{p_l} = 0.008\,, \qquad \frac{M_r}{M_l} = 2.46\,,$$

because a strong sonic rarefaction is present in the exact solution (ρ denotes the density, p the pressure, and M the Mach number). First order solutions have been computed on different grids (from 100 to 1600 cells). A non-physical expansion shock is present when Roe's method is used without any correction. In comparison with the exact solution, computations with the entropy correction show a small discrepancy in the vicinity of the sonic point. This phenomenon is also present when using the Godunov or Osher schemes (see [29] and [2]) and tends to disappear when the mesh is refined and when the entropy correction is applied (see the density plots on figure 4). The evolution of the L^1, L^2, and L^∞ residual norms of the density is plotted in figure 5. The L^1 and L^2 error norms behave approximately like $O(h)$ while solutions computed with Roe's unmodified scheme exhibit an expansion shock which behaves like $O(1)$ (see figure 4).

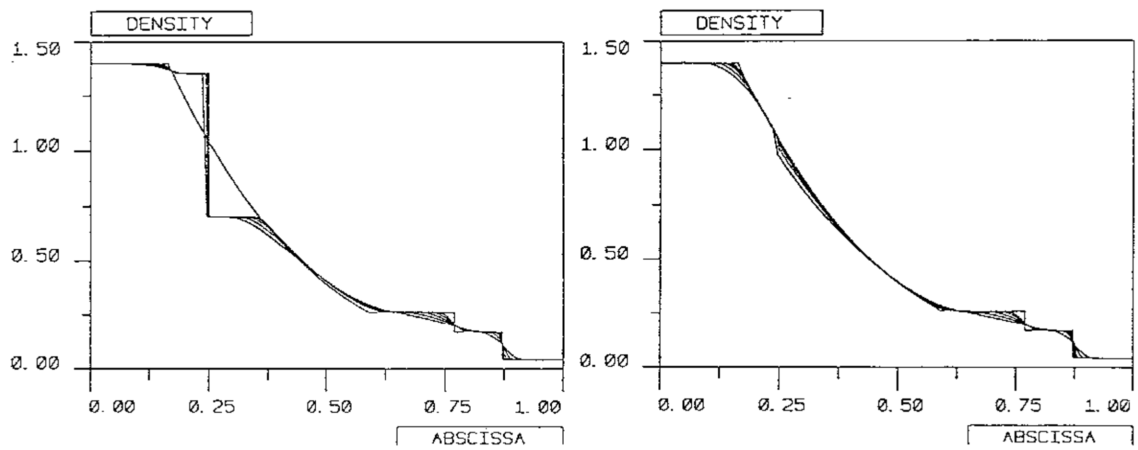

Figure 4: Density profiles obtained for different meshes with and without entropy correction compared with the exact solution for the shock-tube problem.

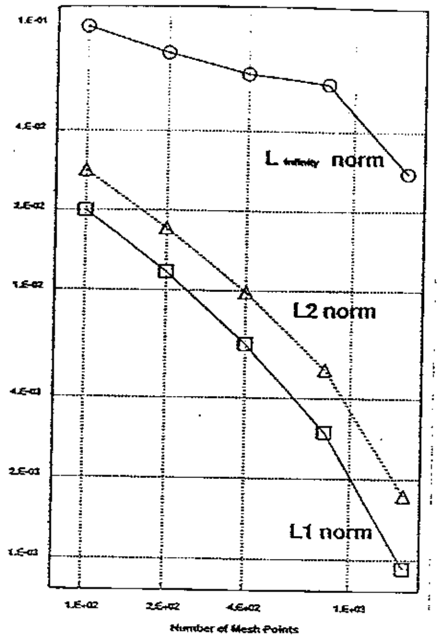

Figure 5: Logarithm of the L^1, L^2, and L^∞ residual norms of the density.

The second problem considered is a supersonic flow at Mach 3 arriving on a step of height 0.2 in a channel of total height 1 (see the survey paper [30]). A cartesian grid of step $\Delta x = \Delta y = 1/40$ was used (see figure 6). At time $t = 0$ the step is introduced in

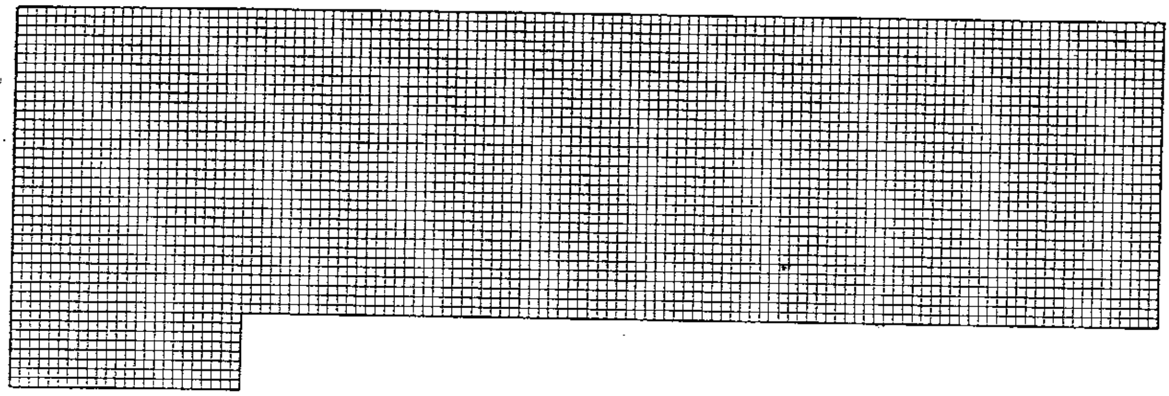

Figure 6: Grid used for the supersonic forward facing step ($\Delta x = \Delta y = 1/40$).

the upstream flow. The interaction of the flow with the step creates a detached shock wave ahead of the step which reflects on the upper wall of the channel and then again downstream on the lower wall. The flow behind the detached shock wave expands around the corner of the step and is transonic in this region (see figure 39). The sonic line which is attached to the corner is a good test for our entropy correction (see also [24]). In contrast with the approach in [30], no special treatment based on the physical phenomenology has been applied to the cells near the corner. All of the results for this problem have been computed with the second order accurate scheme. The density and pressure distributions (figures 7 to 38) from times $t = 0$ to $t = 4$ every half time unit, the timescale being defined as the ratio of the unit length to the upstream sound velocity, illustrate the main features of the unstationnary phase of the flow described previously. Thirty equally spaced contours from one extreme contour level to another are shown in each plot. When no correction is applied, the sonic line attached to the corner of the step appears clearly like a discontinuity (see e.g., figures 7, 18, 23, 34). On the other hand, the plots obtained with the entropy correction give a better description of the transonic expansion fan in this region.

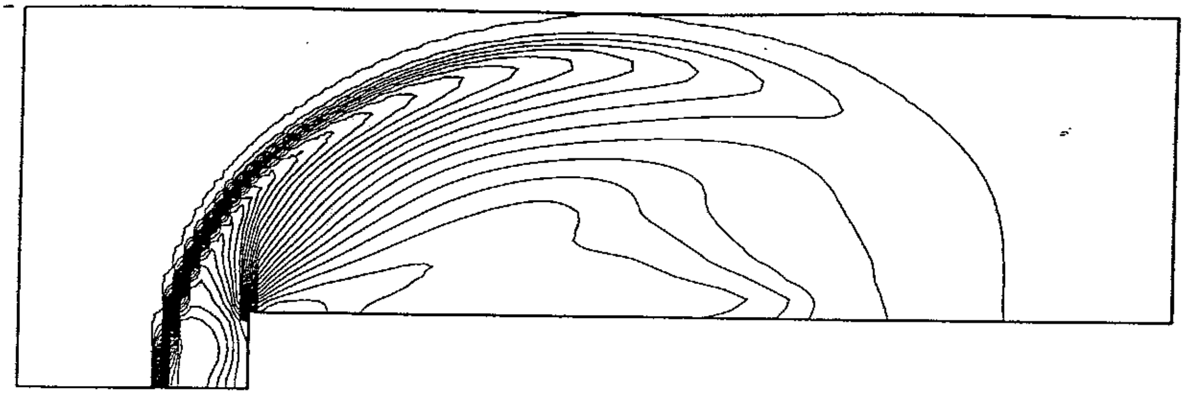

Figure 7: Density contours at time t = 0.5 without entropy correction.

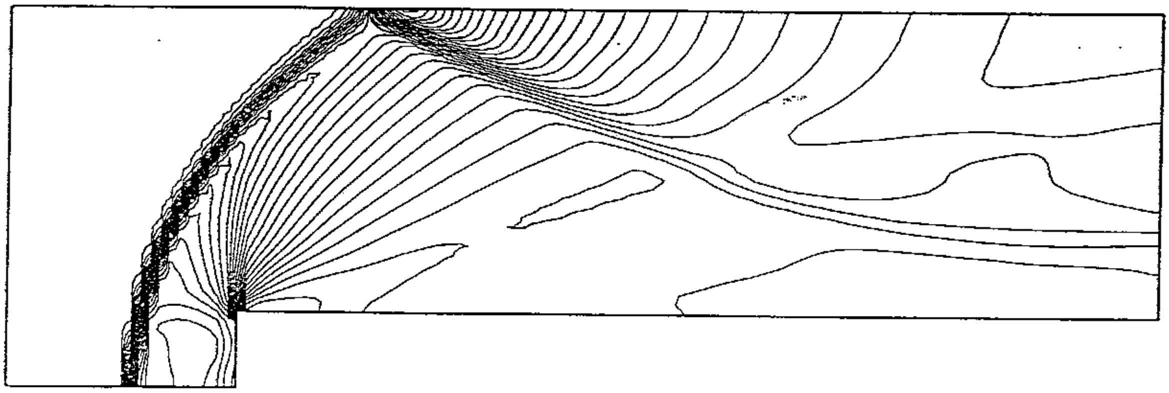

Figure 8: Density contours at time t = 1.0 without entropy correction.

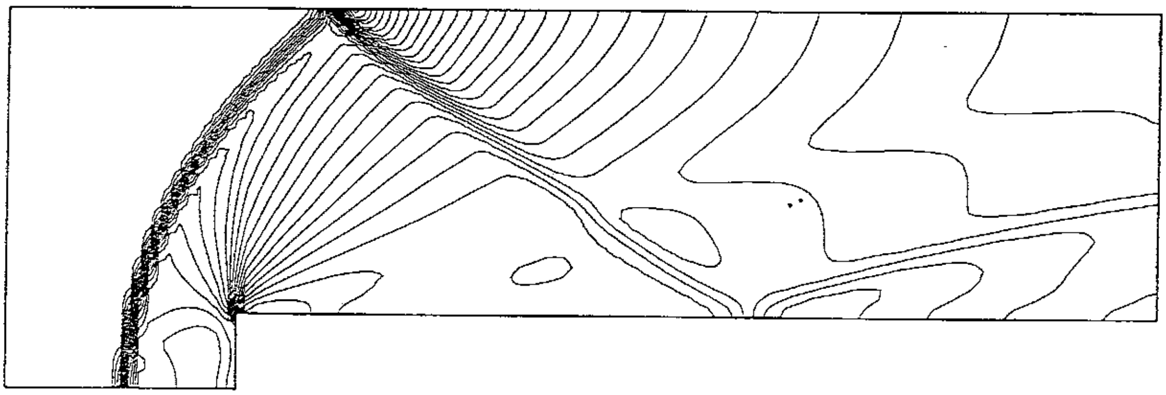

Figure 9: Density contours at time t = 1.5 without entropy correction.

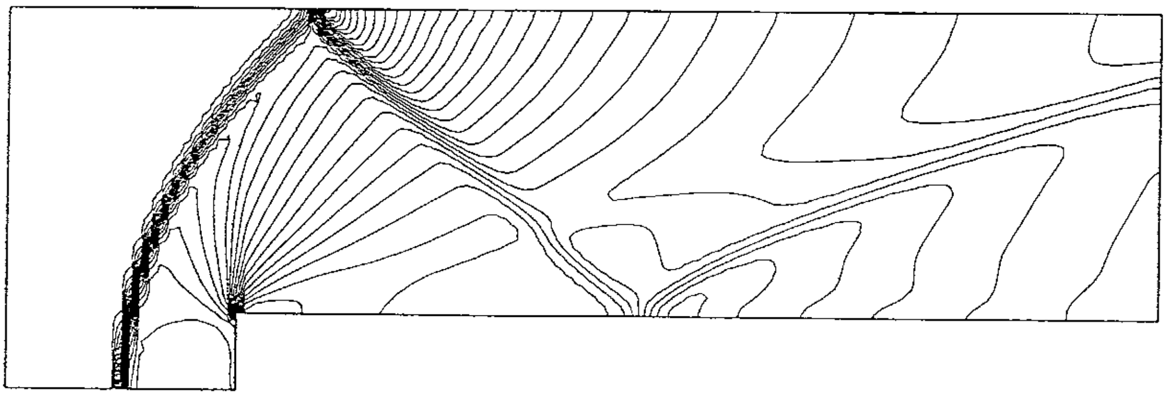

Figure 10: Density contours at time t = 2.0 without entropy correction.

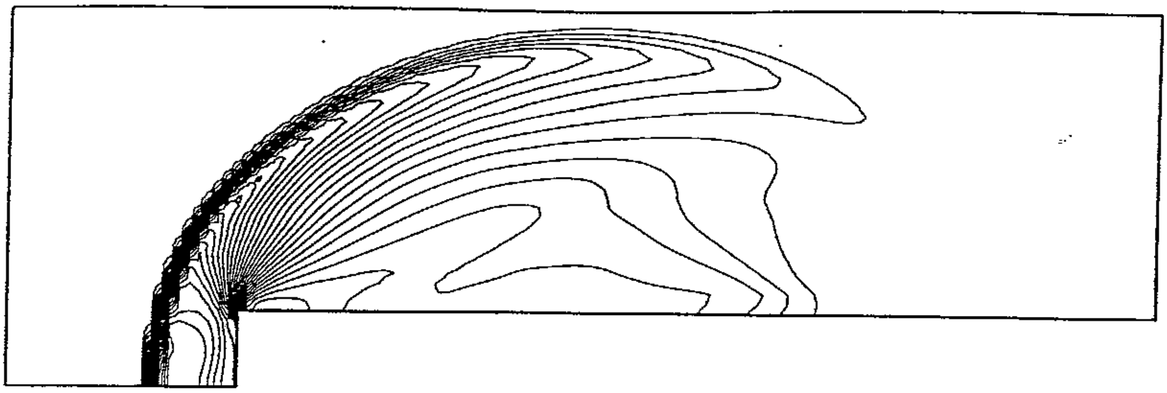

Figure 11: Density contours at time t = 0.5 with entropy correction.

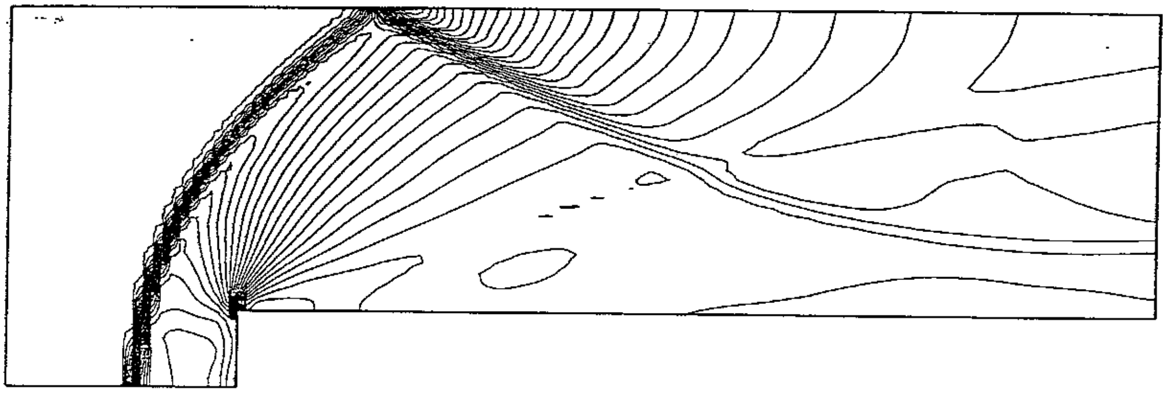

Figure 12: Density contours at time t = 1.0 with entropy correction.

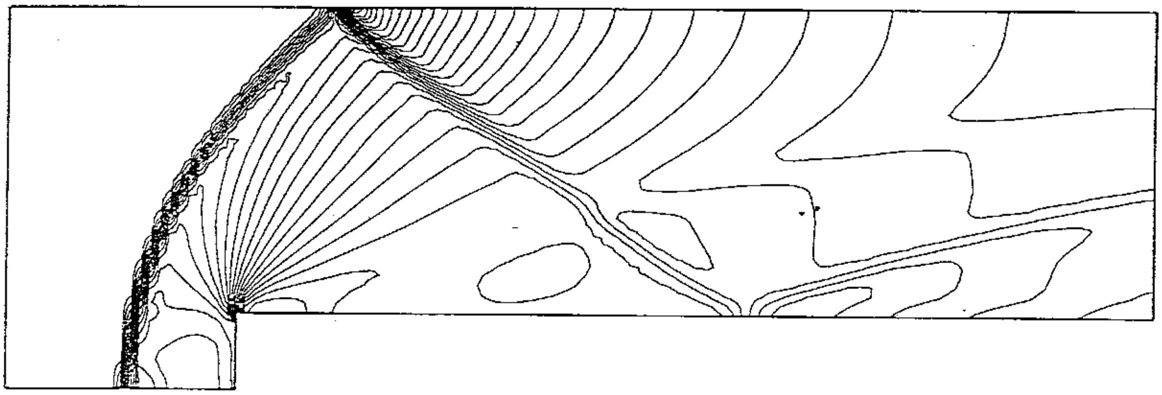

Figure 13: Density contours at time t = 1.5 with entropy correction.

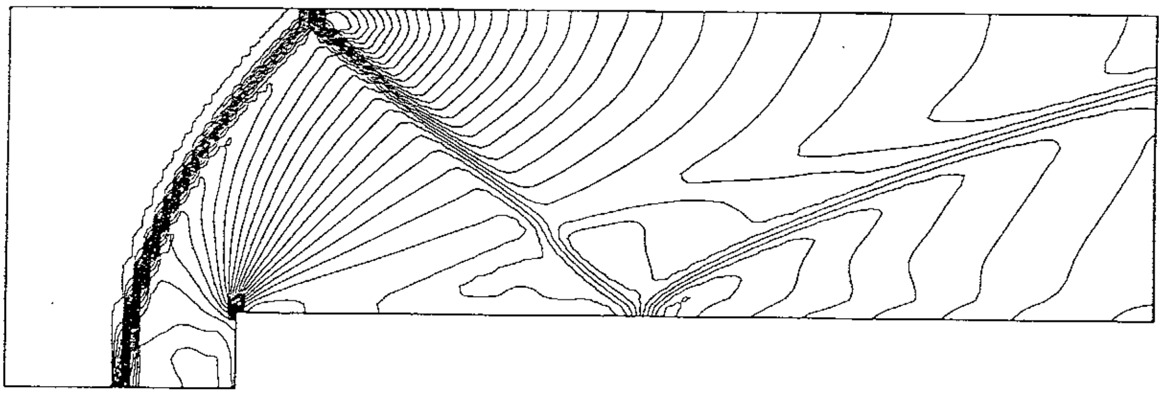

Figure 14: Density contours at time t = 2.0 with entropy correction.

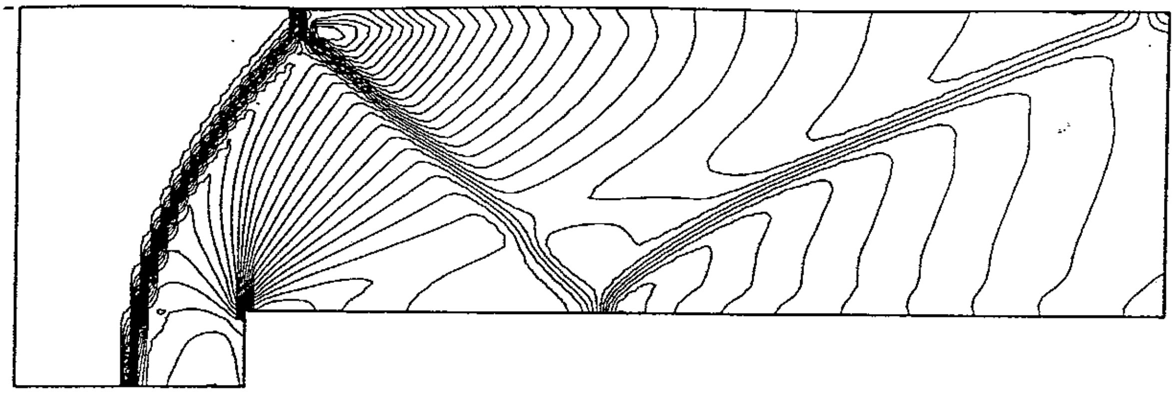

Figure 15: Density contours at time t = 2.5 without entropy correction.

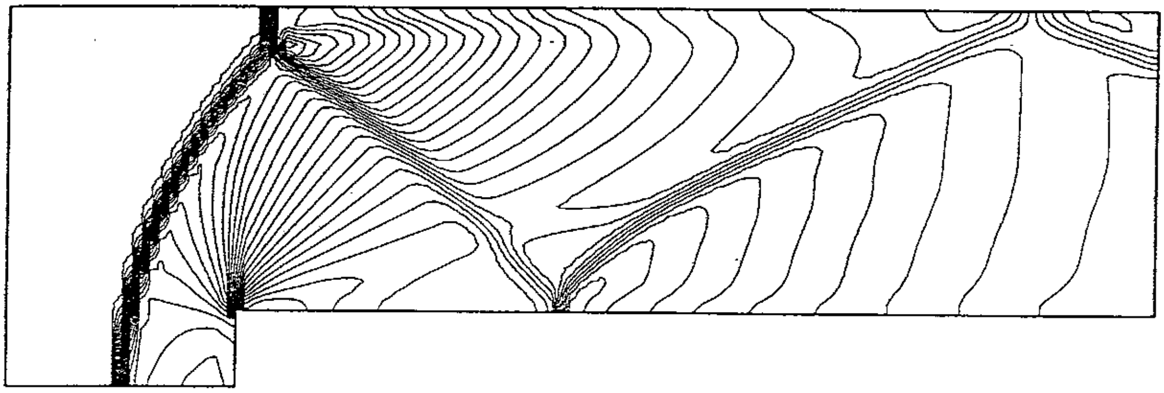

Figure 16: Density contours at time t = 3.0 without entropy correction.

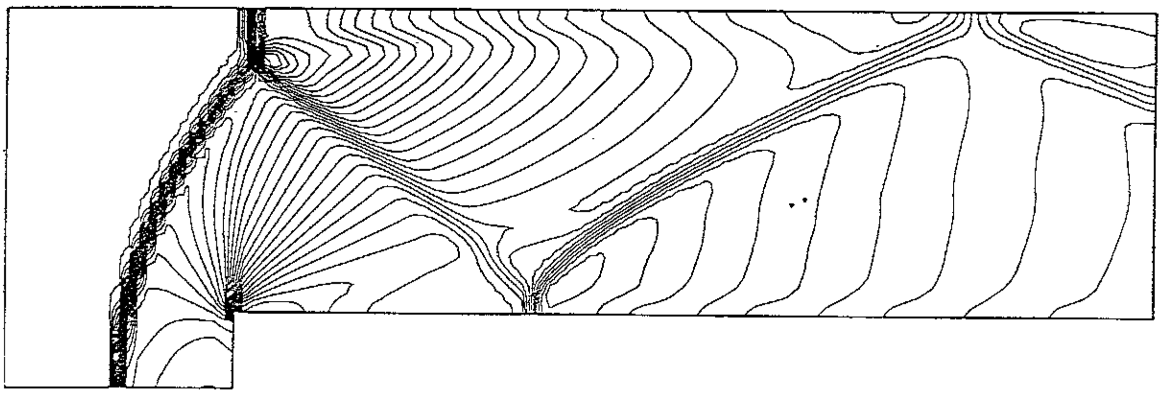

Figure 17: Density contours at time t = 3.5 without entropy correction.

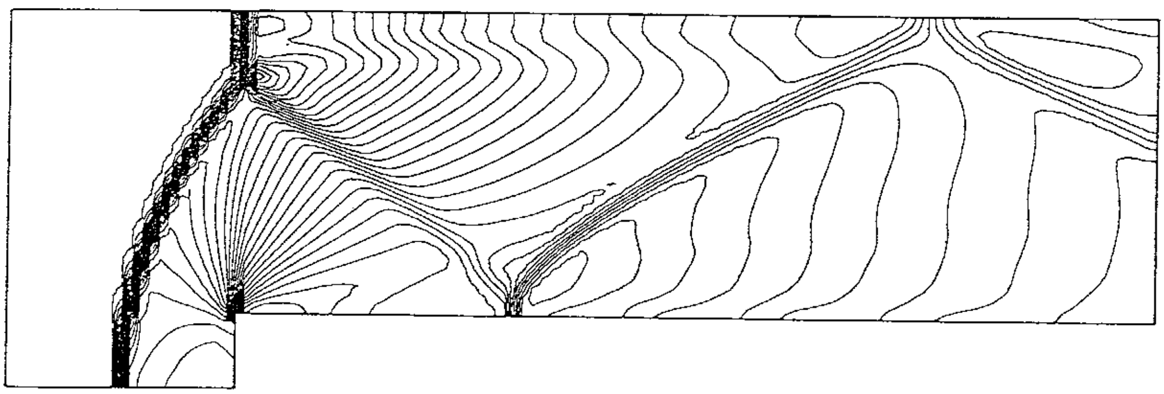

Figure 18: Density contours at time t = 4.0 without entropy correction.

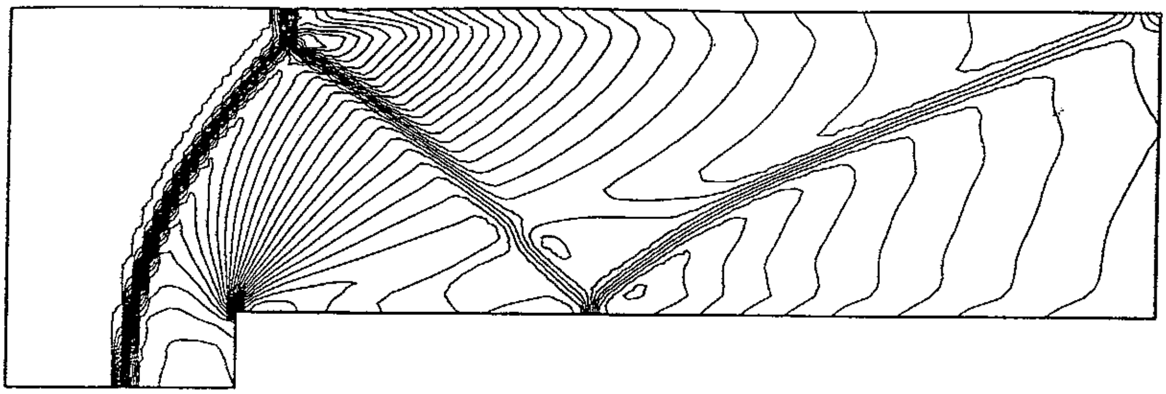

Figure 19: Density contours at time t = 2.5 with entropy correction.

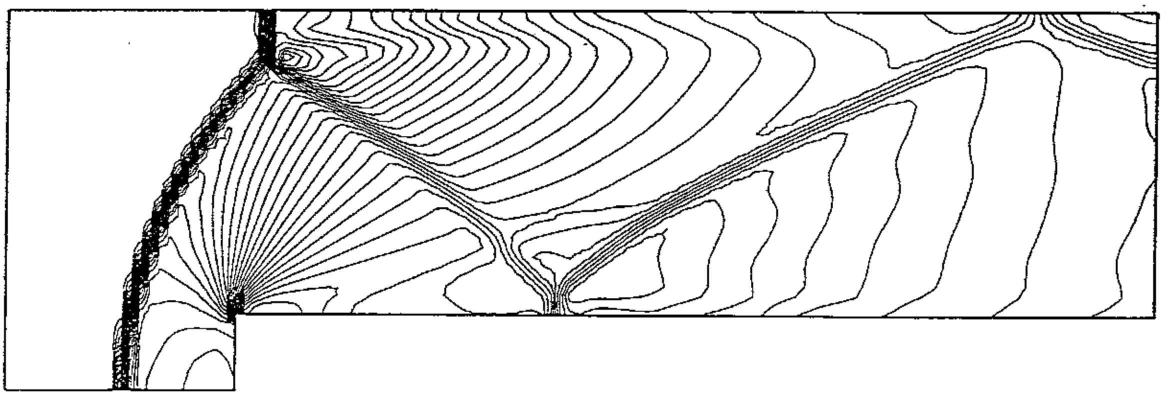

Figure 20: Density contours at time t = 3.0 with entropy correction.

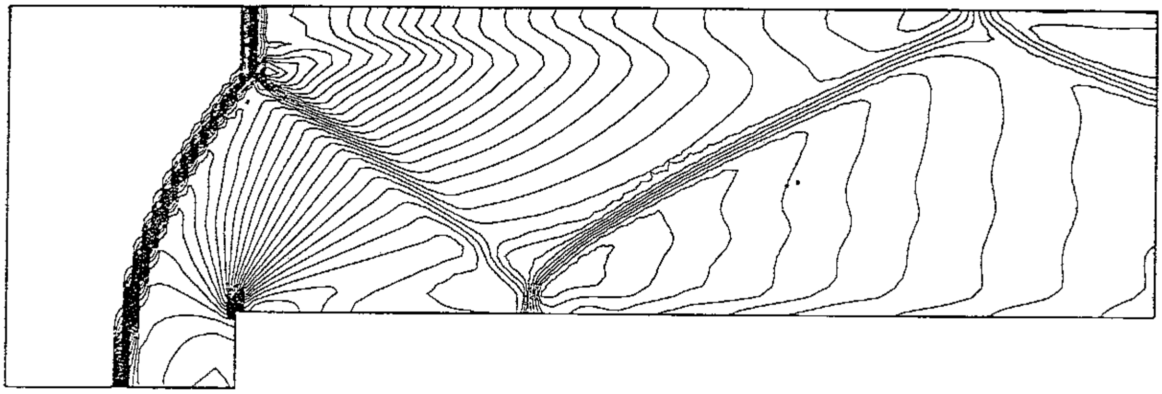

Figure 21: Density contours at time t = 3.5 with entropy correction.

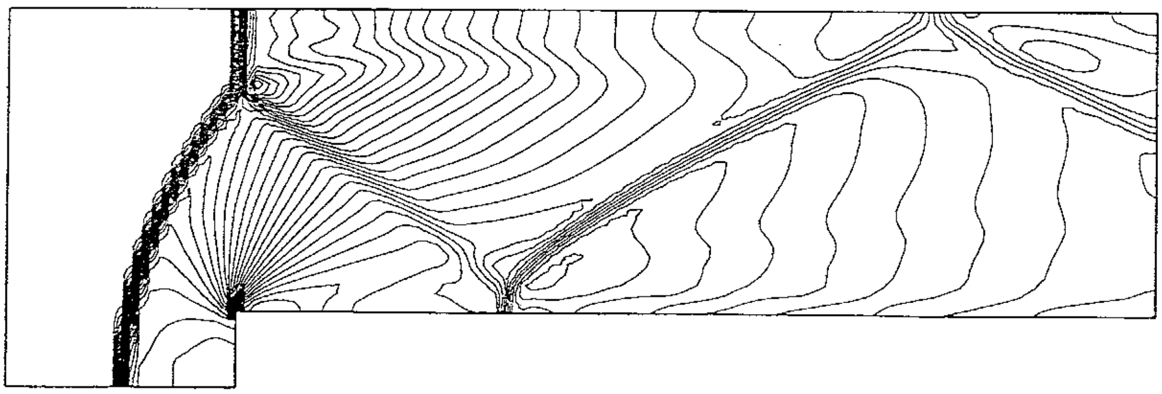

Figure 22: Density contours at time t = 4.0 with entropy correction.

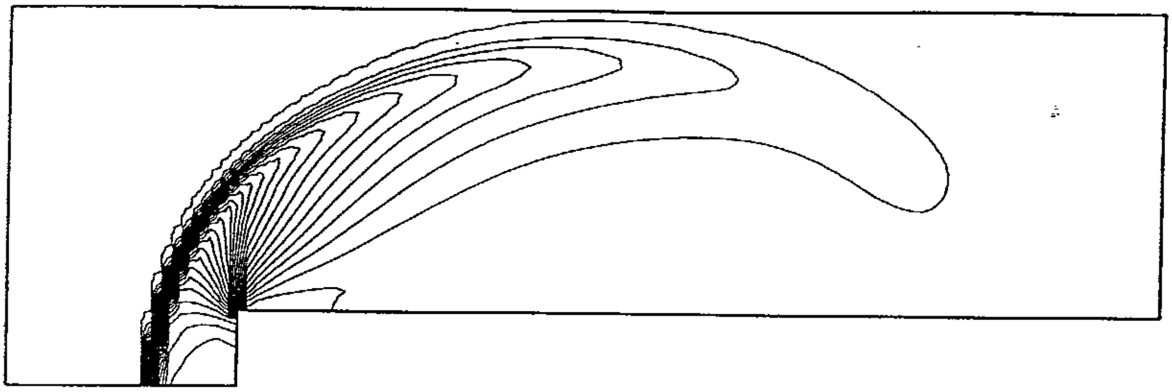

Figure 23: Pressure contours at time t = 0.5 without entropy correction.

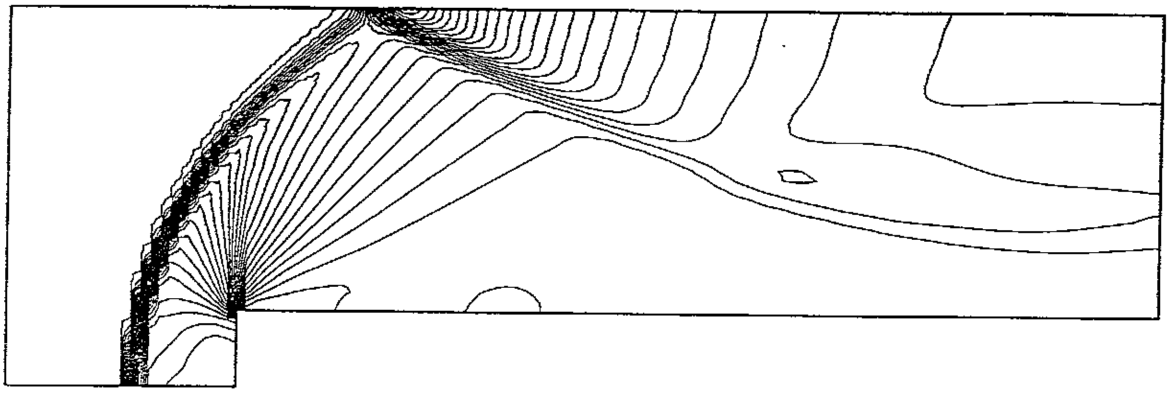

Figure 24: Pressure contours at time t = 1.0 without entropy correction.

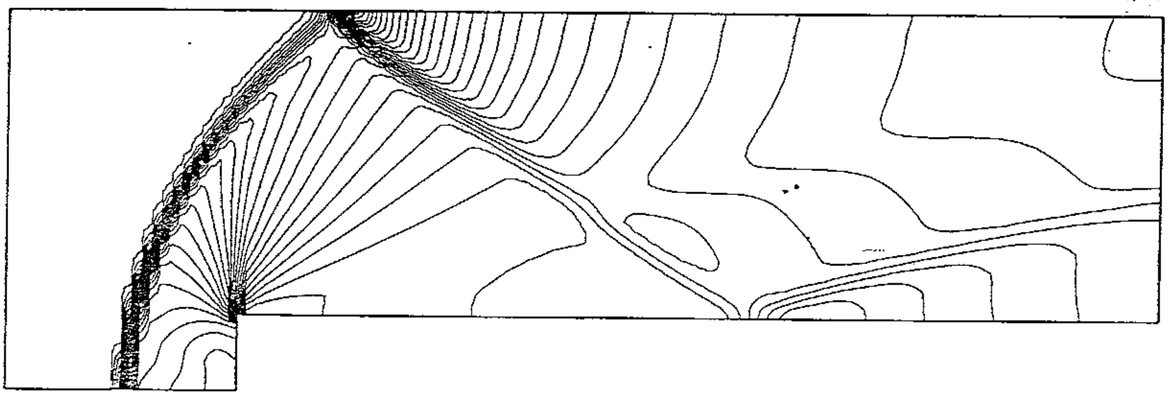

Figure 25: Pressure contours at time t = 1.5 without entropy correction.

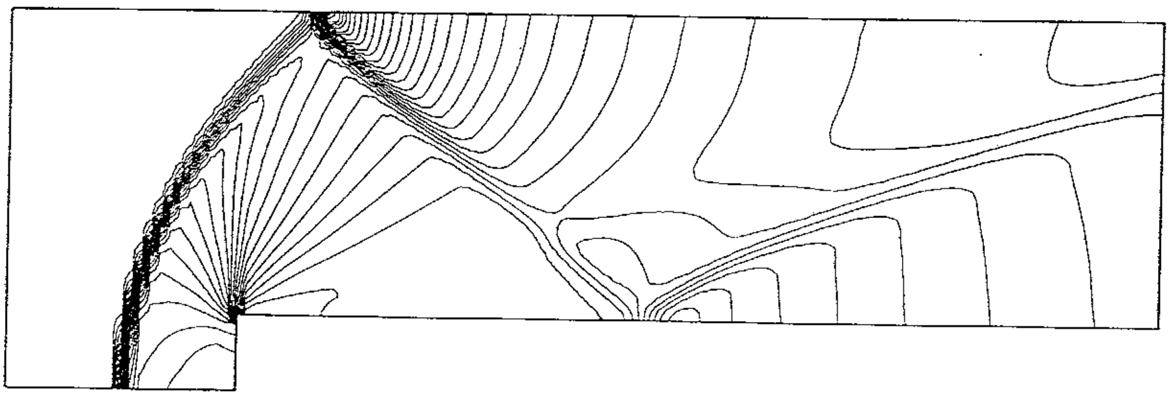

Figure 26: Pressure contours at time t = 2.0 without entropy correction.

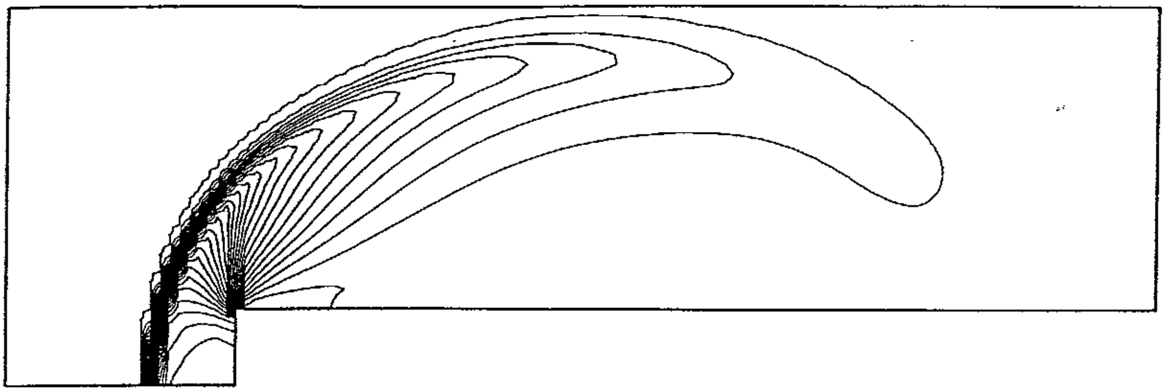

Figure 27: Pressure contours at time t = 0.5 with entropy correction.

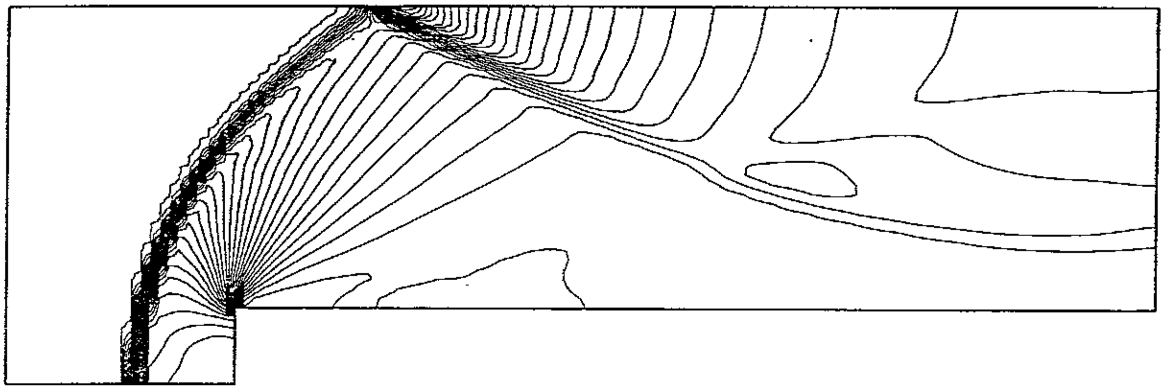

Figure 28: Pressure contours at time t = 1.0 with entropy correction.

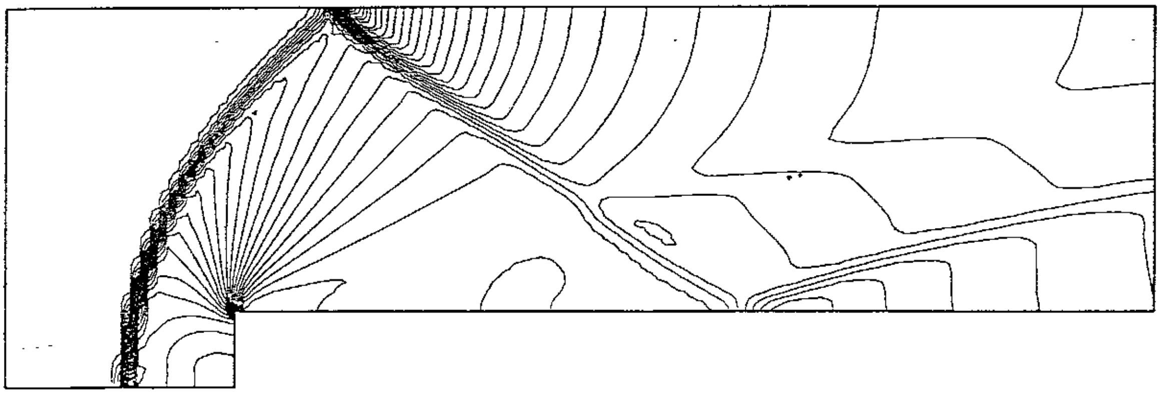

Figure 29: Pressure contours at time t = 1.5 with entropy correction.

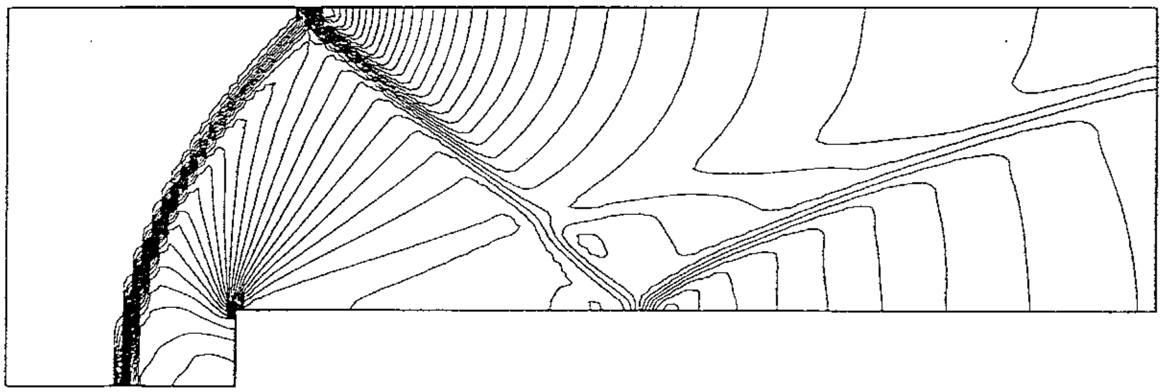

Figure 30: Pressure contours at time t = 2.0 with entropy correction.

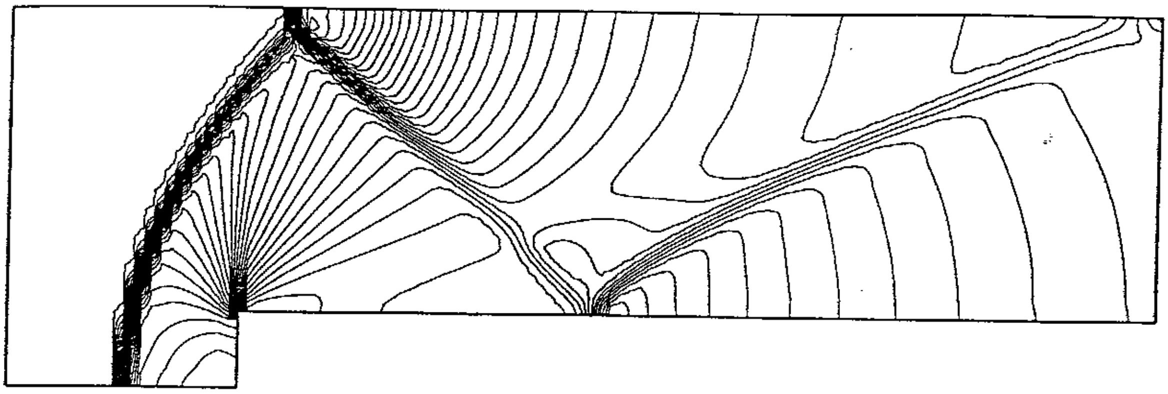

Figure 31: Pressure contours at time t = 2.5 without entropy correction.

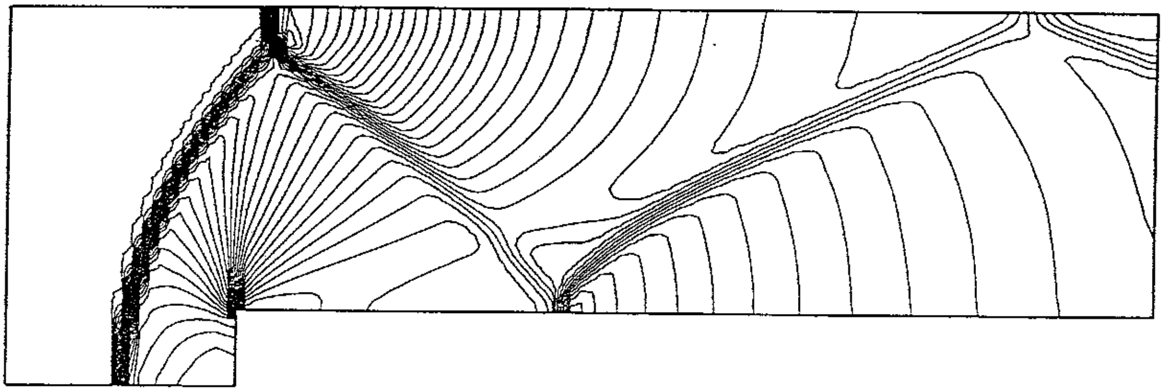

Figure 32: Pressure contours at time t = 3.0 without entropy correction.

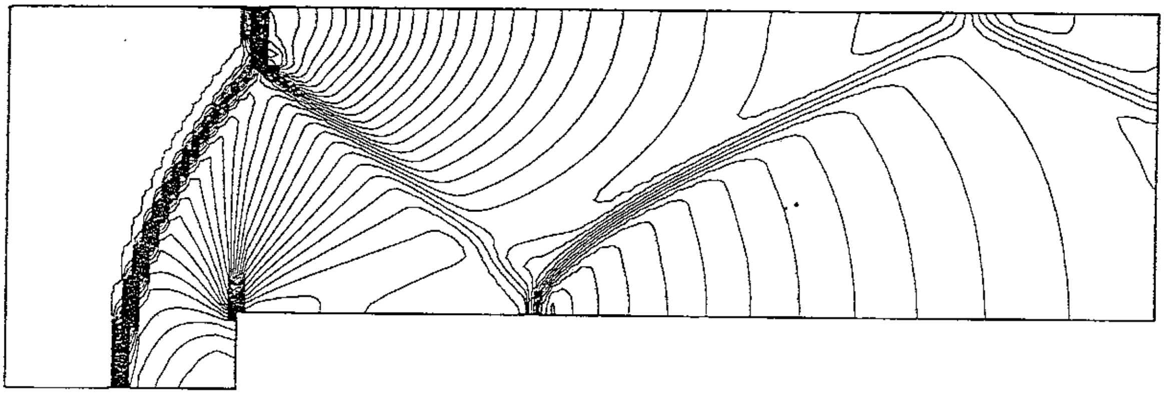

Figure 33: Pressure contours at time t = 3.5 without entropy correction.

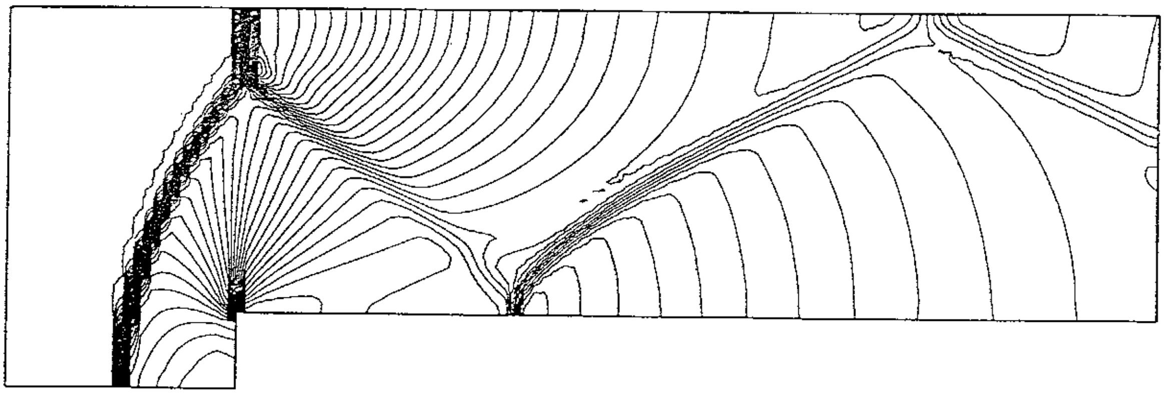

Figure 34: Pressure contours at time t = 4.0 without entropy correction.

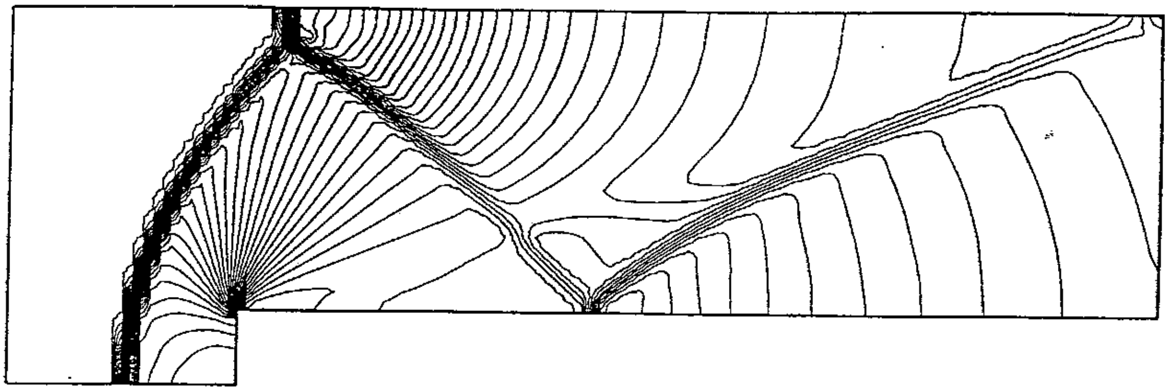

Figure 35: Pressure contours at time t = 2.5 with entropy correction.

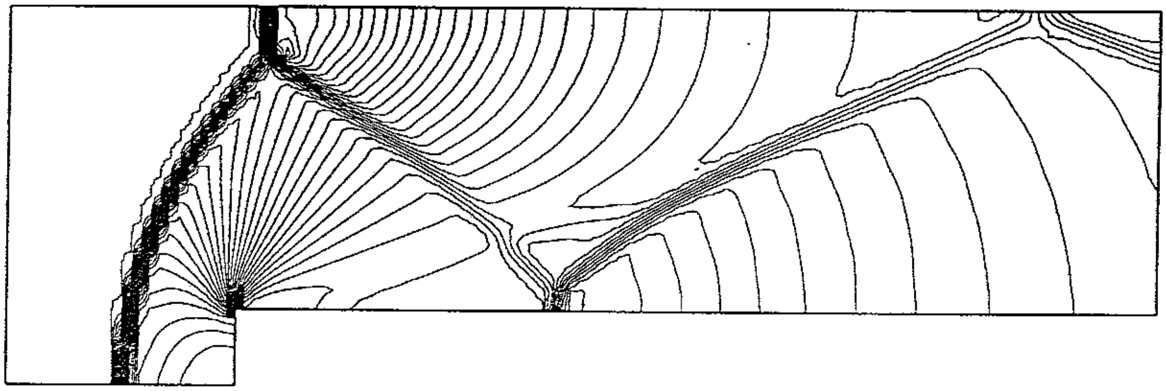

Figure 36: Pressure contours at time t = 3.0 with entropy correction.

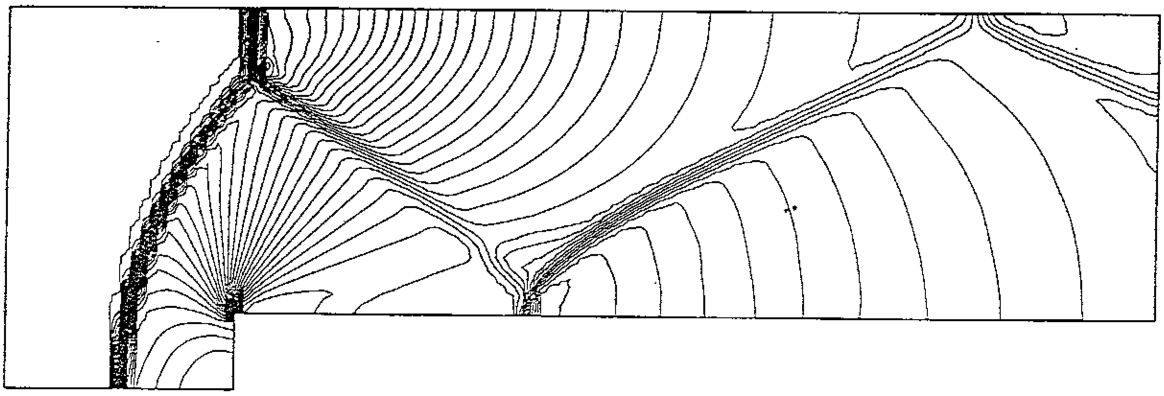

Figure 37: Pressure contours at time t = 3.5 with entropy correction.

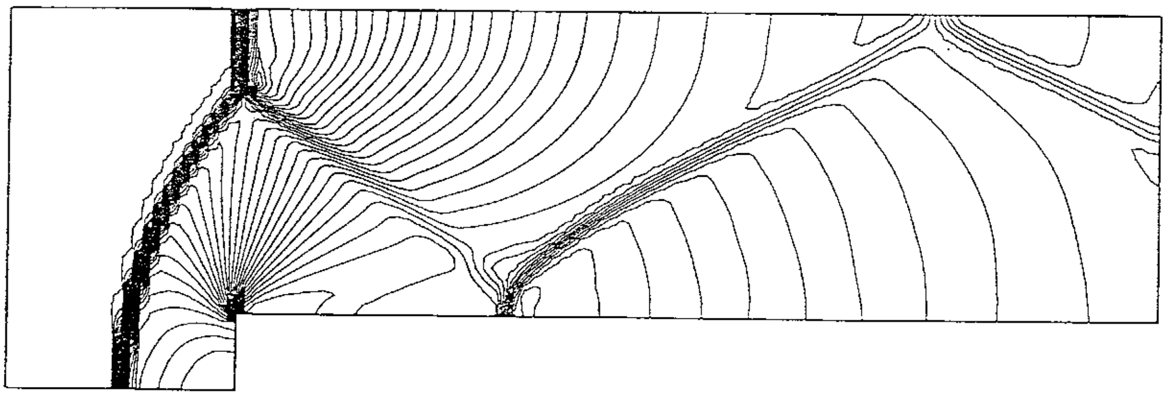

Figure 38: Pressure contours at time t = 4.0 with entropy correction.

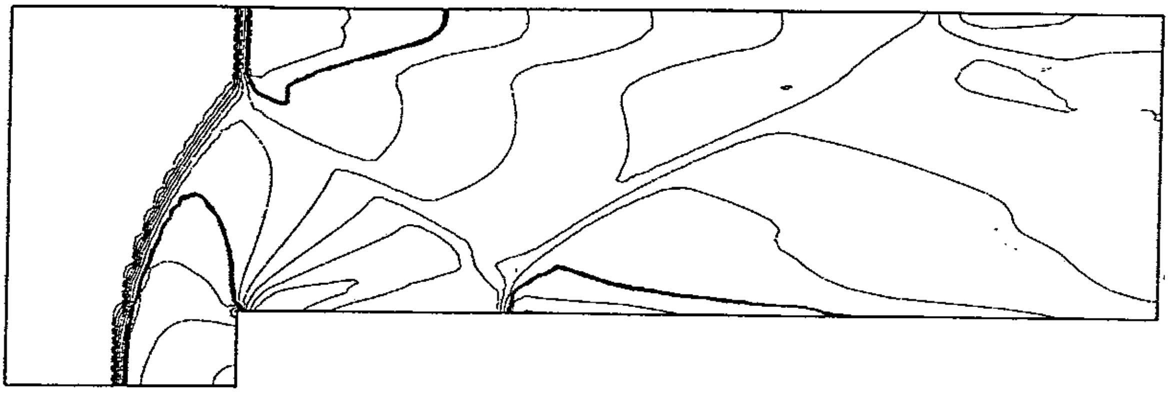

Figure 39: Mach 1 iso-values (bold lines) at time $t = 4$ for the supersonic step problem.

8 Conclusion

We have proposed a non-parameterized approach to entropy-enforcement for Roe-type schemes. It is based on the exact resolution of a Riemann problem associated with a Hermite interpolation of the physical flux rather than an a priori representation of the solution as proposed in previous work. In the scalar convex case, we have proved convergence of the method of lines to the unique entropy solution. We have also proved that this method is consistent in the small with the entropy inequality for hyperbolic systems of conservation laws. Numerical results for the Euler equations show that on a given mesh non-physical expansion shocks disappear. Moreover, the implementation of this entropy correction is easy and inexpensive in existing computer codes.

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