Raviart Thomas Petrov-Galerkin Finite Elements

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Test problem: $-\Delta u = f$, $u = 0$ on $\partial\Omega$.

For mixed Raviart Thomas space RT: $\mathrm{div}: \; RT \longrightarrow P^0$, P^0= piecewise constant.

Idea: We build a local test function space $RT^\star \;\perp\; RT$. Petrov Galerkin approach

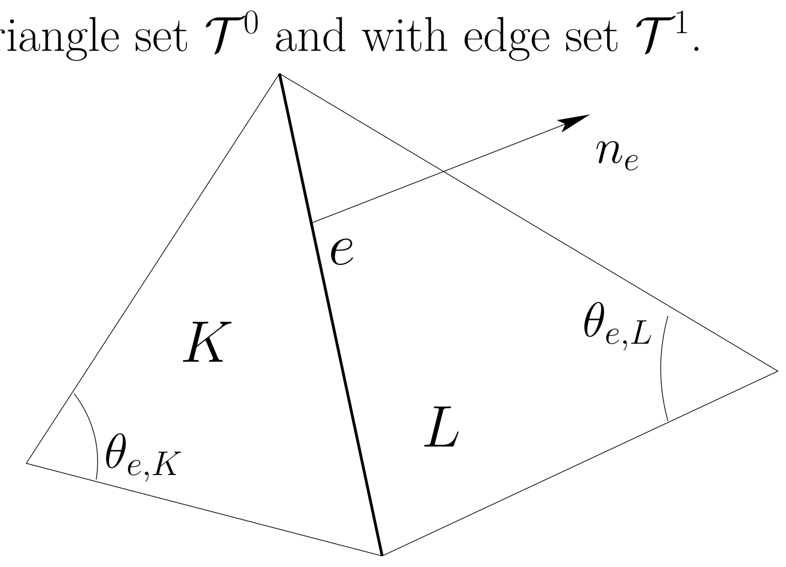
Duality $\Longrightarrow$ local discrete gradient, $\nabla_{\mathcal{T}} \;:\; P^0 \longrightarrow RT$. 2 point flux formula.

Stability conditions $\Downarrow$ 4-point finite volumes [1] and error estimates.

Problematic

Mesh and spaces

- $\mathcal{T}$ = triangle mesh of a domain Ω,
- with triangle set $\mathcal{T}^0$ and with edge set $\mathcal{T}^1$.



- $\mathbb{1}_K$ = the indicator function of K,
$$P^0 = \mathrm{Span}\left(\mathbb{1}_K, \quad K \in \mathcal{T}^0\right).$$
- φ_e = the Raviart Thomas basis function for e (of degree 0), $\quad RT = \mathrm{Span}\left(\varphi_e, \quad e \in \mathcal{T}^1\right)$.

Cell centered finite volumes

Definition of:

- a discrete space of fluxes $\mathcal{F}_{\mathcal{T}}$,
- a discrete divergence $\quad \mathrm{div}_{\mathcal{T}}: \quad \mathcal{F}_{\mathcal{T}} \longrightarrow P^0$,
- a discrete gradient $\quad \nabla_{\mathcal{T}} \quad : \quad P^0 \longrightarrow \mathcal{F}_{\mathcal{T}}$,

Discrete duality property: $\nabla_{\mathcal{T}} = -(\mathrm{div}_{\mathcal{T}})^\star$.
But $\mathcal{F}_{\mathcal{T}}$ is not a functional space.
Questions: what relationships between

- $\mathcal{F}_{\mathcal{T}}$ and $H(\mathrm{div},\Omega)$?
- the discrete operators and the differential ones ?

Mixed Finite Elements

- The flux space is $RT \subset H(\mathrm{div},\Omega)$,
- $\mathrm{div}_{\mathcal{T}} = \mathrm{div}: \; RT \longrightarrow P^0$

Try to define $\nabla_{\mathcal{T}}: \; P^0 \longrightarrow RT$ as $\nabla_{\mathcal{T}} = -\mathrm{div}^\star$
Problem:

- the $\varphi_e \in RT$ are not orthogonal,
- orthogonalisation $\Longrightarrow$ a **non local** discrete gradient.

Petrov Galerkin approach

- Definition of a dual space $RT^\star \neq RT$,
$$RT^\star = \mathrm{Span}\left(\varphi_e^\star, \quad e \in \mathcal{T}^1\right)$$
i. e. satisfying the orthogonality property,
$$\int_\Omega \varphi_e^\star \cdot \varphi_f \, dx = 0, \quad e, \; f \quad \text{distinct edges.}$$
- Discrete gradient definition,
$$\nabla_{\mathcal{T}} = -\Pi^{-1}\,\mathrm{div}^\star \quad : \; P^0 \to RT,$$
following the diagram,
$$\begin{array}{ccc} RT & \xrightarrow{\mathrm{div}} & P_0 \\ \Pi\downarrow & & \downarrow id \\ RT^\star & \xleftarrow[\mathrm{div}^\star]{} & P_0 \end{array}, \qquad \varphi_e^\star = \Pi\varphi_e.$$

RT Dual basis

Definition

$$RT^\star := \Pi(RT) \quad \text{with} \quad \Pi\varphi_e = \varphi_e^\star.$$

General contraints:

1. Orthogonality: $\quad (\varphi_e^\star, \varphi_f)_0 = 0 \;$ if $\; e \neq f$
2. Conformity: $\quad \varphi_e^\star \in H(\mathrm{div},\Omega)$
3. Localisation: $\quad \mathrm{Supp}\,\varphi_e^\star = K \cup L = \mathrm{Supp}\,\varphi_e$
4. Flux normalisation: $\quad \int_e \varphi_e^\star \cdot n_e \, dl = 1 = \int_e \varphi_e \cdot n_e \, dl$

Discrete gradient

Set $u \in P^0$: $\qquad u = \sum_{K\in\mathcal{T}^0} u_K \mathbb{1}_K$.

Then $\qquad \nabla_{\mathcal{T}} u = \sum_{e\in\mathcal{T}^1} p_e \varphi_e$

with $\qquad p_e = \dfrac{u_L - u_K}{(\varphi_e^\star, \varphi_e)_0}$.

This is a **two point flux formula**.
As for cell centered finite volume methods.

Discretisation of $-\Delta u = f$

Discrete Petrov Galerkin: find $u \in P^0$, $p \in RT$,
$$(p,q)_0 + (u, \mathrm{div}\, q)_0 = 0 \quad \text{and} \quad -(\mathrm{div}\, p, v)_0 = (f,v)_0,$$
for $v \in P^0$ and $q \in RT^\star$.

It is equivalent with:
find $u = \sum_K u_K \mathbb{1}_K \in P^0$ so that,
$$\frac{1}{|K|}\sum_{e=K|L} \frac{u_L - u_K}{(\varphi_e^\star, \varphi_e)_0} = \frac{1}{|K|}\int_K f \, dx. \qquad (P_{\mathcal{T}})$$

- Similar to cell centered finite volumes.
- The coefficients $(\varphi_e^\star, \varphi_e)_0$ define the scheme

Retrieving FV4

Proposition

Construction:

1. $\mathrm{div}\,\varphi_e^\star = \delta_k \;$ on $\; K$,
$\mathrm{div}\,\varphi_e^\star = -\delta_L \;$ on $\; L \;$ and $\; \varphi_e^\star = 0$ otherwise,
2. $\varphi_e^\star \cdot n_e = g \;$ on $\; e \;$ and
$\varphi_e^\star \cdot n_f = 0 \;$ on $\; f \;$ for $\; f \neq e$,

where δ_k and g satisfy (1)-(2).

Consequences:

- $\{\varphi_e^\star, \; e \in \mathcal{T}^1\}$ is a RT dual basis as above,
- The coefficients only depend on the cell angles,
$$(\varphi_e^\star, \varphi_e)_0 = \mathrm{cotan}(\theta_{K,e}) + \mathrm{cotan}(\theta_{L,e}).$$
- Equivalent to the 4-point finite volume scheme [1].

Constraints on δ_K and on g

The function $\delta_K: \; K \longrightarrow \mathbb{R}$ must satisfy,
$$\int_K \delta_K(x)\, dx = 1, \qquad (1)$$
$$\int_K \delta_K(x)\, |x - V_i|^2 \, dx = 0,$$
for V_i= vertexes of K, i =1, 2, 3.

The function $g: \; (0,1) \longrightarrow \mathbb{R}$ must satisfy,
$$\int_0^1 g(s) ds = 1, \quad g(s) = g(1-s) \quad \text{and} \qquad (2)$$
$$\int_0^1 g(s)\, s^2 \, ds = 0$$

Numerical analysis

Stability - error estimates

Proposition: with the uniform angle condition,
$$0 < \theta_\star \le \theta_{K,e} \le \theta^\star < \pi/2.$$
there are RT dual basis verifying (S1)-(S4).

Stability: $u_{\mathcal{T}}$ = solution of the discrete problem $(P_{\mathcal{T}})$:
$$\|u_{\mathcal{T}}\|_0 + \|\,\nabla_{\mathcal{T}} u_{\mathcal{T}}\|_{H(\mathrm{div},\Omega)} \le C\|f\|_0.$$
Convergence: u = exact solution,
$$\|u - u_{\mathcal{T}}\|_0 + \|\nabla u - \nabla_{\mathcal{T}} u_{\mathcal{T}}\|_{H(\mathrm{div},\Omega)} \le C h_{\mathcal{T}} \|f\|_0$$
with $h_{\mathcal{T}}$ the mesh size.

Stability conditions

Consider $\Pi: \; \varphi_e \in RT \longrightarrow \varphi_e^\star \in RT^\star$.
The stability condition are: for all $p \in RT$,

$$(p, \Pi p)_0 \;\ge\; C\|p\|_0^2, \qquad (S1)$$
$$\|\Pi p\|_0 \le C\,\|p\|_0, \qquad (S2)$$
$$(\mathrm{div}\,p, \mathrm{div}\,\Pi p)_0 \;\ge\; C\,\|\,\mathrm{div}\,p\|_0^2, \qquad (S3)$$
$$\|\,\mathrm{div}\,\Pi p\|_0 \;\le\; C\,\|\,\mathrm{div}\,p\|_0 \qquad (S4)$$

for C independent on $\mathcal{T}$.

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