Introduction to Contact Homology

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Lecture 1

Contact structures and Reeb dynamics

1.1 Contact structures

In these lectures, M will denote a compact, orientable manifold of dimension $2n-1$.

Definition 1. *A contact structure ξ on M is a maximally non-integrable hyperplane distribution. In other words, if $\xi = \ker \alpha$ locally, then $\alpha \wedge (d\alpha)^{n-1}$ is a non-vanishing top form.*

The aim of these lectures is to introduce contact homology $HC_*(M,\xi)$, a powerful invariant of (M,ξ). It is a part of the Symplectic Field Theory [6] of Eliashberg, Givental and Hofer. But first, let us discuss other contact invariants that are easier to define and compute.
Let

$$\mathcal{J} = \{J : \xi \to \xi \mid J^2 = -I, d\alpha(J\cdot, J\cdot) = d\alpha(\cdot,\cdot), d\alpha(\cdot, J\cdot) > 0\}$$

be the set of compatible complex structures on ξ. It is non empty and contractible.
The "classical invariants" of ξ are the topological invariants of the complex vector bundle (ξ, J), for some $J \in \mathcal{J}$.

Example 1. We will be particularly interested in the following classical invariants :

- The Chern class $c(\xi)$ of (ξ, J). In particular, the first Chern class $c_1(\xi)$ will play a role later in these lectures.
- The formal homotopy class of ξ, defined as the homotopy class of the reduction of the structure group of TM to $U(n-1) \times I$. It is sometimes called almost contact structure, and it encodes all the topological information from (ξ, J).

These invariants do not always distinguish non diffeomorphic contact structures.

Example 2. On T^3, consider the contact structures ξ_n, $n \geq 1$, defined as

$$\xi_n = \ker\big(\cos(n\theta)dx + \sin(n\theta)dy\big).$$

Using 3-dimensional techniques, Giroux [9] and Kanda [17] proved independently that these contact structures are not diffeomorphic. However, their classical invariants vanish identically. We shall see in Lecture 4 that contact homology can distinguish these contact structures ξ_n.

1.2 Reeb dynamics

In order to define contact homology, we first have to study the Reeb dynamics on contact manifolds.

Definition 2. *The Reeb vector field R_α that is associated to a contact form α is characterized by*

$$\begin{cases} \imath(R_\alpha)d\alpha &= 0, \\ \alpha(R_\alpha) &= 1. \end{cases}$$

Note that the dynamics of the vector field R_α strongly depend on the choice of a contact form α for a given contact structure ξ on M. We can indeed multiply α by any positive or negative function on M.

The closed trajectories of this vector field naturally appear in the study of the action functional

$$\mathcal{A} : C^\infty(S^1, M) \to \mathbb{R} : \gamma \to \int_\gamma \alpha.$$

Contact homology can be considered intuitively as a variant of Morse theory for the action functional. We are therefore interested in the critical points of $\mathcal{A}$.

Lemma 1. $\gamma \in Crit(\mathcal{A}) \iff \gamma$ *is a closed Reeb orbit of period* $\mathcal{A}(\gamma)$*.*

Proof. Let γ_t, $t \in [0,1]$, be a 1-parameter family of loops in M, with $\gamma_0 = \gamma$. We have

$$\begin{aligned} \frac{d}{dt}\mathcal{A}(\gamma_t)|_{t=0} &= \frac{d}{dt}|_{t=0} \int_{S^1} \gamma_t^* \alpha \\ &= \int_{S^1} \gamma^* \mathcal{L}_X \alpha \qquad \text{where } X = \frac{d}{dt}\gamma_t|_{t=0} \\ &= \int_\gamma \imath(X) d\alpha \\ &= \int_\gamma d\alpha(X, \dot{\gamma}) \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d}{dt}\mathcal{A}(\gamma_t)|_{t=0} = 0 \text{ for all } \gamma_t \quad &\iff \quad \imath(\dot\gamma)d\alpha = 0 \\ &\iff \quad \dot\gamma \text{ is proportional to } R_\alpha \end{aligned}$$

Moreover, since $\alpha(R_\alpha) = 1$, we see that $\int_\gamma \alpha$ is the period of γ when it is parametrized so that $\dot{\gamma} = R_\alpha$. ☐

As in Morse theory, we need to study the Hessian of $\mathcal{A}$ at the critical points. This corresponds to the linearized Reeb flow near a periodic orbit. Let γ be a closed Reeb orbit of period T, and let $p \in \gamma$. Let $\varphi_t : M \to M$ be the Reeb flow after time t; note that this map preserves ξ . The linearized return map $\Psi_\gamma : \xi_p \to \xi_p$ is the restriction to ξ_p of the differential at p of the map $\varphi_T : M \to M$. Since the Reeb flow preserves $d\alpha$, this map is symplectic.

Definition 3. *The closed Reeb orbit γ is non-degenerate if the map $\Psi_\gamma : \xi_p \to \xi_p$ has no eigenvalue equal to* 1.

Note that γ is non-degenerate if and only if γ is a non-degenerate critical point of $\mathcal{A}$, modulo reparametrization. Non-degeneracy can always be achieved by a small perturbation.

Lemma 2. *For any contact structure ξ on M, there exists a contact form α for ξ such that all closed orbits of R_α are non-degenerate.*

Proof. Consider the graph Γ_α of the Reeb flow in $\mathbb{R}^+ \times M \times M$: $(t, x, y) \in \Gamma_\alpha$ iff $\varphi_t(x) = y, t \geq 0$. Fix $T > 0$. We want to find an arbitrarily small perturbation α_T of α so that the perturbed graph Γ_{α_T} is transverse to the graph $\mathbb{R}^+ \times \Delta_M$ of the identity in the compact set $[0, T] \times M \times M$. Let $p \in \Gamma_\alpha \cap [0, T] \times \Delta_M$ be a point of non-transversal intersection, and let γ be the degenerate closed Reeb orbit through p. Given a function f_p such that $df_p = 0$ along γ and supported in a small tubular neighborhood of γ and $\epsilon > 0$ very small, we can replace α with $(1 + \epsilon f_p)\alpha$. Then the linearized Reeb flow along γ has the additional term

$$\frac{\epsilon}{(1 + \epsilon f_p)^2} J\mathrm{Hess}(f_p) \qquad \text{for } J \in \mathcal{J},$$

where $\mathrm{Hess}(f_p)$ is the Hessian of f_p with respect to the metric $d\alpha(\cdot, J\cdot)$. Hence, we can choose f_p so that the perturbed linearized Reeb flow along γ is non-degenerate. If $\epsilon > 0$ is sufficiently small, no closed orbit of period $\leq T$ will be created by this perturbation. Therefore, there is a neighborhood U_p of p so that all closed Reeb orbits with period $\leq T$ passing through U_p are non-degenerate. We can do this for all $p \in \Gamma_\alpha \cap [0, T] \times \Delta_M$. Extract a finite covering for the compact set $\Gamma_\alpha \cap [0, T] \times \Delta_M$ consisting of open sets U_{p_i}, $i = 1, \ldots, N$. Let R be the vector space generated by the corresponding functions f_{p_i}, $i = 1, \ldots, N$. By construction, the manifold

$$\Gamma_{\alpha,R} = \{(t, x, y) \in \mathbb{R}^+ \times M \times M \mid \varphi_t^f(x) = y, f \in R, \|f\| \text{ small}\}$$

is transverse to $[0, T] \times \Delta_M$, where φ^f is the perturbed Reeb flow corresponding to $(1 + f)\alpha$. By Sard theorem, we can pick a regular value $f_T \in R$, arbitrarily close to 0, of the projection map $\Gamma_{\alpha,R} \cap [0, T] \times \Delta_M \to R$. Then, the closed Reeb orbits with period $\leq T$ for the contact form $\alpha_T = (1 + f_T)\alpha$ are all non-degenerate.

The above discussion shows hat the set $\mathcal{C}_T$ of contact forms for ξ such that all closed orbits of R_α with period $\leq T$ are non-degenerate, is open and dense. By Baire theorem, $\underset{T\geq 0}{\cap}\mathcal{C}_T$ is not empty. $\square$

In order to define contact homology, we have to assume that α is chosen generically, so that all closed Reeb orbits are non-degenerate.

Let γ be a non-degenerate closed Reeb orbit of period T. Fix a symplectic trivialization of ξ along γ. Then the path $d\varphi_t : \xi_p \to \xi_{\varphi_t(p)}$, $t \in [0,T]$, is represented by a path $\Psi_\gamma(t)$, $0 \leq t \leq T$, of symplectic matrices, such that $\Psi_\gamma(0) = I$ and $\det(\Psi_\gamma(T) - I) \neq 0$. A number $t \in [0,T]$ is called a crossing if $\det(\Psi_\gamma(t) - I) = 0$. We denote $\ker(\Psi_\gamma(t) - I)$ by E_t. For a crossing $t \in [0,T]$, the crossing form $\Gamma(\Psi_\gamma, t)$ is the quadratic form on E_t defined by

$$\Gamma(\Psi_\gamma, t)v = d\alpha(v, \dot{\Psi}_\gamma v) \qquad \text{for } v \in E_t.$$

A crossing t is regular if the crossing form at t is non-degenerate. If the path Ψ_γ has regular crossings, we define

$$\mu_{CZ}(\Psi_\gamma) = \frac{1}{2}\text{sign }\Gamma(\Psi_\gamma, 0) + \sum_{t \text{ crossing } \neq 0} \text{sign }\Gamma(\Psi_\gamma, t).$$

The next Lemma [24] shows that the Conley-Zehnder index is invariant under some deformations.

Lemma 3. (Robbin, Salamon) *Two paths of symplectic matrices Ψ_0 and Ψ_1, that are homotopic with fixed ends and have regular crossings, satisfy $\mu_{CZ}(\Psi_0) = \mu_{CZ}(\Psi_1)$.*

Therefore, we can make the following definition for more general paths.

Definition 4. *The Conley-Zehnder index of a path of symplectic matrices $\Psi(t)$, $t \in [0,T]$, satisfying $\Psi(0) = I$ and $\det(\Psi(T) - I) \neq 0$, is defined as*

$$\mu_{CZ}(\tilde{\Psi}) = \frac{1}{2}sign\ \Gamma(\tilde{\Psi}, 0) + \sum_{t\ crossing\ \neq 0} sign\ \Gamma(\tilde{\Psi}, t)$$

for a small pertubation $\tilde{\Psi}$ of Ψ with fixed ends, having regular crossings.

Note that there are other ways to define the Conley-Zehnder index (see for example [24]), but this approach is useful for computations.

For any closed Reeb orbit γ in M^{2n-1}, we define $|\gamma| = \mu_{CZ}(\Psi_\gamma) + n - 3$. Of course, the index $\mu_{CZ}(\Psi_\gamma)$ depends on the choice of the trivialization. In particular, if γ is homologically trivial, we choose a spanning surface S_γ for γ and trivialize ξ on S_γ. Given $A \in H_2(M, \mathbb{Z})$, we can form the connected sum $S_\gamma \sharp A$. To this new spanning surface corresponds a new trivialization of ξ. We have

$$\mu_{CZ}(\gamma; S_\gamma \sharp A) = \mu_{CZ}(\gamma; S_\gamma) + 2\langle c_1(\xi), A\rangle.$$

It is therefore natural to introduce the following grading on $H_2(M,\mathbb{Z})$:

$$|A| = -2\langle c_1(\xi), A\rangle \qquad \text{for } A \in H_2(M,\mathbb{Z}).$$

Let $\mathcal{R}$ be a submodule of $H_2(M,\mathbb{Z})$ with zero grading. Then the quotient $H_2(M,\mathbb{Z})/\mathcal{R}$ has a well-defined grading.
Let $\gamma = \gamma_1$ be a closed Reeb orbit with minimal period T. We are also interested in the multiples covers γ_m of γ with period mT, $m \geq 2$.
There are 2 ways the grading of γ_m can behave :

(i) the parity of $|\gamma_m|$ is the same for all $m \geq 1$.

(ii) the parity for the even multiples $|\gamma_{2k}|$, $k \geq 1$, disagrees with the parity for the odd multiples $|\gamma_{2k-1}|$, $k \geq 1$.

Definition 5. *In the second case, the even multiples $|\gamma_{2k}|$, $k \geq 1$, are called bad orbits. An orbit that is not bad is called good.*

Note that this notion is well-defined, since $|\gamma| \mod 2$ is independent of te choice of a trivialization of ξ along γ.

Definition 6. *Let C_* be the graded module freely generated by all good closed Reeb orbits γ, over the graded ring $\mathbb{Q}[H_2(M,\mathbb{Z})/\mathcal{R}]$.*

Elements of $\mathbb{Q}[H_2(M,\mathbb{Z})/\mathcal{R}]$ will be written as $\sum_{i=1}^{k} q_i e^{A_i}$, where $q_i \in \mathbb{Q}$ and $A_i \in H_2(M,\mathbb{Z})/\mathcal{R}$. The module C_* splits as the direct sum $\bigoplus_{\bar{a}} C_*^{\bar{a}}$, where $\bar{a}$ denotes a free homotopy class for the closed Reeb orbits γ.

Definition 7. *Let A be the graded supercommutative algebra with a unit, freely generated by all good closed Reeb orbits γ, over the graded ring $\mathbb{Q}[H_2(M,\mathbb{Z})/\mathcal{R}]$.*

Here supercommutative means that $\gamma_1\gamma_2 = (-1)^{|\gamma_1||\gamma_2|}\gamma_2\gamma_1$. Note that the coefficient ring is commutative since its grading is even.
Next, we need to introduce a differential d on C_* and A, counting J-holomorphic curves in the symplectization of M.

Lecture 2

Holomorphic curves in symplectizations

Pseudo-holomorphic curves were introduced in symplectic geometry by Gromov [10]. The construction of the moduli spaces of pseudo-holomorphic curves in closed symplectic manifolds is nicely explained by McDuff and Salamon [22]. On the other hand, pseudo-holomorphic curves were not considered in symplectizations of contact manifolds for some time, because their behavior near the ends of the symplectization was not understood. Hofer [12] was the first to use such holomorphic curves, in order to prove the Weinstein conjecture on S^3. Since then, they have been the object of important analytical work, as in the papers [13], [14], [15] and [16].

Definition 8. *The symplectization of the contact manifold (M, ξ) with contact form α is the symplectic manifold $(\mathbb{R} \times M, d(e^t\alpha))$, where t is the coordinate of $\mathbb{R}$.*

Given $J \in \mathcal{J}$, we can extend J to an almost complex structure (that we still denote by J) on $\mathbb{R} \times M$, compatible with $\omega = d(e^t\alpha)$: $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$ and $\omega(\cdot, J\cdot) > 0$. We define this extension by letting $J\frac{\partial}{\partial t} = R_\alpha$.
We will be interseted in J-holomorphic curves in the symplectization, i.e. maps $F : (\Sigma, j) \to (\mathbb{R} \times M, J)$ defined on a Riemann surface (Σ, j) and satisfying $dF \circ j = J \circ dF$.
We restrict ourselves to rational curves $\Sigma = S^2 \setminus \{x, y_1, \ldots, y_s\}$ with several punctures. We require the following behavior near the punctures : let (ρ, θ) be polar coordinates centered on a puncture; we write $F = (a, f) \in \mathbb{R} \times M$. We want

$$\begin{aligned}
\lim_{\rho \to 0} a(\rho, \theta) &= +\infty \qquad \text{for puncture } x, \\
&= -\infty \qquad \text{for punctures } y_1, \ldots, y_s, \\
\lim_{\rho \to 0} f(\rho, \theta) &= \gamma\left(-\frac{T}{2\pi}\theta\right) \qquad \text{for puncture } x, \\
&= \gamma_i\left(\frac{T_i}{2\pi}\theta\right) \qquad \text{for puncture } y_i \ (i = 1, \ldots, s),
\end{aligned}$$

for some closed orbits γ of period T and γ_i of period T_i, $i = 1, \dots, s$. In other words, F must converge at $t = \pm\infty$ to vertical cylinders over closed Reeb orbits. Note that these are clearly J-holomorphic, since $J\frac{\partial}{\partial t} = R_\alpha$.

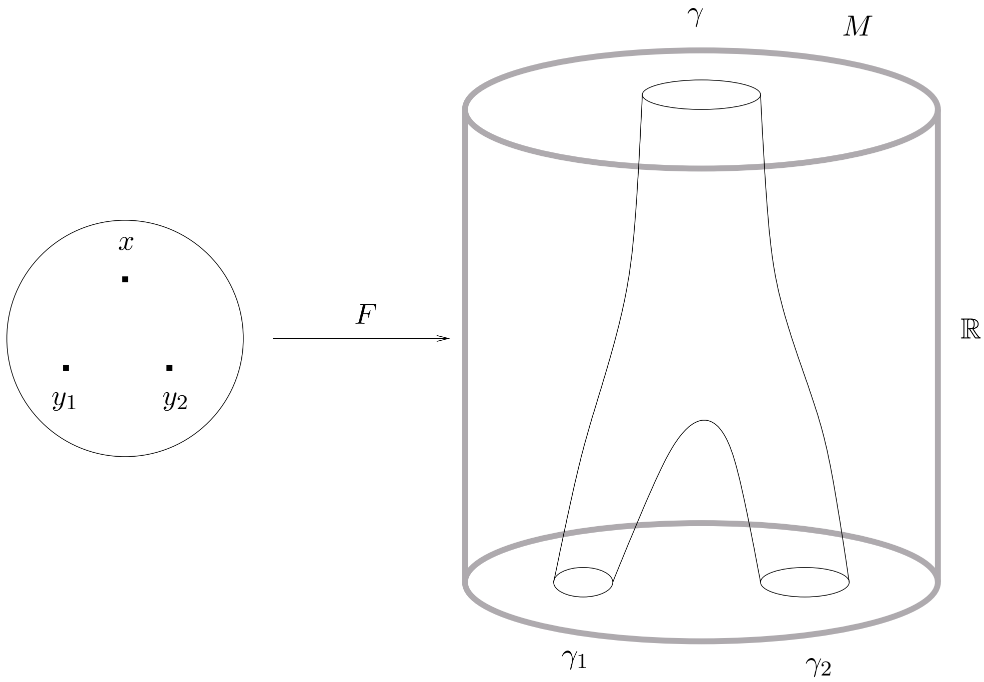

Figure 2.1: Holomorphic map F in $\mathbb{R} \times M$ with $s = 2$.

We denote by $\mathcal{H}ol(\gamma; \gamma_1, \dots, \gamma_s)$ the set of J-holomorphic curves $F : (S^2 \setminus \{x, y_1, \dots, y_s\}, j) \to (\mathbb{R} \times M, J)$ that are asymptotic to γ near x at $+\infty$ and to γ_i near y_i, $i = 1, \dots, s$ at $-\infty$.
We introduce on this set the following equivalence relation : the holomorphic maps $F : (S^2 \setminus \{x, y_1, \dots, y_s\}, j) \to (\mathbb{R} \times M, J)$ and $F' : (S^2 \setminus \{x', y'_1, \dots, y'_s\}, j') \to (\mathbb{R} \times M, J)$ are equivalent if and only if there exists a biholomorphism $h : (S^2, j) \to (S^2, j')$ so that $h(x) = x'$, $h(y_i) = y'_i$ for $i = 1, \dots, s$, and $F = F' \circ h$.

Definition 9. *The moduli space of J-holomorphic curves $\mathcal{M}(\gamma; \gamma_1, \dots, \gamma_s)$ is the set of equivalence classes in $\mathcal{H}ol(\gamma; \gamma_1, \dots, \gamma_s)$. It has an $\mathbb{R}$-action induced by the translation $t \to t + \Delta t$ in $\mathbb{R} \times M$.*

We would like $\mathcal{M}$ to have a geometric structure that is as nice as possible. We can achieve this by perturbing the Cauchy-Riemann equation $\overline{\partial} F = \frac{1}{2}(dF + J \circ dF \circ j) = 0$. We use a multi-valued perturbation $\overline{\partial} F = \nu(F) \in \Gamma(F)$ that is invariant under the action of biholomorphisms. The perturbation of the moduli spaces follows the approach of Liu and Tian [20]; additional details about this construction can be found in the exposition [21] by McDuff. Note that there are other approaches to the perturbation of the moduli spaces; these are due to Fukaya and Ono [8], Li and Tian [19], Ruan [23], and Siebert [26]. After this delicate process, we obtain the following structure on $\mathcal{M}$.

Proposition 1. *$\mathcal{M}(\gamma;\gamma_1,\ldots,\gamma_s)/\mathbb{R}$ is a compact branched labeled manifold with corners, i.e. the union of manifolds with corners along a codimension 1 branching locus, with each manifold having a rational weight, so that near each branching point, the sum of all entering weights equals the sum of all exiting weights. Each manifold with corner has dimension*

$$(n-3)(1-s)+\mu_{CZ}(\gamma)-\sum_{i=1}^{s}\mu_{CZ}(\gamma_i)+2c_1^{rel}(\xi,\Sigma)-1$$

where $c_1^{rel}(\xi,\Sigma)$ is the relative first Chern class of ξ on Σ, relative to the fixed trivializations of ξ along the closed Reeb orbits at the punctures.

The rational weights take into account the automorphisms of J-holomorphic curves : if F is a rigid element in $\mathcal{M}/\mathbb{R}$ of dimension 0, then the weight of F is $\frac{1}{k}$, where k is the order of the automorphism group of F.

According to the compactness theorem [4], the boundary of $\mathcal{M}/\mathbb{R}$ consists of "broken" J-holomorphic curves, i.e. finite collections $C_j=\{F_{1,j},\ldots,F_{l,j}\}$, $j=1,\ldots,k$ of J-holomorphic curves as above, such that

(i) the closed orbits at $+\infty$ in C_j coincide exactly with the orbits at $-\infty$ for C_{j-1}, $j=2,\ldots,k$,

(ii) the closed orbit at $+\infty$ for C_1 is γ,

(iii) the closed orbits at $-\infty$ for C_k are $\gamma_1,\ldots,\gamma_s$.

This is illustrated by Figure 2.2. The codimension of such broken J-holomorphic curves in $\mathcal{M}/\mathbb{R}$ is given by $k-1$. Conversely, any such "broken" J-holomorphic curve lies in the boundary of $\mathcal{M}/\mathbb{R}$.

We now explain why this degeneration process does not involve J-holomorphic curves with several closed orbits at $+\infty$. In order to approach such a limiting curve, a J-holomorphic curve would have to develop a maximum for the t coordinate in $\mathbb{R}\times M$ and have this maximum diverge to $+\infty$. However, the existence of such a maximum is prohibited by the following lemma.

Lemma 4. *For any J-holomorphic curve $F=(a,f):(\Sigma,j)\to(\mathbb{R}\times M,J)$, the function a has no local maximum.*

Proof. Note that $\Delta e^a=F^*(-dd^ce^t)$, where $d^c\beta=d(\beta\circ J)$. But

$$\begin{aligned}-dd^ce^t &= -d(de^t\circ J)\\ &= -d(e^tdt\circ J)\\ &= d(e^t\alpha)\\ &= \omega.\end{aligned}$$

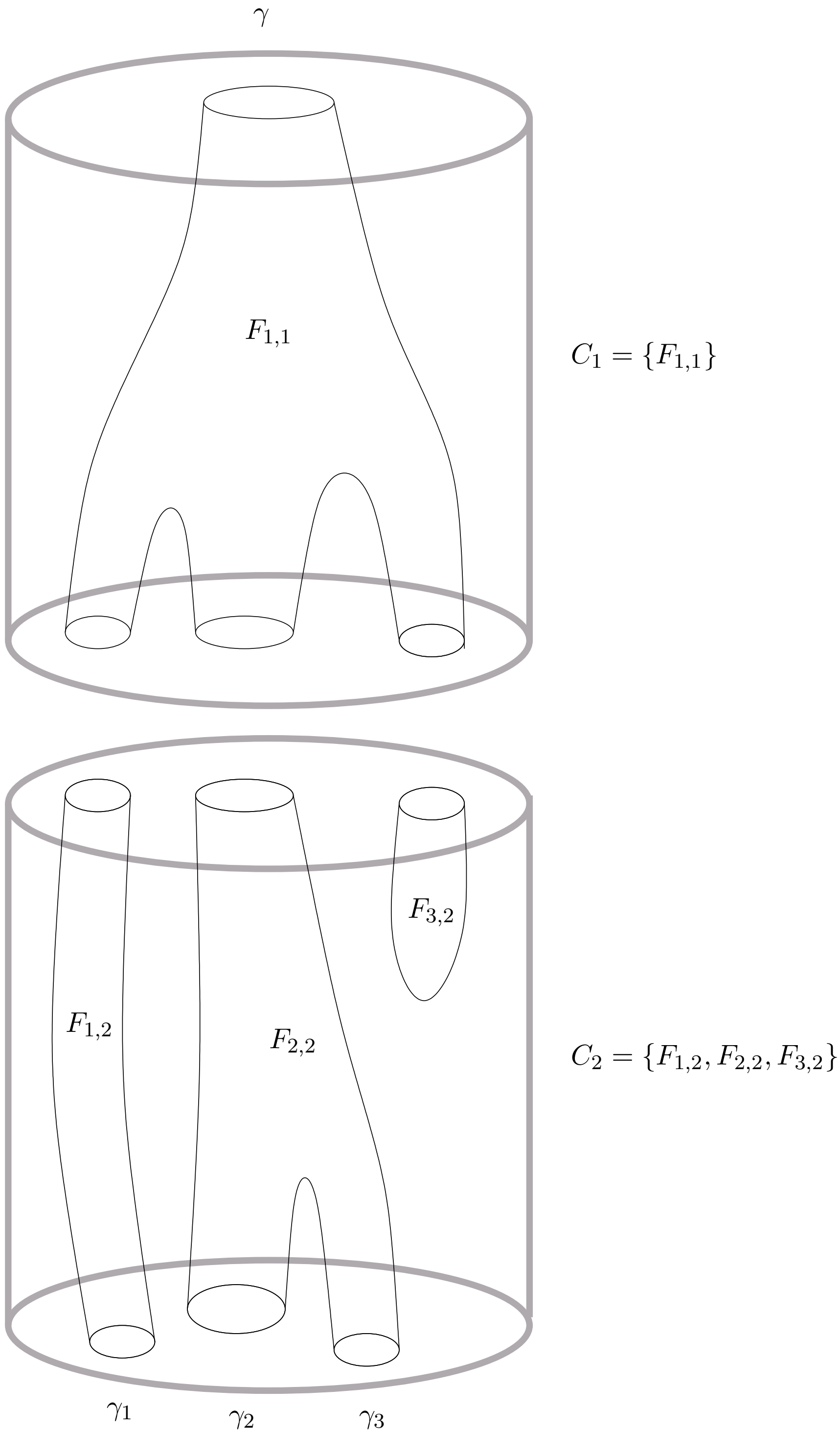

Figure 2.2: "Broken" holomorphic map with $k = 2$.

Hence, $\Delta e^a = F^*\omega > 0$ on Σ and the function e^a can have no maximum. In particular, the same restriction applies to a. $\square$

The moduli spaces $\mathcal{M}$ can be oriented in a coherent fashion, if we fix an ordering of the orbits $\gamma_1, \dots, \gamma_s$ at $-\infty$.
These orientations satisfy the following properties :

(i) the gluing map

$$\mathcal{M}(\gamma; \gamma_1, \dots, \gamma_s) \times \mathcal{M}(\gamma_s; \gamma'_1, \dots, \gamma'_{s'}) \to \mathcal{M}(\gamma; \gamma_1, \dots, \gamma_{s-1}, \gamma'_1, \dots, \gamma'_{s'})$$

preserves the coherent orientations,

(ii) exchanging the order of the orbits γ_i and γ_{i+1} in $\mathcal{M}(\gamma; \gamma_1, \dots, \gamma_s)$ changes the coherent orientation by $(-1)^{|\gamma_i||\gamma_{i+1}|}$.

These coherent orientations are constructed in [5]; another approach is outlined in [6].
Note that, if one of the orbits $\gamma, \gamma_1, \dots, \gamma_s$ is bad, there is no well-defined coherent orientation on $\mathcal{M}(\gamma; \gamma_1, \dots, \gamma_s)$. This is why we had to ignore the bad orbits in the definition of C_* and A.
To each holomorphic curve in $\mathcal{M}$, we want to associate a homology class in $H_2(M, \mathbb{Z})$. In order to do this, we must make additional choices, for each closed Reeb orbit. We write $H_1(M, \mathbb{Z}) = F \oplus T$, where F is a free module and T is a torsion module. Pick representatives $C_1, \dots, C_k$ in M for a basis of F, together with a trivialization of ξ along each representative C_i, $i = 1, \dots, k$. Pick representatives $D_1, \dots, D_l$ in M for a minimal set generating T, together with a trivialization of ξ along each representative D_j, $j = 1, \dots, l$. Now, for a closed Reeb orbit γ, we distinguish the following cases :

(i) $[\gamma] = 0 \in H_1(M, \mathbb{Z})$. As in the previous lecture, we choose a spanning surface S_γ and use it to trivialize ξ along γ.

(ii) $0 \neq [\gamma] \in F$. We choose a surface S_γ realizing a homology between γ and a linear combination of the representatives C_i, $i = 1, \dots, k$. We then use S_γ to extend the chosen trivializations of ξ along the C_i, $i = 1, \dots, k$ to γ.

(iii) $0 \neq [\gamma] \notin F$. Choose a surface S_γ realizing a homology between γ and a linear combination of the representatives C_i, $i = 1, \dots, k$ and D_j, $j = 1, \dots, l$, with minimal nonnegative coefficients for the representatives D_j, $j = 1, \dots, l$. We then use S_γ to extend the chosen trivializations of ξ along the C_i, $i = 1, \dots, k$ and the D_j, $j = 1, \dots, l$, to γ. Let γ_m be the smallest multiple of γ so that $[\gamma_m] \in F$. We have 2 different trivializations of ξ along γ_m : one obtained from case (ii) above, and the other obtained by pull-back from γ. Let $w \in \mathbb{Z}$ be the rotation number of the second trivialization with respect to the first one. We define

$$|\gamma| = \mu_{CZ}(\gamma) - 2\frac{w}{m} + n - 3 \in \frac{1}{m}\mathbb{Z}.$$

To a J-holomorphic curve in $\mathcal{M}(\gamma; \gamma_1, \dots, \gamma_s)$, we can glue the chosen surfaces $S_\gamma, S_{\gamma_1}, \dots, S_{\gamma_s}$ and obtain a closed surface in M. Let $A \in H_2(M, \mathbb{Z})$ be its homology class; we can use it to decorate the corresponding connected component $\mathcal{M}^A(\gamma; \gamma_1, \dots, \gamma_s)$ of the moduli space.
If we define the grading of the orbits γ using the trivializations of ξ defined above, then the dimension formula for the moduli spaces takes the convenient form

$$\dim \mathcal{M}^A(\gamma; \gamma_1, \dots, \gamma_s) = |\gamma| - \sum_{i=1}^{s} |\gamma_i| + 2\langle c_1(\xi), A\rangle.$$

Note that the right hand side if always an integer, even if the grading is fractional as in case (iii) above.
We are now in position to define the differential for our chain complexes C_* and A.

Differential d on C_*

The numbers

$$n^A_{a,b} = \begin{cases} 0 & \text{if } \dim \mathcal{M}^A(\gamma_a; \gamma_b) \neq 1 \\ \sum_{p \in \mathcal{M}^A(\gamma_a;\gamma_b)/\mathbb{R}} \text{weight}(p) & \text{if } \dim \mathcal{M}^A(\gamma_a; \gamma_b) = 1 \end{cases}$$

count rigid J-holomorphic cylinders joining closed orbits γ_a and γ_b, with homology class $A \in H_2(M, \mathbb{Z})$. Note that the rational numbers $n^A_{a,b}$ are always finite, and are nonzero for finitely many homology classes $A \in H_2(M, \mathbb{Z})$, since the moduli spaces $\mathcal{M}/\mathbb{R}$ are compact.
We combine this information in the coefficients

$$n_{a,b} = \sum_{A \in H_2(M,\mathbb{Z})} n^A_{a,b} e^{\pi(A)} \in \mathbb{Q}[H_2(M, \mathbb{Z})/\mathcal{R}]$$

where $\pi : H_2(M, \mathbb{Z}) \to H_2(M, \mathbb{Z})/\mathcal{R}$ is the natural projection.
We finally define

$$d\gamma_a = m_a \sum_b n_{a,b} \gamma_b$$

where m_a is the multiplicity of γ_a.
By the dimension formula for the moduli spaces, the grading of the differential is -1 :

$$d : C_k \to C_{k-1} \qquad \text{for all } k \in \mathbb{Z}.$$

Differential d on A

Let $\bar{b} = (b_1, \dots, b_s)$. The numbers

$$n^A_{a,\bar{b}} = \begin{cases} 0 & \text{if } \dim \mathcal{M}^A(\gamma_a; \gamma_{b_1}, \dots, \gamma_{b_s}) \neq 1 \\ \sum_{p \in \mathcal{M}^A(\gamma_a;\gamma_{b_1},\dots,\gamma_{b_s})/\mathbb{R}} \text{weight}(p) & \text{if } \dim \mathcal{M}^A(\gamma_a; \gamma_{b_1}, \dots, \gamma_{b_s}) = 1 \end{cases}$$

count rigid J-holomorphic curves converging to the closed orbit γ_a at $+\infty$ and to the closed orbits $\gamma_{b_1}, \dots, \gamma_{b_s}$ at $-\infty$, with homology class $A \in H_2(M, \mathbb{Z})$. As before, the rational numbers $n^A_{a,\bar{b}}$ are always finite, and are nonzero for finitely many homology classes $A \in H_2(M, \mathbb{Z})$, since the moduli spaces $\mathcal{M}/\mathbb{R}$ are compact.

We combine this information in the coefficients

$$n_{a,\bar{b}} = \sum_{A \in H_2(M,\mathbb{Z})} n^A_{a,\bar{b}} e^{\pi(A)} \in \mathbb{Q}[H_2(M, \mathbb{Z})/\mathcal{R}]$$

where $\pi : H_2(M, \mathbb{Z}) \to H_2(M, \mathbb{Z})/\mathcal{R}$ is the natural projection.

We finally define

$$d\gamma_a = m_a \sum_{\bar{b}} n_{a,\bar{b}} \gamma_{b_1} \dots \gamma_{b_s}$$

where m_a is the multiplicity of γ_a. The product $\gamma_{b_1} \dots \gamma_{b_s}$ is understood to be the unit $1 \in \mathsf{A}$ if $s = 0$.

Note that the above expression is well-defined : if we permute b_i and b_{i+1}, we pick a sign $(-1)^{|\gamma_{b_i}||\gamma_{b_{i+1}}|}$ due to the supercommutativity of A, but the coefficient $n_{a,\bar{b}}$ is multiplied by the same sign, due to the change in the coherent orientation of $\mathcal{M}$.

By the dimension formula for the moduli spaces, the grading of the differential is again -1.

Cylindrical contact homology

Eliashberg-Hofer : If $C^0_k = 0$ for $k = -1, 0, 1$, then for every free homotopy class $\bar{a}$ in M,

(i) $d^2 = 0$,

(ii) $H_*(C^{\bar{a}}_*, d)$ is independent of the contact form α for ξ, the complex structure J and the perturbation ν for the moduli spaces.

Definition 10. *The cylindrical contact homology $HC^{\bar{a}}_*(M, \xi)$ in the free homotopy class $\bar{a}$ is the homology $H_*(C^{\bar{a}}_*, d)$ of the chain complex $(C^{\bar{a}}_*, d)$.*

Contact homology

Eliashberg-Hofer :

(i) $d^2 = 0$, so that (A, d) is a differential graded algebra (DGA),

(ii) $H_*(\mathsf{A}, d)$ is independent of the contact form α for ξ, the complex structure J and the perturbation ν for the moduli spaces.

Definition 11. *The contact homology $HC_*(M, \xi)$ is the homology $H_*(\mathsf{A}, d)$ of the DGA (A, d).*

Idea of the proof for cylindrical contact homology

Since cylindrical contact homology is analogous to the symplectic Floer homology [7], it is not surprising that the proof for invariance is very similar to the Floer case (see for example [25]). There are, however, a few additional features in the contact case.

(i) d^2 counts "broken" J-holomorphic cylinders with 2 levels. These are in the boundary of the moduli spaces $\mathcal{M}/\mathbb{R}$ of dimension 1. But we have

$$\sum_{p\in\partial\mathcal{M}/\mathbb{R}} \text{weight}(p) = 0.$$

These weights are the products of the weights of the rigid J-holomorphic cylinders in each level and of the number of ways such J-holomorphic cylinders can be glued to each other. The latter is precisely the multiplicity of the closed Reeb orbit where we glue; this is why we need the factor m_a in the expression of $d\gamma_a$.

Therefore, we just need to check that nothing else sits in the boundary of these moduli spaces. However, a J-holomorphic cylinder of index 2 might also degenerate into a pair of pants of index 1 and a plane of index 1, see Figure 2.3. Since the closed Reeb orbit at $+\infty$ for the plane has degree 1 and is contractible, such a configuration is impossible if we assume that $C_1^0 = 0$.

(ii) Given different choices (α_0, J_0) and (α_1, J_1), we can find a homotopy (α_s, J_s), $s \in [0,1]$, joining them. Let $f : [0,1] \to \mathbb{R}$ be a smooth increasing function such that $f(t) = 0$ is t is sufficiently small and $f(t) = 1$ if t is sufficiently large. If f increases slowly enough, we can use these data to construct a symplectic cobordism $(\mathbb{R}\times M, d(e^t\alpha_{f(t)}))$, equipped with an almost complex structure J such that $J|_{\{t\}\times M} = J_{f(t)}$, $t \in \mathbb{R}$.

Counting rigid J-holomorphic cylinders in this symplectic cobordism, we get a map $\Phi : C_{1,*}^{\bar{a}} \to C_{0,*}^{\bar{a}}$ of degree 0 (and not -1), since the symplectic cobordism is not invariant under translations $t \to t + \Delta t$.

The boundary of moduli spaces of dimension 1 contains "broken" J-holomorphic cylinders with 2 levels : one in the symplectic cobordism, one in the symplectization with contact form α_0 or α_1. If nothing else sits in the boundary of these moduli spaces, we obtain an algebraic relation $\Phi \circ d_1 = d_0 \circ \Phi$. However, a J-holomorphic cylinder of index 1 might also degenerate into a pair of pants of index 1 in the symplectization for α_1 and a plane of index 0 in the symplectic cobordism. This time, such a configuration is impossible if we assume that $C_0^0 = 0$.

Hence, the map $\Phi : (C_{1,*}^{\bar{a}}, d_1) \to (C_{0,*}^{\bar{a}}, d_0)$ is a chain map and induces a map $\widetilde{\Phi} : H_*(C_{1,*}^{\bar{a}}, d_1) \to H_*(C_{0,*}^{\bar{a}}, d_0)$. We want to show that $\widetilde{\Phi}$ is an isomorphism. In order to do this, we need to construct an inverse for $\widetilde{\Phi}$.

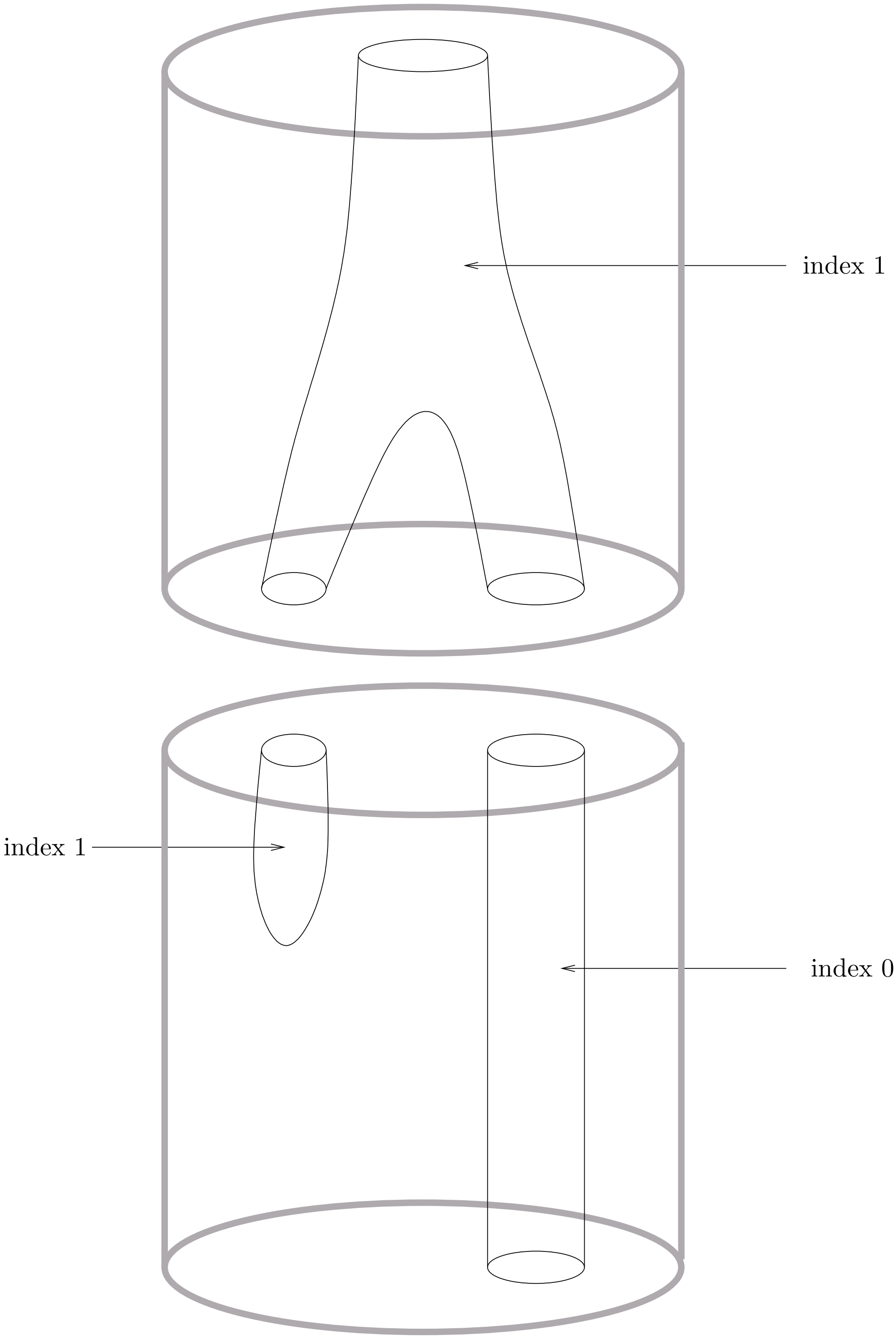

Figure 2.3: Splitting into a pair of pants and a plane.

A natural candidate is the map $\widetilde{\Psi} : H_*(C^{\bar{a}}_{0,*}, d_0) \to H_*(C^{\bar{a}}_{1,*}, d_1)$ that is obtained using the same construction as above, after switching (α_0, J_0) and (α_1, J_1). The composition $\Psi \circ \Phi$ counts rigid J-holomorphic cylinders in the symplectic cobordism obtained after gluing the above 2 symplectic cobordisms along (M, α_0). This glued symplectic cobordism can clearly be deformed into the symplectization $(\mathbb{R} \times M, d(e^t \alpha_1))$. But the only J-holomorphic cylinders of index 0 in a symplectization must be invariant under translations $t \to t + \Delta t$, so these are just the vertical cylinders over the closed Reeb orbits. Hence, counting such holomorphic cylinders in the symplectization gives the identity.

In order to understand the relationship between $\Psi \circ \Phi$ and the identity, we have to study J_λ-holomorphic cylinders in a 1-parameter family of symplectic cobordisms $(\mathbb{R} \times M, \omega_\lambda)$, $\lambda \in [0,1]$. For $\lambda = 0$, the symplectic cobordism is the symplectization for α_1, and for $\lambda = 1$, the symplectic cobordism is the glued symplectic cobordism. Counting rigid J_λ-holomorphic cylinders in such a 1-parameter family of symplectic cobordisms, we obtain a map $h : C^{\bar{a}}_{1,*} \to C^{\bar{a}}_{1,*+1}$ of degree 1, because for a finite number of values of $\lambda \in [0,1]$, the almost complex structure J_λ is not generic and admits J_λ-holomorphic cylinders of index -1.

The boundary of the moduli space $\mathcal{M}_{\lambda \in [0,1]}$ of J_λ-holomorphic cylinders of index 0 contains :

- holomorphic cylinders of index 0 for $\lambda = 0$,
- holomorphic cylinders of index 0 for $\lambda = 1$,
- "broken" J-holomorphic cylinders with 2 levels, consisting of a J_λ-holomorphic cylinder of index -1 and a J-holomorphic cylinder of index 1 in the symplectization above or below.

If nothing else sits in the boundary of these moduli spaces, we obtain an algebraic relation $\Psi \circ \Phi - id = d_1 \circ h + h \circ d_1$. However, a J-holomorphic cylinder of index 0 might also degenerate into a pair of pants of index 1 in the symplectization for α_1 and a plane of index -1 in the symplectic cobordism for a fixed value of $\lambda \in [0,1]$. This time, such a configuration is impossible if we assume that $C^0_{-1} = 0$.

The above algebraic relation means that the map $\Psi \circ \Phi$ is chain homotopic to the identity, so that the map induced by $\Psi \circ \Phi$ on $H_*(C^{\bar{a}}_{1,*}, d_1)$ is the identity. Similarly, $\Phi \circ \Psi$ induces the identity on $H_*(C^{\bar{a}}_{0,*}, d_0)$. Hence, $\Phi : H_*(C^{\bar{a}}_{1,*}, d_1) \to H_*(C^{\bar{a}}_{0,*}, d_0)$ is an isomorphism.

Lecture 3

Morse-Bott techniques

It is generally very difficult to compute contact homology using the definition from the previous lectures. The main reason for these difficulties is that we have to choose the contact form α generically, so that all the closed Reeb orbits are non-degenerate. This implies that the Reeb dynamics will be rather chaotic, and the almost complex structure J satisfying $J\frac{\partial}{\partial t} = R_\alpha$ will be very complicated, so that the Cauchy-Riemann equation is extremely hard to solve.
Thinking of contact homology as a variant of Morse theory for the action functional $\mathcal{A}$, the contact form α must be generic so that $\mathcal{A}$ is a Morse functional. We can relax this assumption by just requiring that $\mathcal{A}$ be a Morse-Bott functional. The appropriate conditions on the contact form α can be expressed as follows.

Definition 12. *A contact form α for (M, ξ) is of Morse-Bott type if :*

(i) the action spectrum $\sigma(\alpha) = \{\int_\gamma \alpha \mid \gamma$ closed Reeb orbit$\}$ is discrete,

(ii) the sets $N_T = \{p \in M \mid \varphi_T(p) = p\}$ are closed submanifolds of M, such that rank $d\alpha|_{N_T}$ is locally constant and $T_pN_T = \ker(d\varphi_T(p) - I)$.

In that case, the Reeb flow induces a circle action on N_T and we denote the quotient N_T/S^1 by S_T. These are orbifolds with singularity groups $\mathbb{Z}_m$; the singular loci consist of closed orbits of period $\frac{T}{m}$ covered m times, see Figure 3.1. We will denote the connected components of these orbifolds by S_i.

Example 3. Consider the lens space $L_{p,q} = S^3/\mathbb{Z}_p$ where $\mathbb{Z}_p$ acts on $S^3 \subset \mathbb{C}^2$ by $(z_1, z_2) \to (e^{i\frac{2\pi}{p}} z_1, e^{i\frac{2\pi q}{p}} z_2)$. The 1-form $\frac{i}{2}\sum_{j=1}^2 (z_j d\bar{z}_j - \bar{z}_j dz_j)$ on $\mathbb{C}^2$ descends to a contact form α on $L_{p,q}$. The corresponding Reeb flow is given by $(z_1, z_2) \to (e^{it}z_1, e^{it}z_2)$. It follows that all Reeb orbits are closed. Let m be the smallest positive integer such that p divides $m(q-1)$. The orbit spaces are then $S_{\frac{2\pi}{p}k}$, for $k = 1, 2, \ldots$ If m does not divide k, the orbit space consists of a pair of points, corresponding to the orbits $\{z_1 = 0\}$ and $\{z_2 = 0\}$. If m divides k, the orbit space is topologically a 2-sphere, but with 2 orbifold singularities with group $\mathbb{Z}_k$.

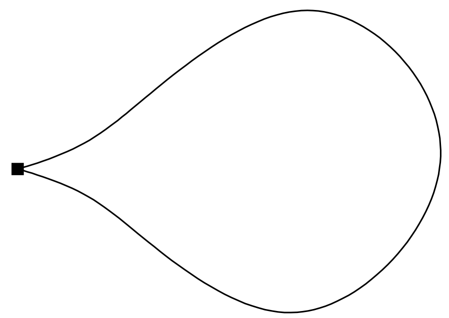

Figure 3.1: Orbit spaces $S_T \subset S_{mT}$.

We can extend the definition of the Conley-Zehnder index to the closed Reeb orbits in S_T. This time, with a trivialization of ξ along the orbit γ, we obtain a path of symplectic matrices $\Psi_\gamma(t)$, $t \in [0, T]$, such that $\Psi_\gamma(0) = I$ but $\Psi_\gamma(T)$ may have 1 as an eigenvalue. Actually, the corresponding eigenspace is the tangent space of S_T at γ. Following Robbin and Salamon [24], we define

$$\mu(\Psi_\gamma) = \frac{1}{2}\text{sign } \Gamma(\tilde{\Psi}, 0) + \sum_{t \text{ crossing } \neq 0} \text{sign } \Gamma(\tilde{\Psi}, t) + \frac{1}{2}\text{sign } \Gamma(\tilde{\Psi}, T)$$

for a small pertubation $\tilde{\Psi}$ of Ψ_γ with fixed ends, having regular crossings. In other words, we simply add the contribution of the last crossing at $t = T$. We will call this generalized index $\mu(\Psi_\gamma)$ the Maslov index, in order to distinguish it from the Conley-Zehnder index. Note that the Maslov index is a half integer : $\mu(\Psi_\gamma) \in \frac{1}{2}\mathbb{Z}$. However, the combination $\mu(\Psi_\gamma) \pm \frac{1}{2}\dim S_T$ is always an integer. Since the Maslov index is continuous, its value does not depend on the choice of $\gamma \in S_T$; we will therefore denote it by $\mu(S_T)$.

Moduli spaces

From now on, we assume that J is an almost complex structure on $(\mathbb{R} \times M, d(e^t\alpha))$ that is S^1-invariant along the submanifolds N_T.
The moduli spaces of J-holomorphic curves will be decorated with connected components S_i of the orbit spaces, instead of individual orbits γ_i.
The moduli space $\mathcal{M}(S^+; S_1^-, \ldots, S_s^-)$ is equipped with evaluation maps $ev^+ : \mathcal{M} \to S^+$ and $ev_i^- : \mathcal{M} \to S_i^-$, $i = 1, \ldots, s$.

Proposition 2. $\mathcal{M}(S^+; S_1^-, \ldots, S_s^-)/\mathbb{R}$ *is a compact branched labeled manifold with corners, i.e. the union of manifolds with corners along a codimension* 1 *branching locus, with each manifold having a rational weight, so that near each branching point, the sum of all entering weights* *equals the sum of all exiting weights. Each manifold with corner has dimension*

$$(n-3)(1-s) + \mu(S^+) + \frac{1}{2}\dim S^+ - \sum_{i=1}^{s} \mu(S_i^-) + \frac{1}{2}\sum_{i=1}^{s} \dim S_i^- + 2c_1^{rel}(\xi, \Sigma) - 1$$

where $c_1^{rel}(\xi, \Sigma)$ is the relative first Chern class of ξ on Σ, relative to the fixed trivializations of ξ along the closed Reeb orbits at the punctures.

As in the previous lecture, we want to associate a homology class in $H_2(M, \mathbb{Z})$ to a holomorphic curve in $\mathcal{M}(S^+; S_1^-, \dots, S_s^-)$. In order to do this, we fix a base point in each orbit space S_i and make the same choices of spanning surfaces and trivializations of ξ along the corresponding orbits. Then, given a holomorphic curve $F \in \mathcal{M}(S^+; S_1^-, \dots, S_s^-)$, we connect the orbits $ev^+(F) \in S^+$ and $ev_i^-(F) \in S_i^-$, $i = 1, \dots, s$, to the corresponding base points using curves in the orbit spaces. We glue the cylinders in M lying above these curves in the orbit spaces S_i to the holomorphic curve F, then we glue the appropriate spanning surfaces at the base points of the orbit spaces S_i. We obtain a surface in M representing a homology class in $H_2(M, \mathbb{Z})$.
However, the obtained homology class depends on the choice of the connecting curves in the orbit spaces. The homology class is well-defined in the quotient space $H_2(M, \mathbb{Z})/\mathcal{R}$, where $\mathcal{R}$ is generated by the homology classes of tori in M above non-contractible loops in the orbit spaces S_i. Note that $\mathcal{R} \subset \ker c_1(\xi)$, so that the grading is well-defined.
The boundary of the moduli spaces $\mathcal{M}(S^+; S_1^-, \dots, S_s^-)/\mathbb{R}$ consists, as before, of "broken" J-holomorphic curves. Recall that, given continuous maps $f : A \to C$ and $g : B \to C$, the fibered product $A \times_C B$ is defined as $\{(a, b) \in A \times B \mid f(a) = g(b)\}$. Therefore, the codimension 1 boundary of $\mathcal{M}/\mathbb{R}$ consists of the union of fibered products $\mathcal{M}(S^+; S_{i_1}^-, \dots, S_{i_r}^-, S)/\mathbb{R} \times_S \mathcal{M}(S; S_{i_{r+1}}^-, \dots, S_{i_s}^-)/\mathbb{R}$, where $i_1, \dots, i_s$ is a permutation of $1, \dots, s$ and $0 \le r \le s$.
The perturbation of the moduli spaces is performed in such a way that the evaluation maps are transverse to each other. This is necessary in order to make sure that the boundary of $\mathcal{M}/\mathbb{R}$ is regular and has the predicted dimension.

Relation to the generic case

We can perturb a contact form of Morse-Bott type α using the formula

$$\alpha_\epsilon = (1 + \epsilon f)\alpha \qquad \text{for small } \epsilon > 0,$$

where the function f is chosen so that the closed R_{α_ϵ}-orbits are non-degenerate if their period is less than $T(\epsilon)$. This bound $T(\epsilon) \to \infty$ as $\epsilon \to 0$. Since the Reeb field is modified, so is the almost complex structure J : we obtain a 1-parameter family J_ϵ.
We can define contact homology for $\epsilon > 0$ using the previous lectures. We then study the degeneration of the Reeb flow and of the J_ϵ-holomorphic curves as $\epsilon \to 0$. We can then write the chain complex of contact homology using only the geometric data at $\epsilon = 0$.

Computation of contact homology

1. Choose a set of admissible Morse functions $f_T : S_T \to \mathbb{R}$, forming Morse-Smale pairs with the metric induced by $d\alpha(\cdot, J\cdot)$. In other words, if $S_T \subset S_{kT}$, the function f_{kT} extends the function f_T with a positive definite Hessian in the normal directions to S_T in S_{kT}.

2. The chain complex C_* or A is generated by the "good" critical points of the functions f_T, for all orbit spaces S_T, $T \in \sigma(\alpha)$. The grading of a critical point $p \in \mathrm{Crit}(f_T)$ is given by

$$|p| = \mu(S_T) - \frac{1}{2}\dim S_T + \mathrm{index}_p(f_T) + n - 3.$$

   The notion of a good critical point coincides with the notion of a good orbit, using the above grading. Since the Morse function f_T are admissible, we have $\mathrm{index}_p(f_T) = \mathrm{index}_p(f_{kT})$ for all $k \geq 1$. Hence, the even multiples of all orbits in a connected component of S_T are either good or bad, depending on the parities of $\mu(S_T) - \frac{1}{2}\dim S_T$ and $\mu(S_{2T}) - \frac{1}{2}\dim S_{2T}$.

   The coefficient ring of the chain complex is $\mathbb{Q}[H_2(M,\mathbb{Z})/\mathcal{R}']$, where $\mathcal{R} \subset \mathcal{R}' \subset \ker c_1(\xi)$.

3. The differential counts "generalized" holomorphic curves, consisting of a finite collection of J-holomorphic maps $C_j = \{F_{1,j}, \ldots, F_{l,j}\}$, $j = 1, \ldots, k$, and $t_j \in \mathbb{R}^+$, $j = 1, \ldots, k-1$, such that

   (i) The gradient flow of the functions f_T after time t_{j-1}, applied to the closed orbits at $+\infty$ in C_j, gives exactly the orbits at $-\infty$ for C_{j-1}, $j = 2, \ldots, k$,

   (ii) the closed orbit at $+\infty$ for C_1 is in the stable manifold $W^s(p^+)$,

   (iii) the closed orbits at $-\infty$ for C_k are in the unstable manifolds $W^u(p_1^-), \ldots, W^u(p_s^-)$.

   This is illustrated by Figure 3.2.

   Note that, in many cases, the only rigid "generalized" holomorphic curves satisfy $k = 1$. In that case, the differential simply counts elements in the fibered product

$$W^u(p^+) \times_{S^+} \mathcal{M}(S^+; S_1^-, \ldots, S_s^-)/\mathbb{R} \times_{S_1^- \times \ldots \times S_s^-} W^s(p_1^-) \times \ldots \times W^s(p_s^-).$$

The above steps can be used to compute contact homology [2].

Proposition 3. *Assume that α is a contact form of Morse-Bott type for (M, ξ), and that J is an almost complex structure on $(\mathbb{R} \times M, d(e^t\alpha))$ that is S^1-invariant along the submanifolds N_T. Then the homology of the above chain complex is isomorphic to the contact homology $HC_*(M, \xi)$ with coefficient ring $\mathbb{Q}[H_2(M,\mathbb{Z})/\mathcal{R}']$.*

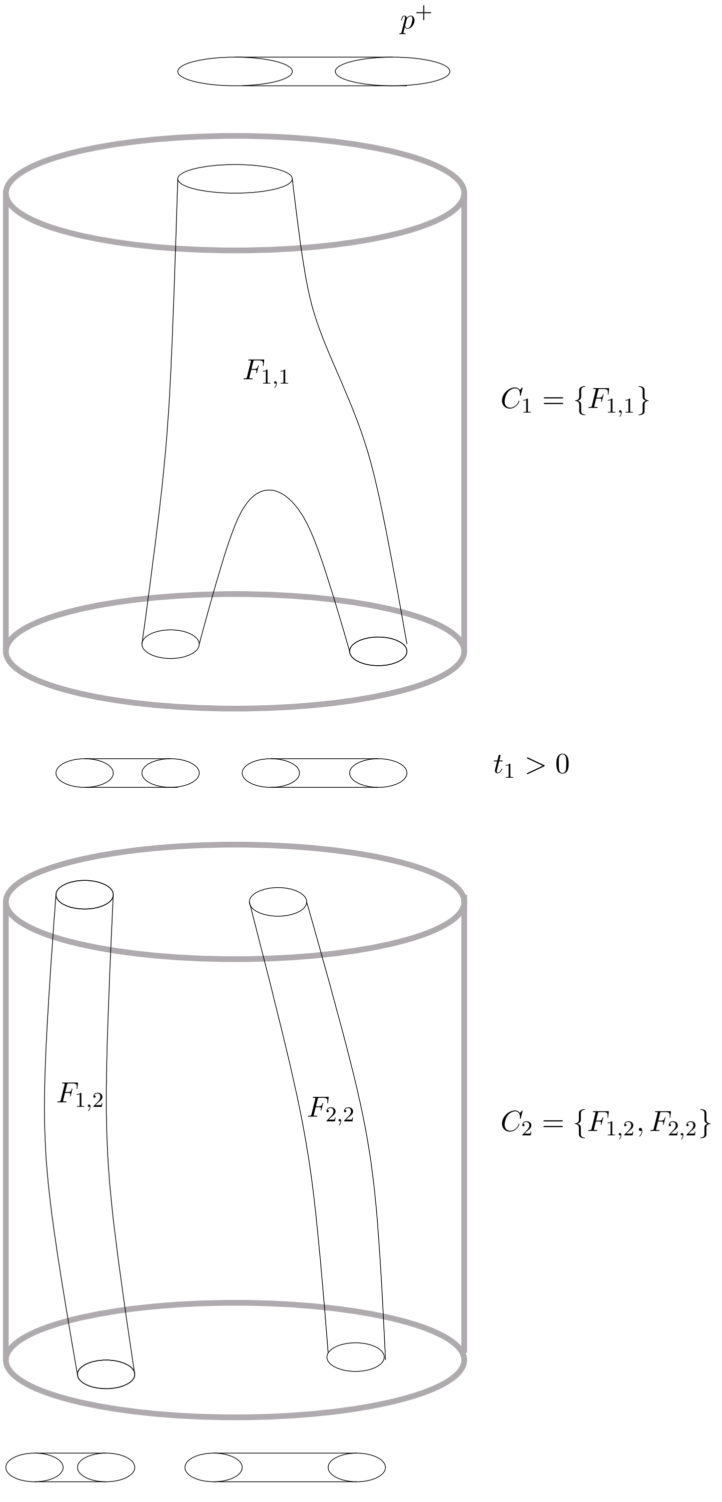

Figure 3.2: Generalized holomorphic map with $k = 2$.

Lecture 4

Examples and applications

4.1 Standard contact sphere

Consider $S^{2n-1} \subset \mathbb{C}^n$ and let $\xi_{std} = TS^{2n-1} \cap iTS^{2n-1}$ be the hyperplane field of complex tangencies. This is a contact structure, that admits the natural contact form

$$\alpha = \frac{i}{2}\sum_{j=1}^{n}(z_j d\bar{z}_j - \bar{z}_j dz_j)|_{S^{2n-1}}.$$

The corresponding Reeb flow is given by

$$(z_1, \dots, z_n) \to (e^{it}z_1, \dots, e^{it}z_n)$$

and hence is completely periodic.

4.1.1 Computation without Morse-Bott

We first need to perturb the contact form α. The simplest way is to use contact form

$$\widetilde{\alpha} = \frac{i}{2}\sum_{j=1}^{n} a_j(z_j d\bar{z}_j - \bar{z}_j dz_j)|_{S^{2n-1}},$$

where the coefficients a_j, $j = 1, \dots, n$, are rationally independent. The corresponding Reeb flow is given by

$$(z_1, \dots, z_n) \to (e^{i\frac{t}{a_1}}z_1, \dots, e^{i\frac{t}{a_n}}z_n).$$

There are n simple closed Reeb orbits, contained in the complex coordinate lines : $\gamma_j \subset \{z_i = 0, i \neq j\}$. We denote the closed orbit covering m times the orbit γ_j by $\gamma_{j,m}$.
Let us compute the Conley-Zehnder index of the orbits $\gamma_{j,m}$. The crossings occur at $t = 2\pi k a_j$, for $j = 1, \dots, n$ and $k = 1, 2, \dots$ The crossing form is the identity on a 2-dimensional eigenspace,

so each crossing brings contribution 2. We obtain

$$\mu_{CZ}(\gamma_{j,m}) = n - 1 + 2\sum_{i\neq j}\lfloor\frac{ma_j}{a_i}\rfloor + 2m.$$

This is a rather complicated formula. To get around this, we choose $a_1 << a_2, \dots a_n$. In that case,

$$\begin{aligned} \mu_{CZ}(\gamma_{1,m}) &= n - 1 + 2m, \\ \mu_{CZ}(\gamma_{j,m}) &>> \mu_{CZ}(\gamma_{1,m}), \end{aligned}$$

if m is sufficiently small. If we reduce the value of $a_1 > 0$, these formulas are valid for a larger range for m.The lowest degree generators have grading $|\gamma_{1,m}| = 2n - 4 + 2m$, $m = 1, 2, \dots$
Since the grading is purely even and $|d| = -1$, we see that the differential d vanishes. Hence, the above generators actually generate contact homology $HC_*^0(S^3, \xi_{std})$.
Using the invariance of contact homology, we can reduce a_1 at will and proceed as if the grading formulas were valid for all $m \in \mathbb{Z}^+$. Hence, we see that $HC_*^0(S^3, \xi_{std})$ has one generator in each even degree that is $\geq 2n - 2$.
Note that, without knowing a priori that $d = 0$, it would be virtually impossible to compute d since $J \neq i$ on $\mathbb{R} \times S^{2n-1}$. For example, the symplectic cobordism interpolating between the contact forms with weights $a_1 < \dots < a_n$ very close to 1 and with weights satisfying $a_1 << a_2, \dots, a_n$ must contain, according to the proof of invariance, J-holomorphic cylinders asymptotic to the orbits $\gamma_{k,1}$ and $\gamma_{1,k}$ respectively, for $k = 1, \dots, n$. For $k = 1$, the cylinder is vertical, but for $k \neq 1$, these cylinders have a rather complicated behavior in $\mathbb{R} \times S^{2n-1}$ and are litterally impossible to find explicitly.

4.1.2 Computation using Morse-Bott

In this case, we can choose $J = i$, so that $(\mathbb{R} \times S^{2n-1}, J) \simeq (\mathbb{C}^n \setminus \{0\}, i)$. Moreover, all Reeb orbits are closed and the orbit spaces are $S_{2\pi m} = \mathbb{C}P^{n-1}$.
The Maslov index of $S_{2\pi m}$ is $\mu(S_{2\pi m}) = 2nm$, because the linearized Reeb flow consists of n blocks

$$\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

The crossings $t = 2\pi k$, $k \in \mathbb{Z}$, give a contribution 2 for each block.
Since the Reeb flow induces a global S^1-action on the symplectization, preserving i, there are no rigid holomorphic curves other than vertical cylinders over closed Reeb orbits. Therefore, the differential d counts only generalized holomorphic curves with $k = 0$, i.e. gradient trajectories on the orbit spaces. Contact homology will therefore contain a copy of $H_*(\mathbb{C}P^{n-1}, \mathbb{Q})$ for each orbit space.

Hence, we obtain one generator of $HC_*^0(S^3, \xi_{std})$ in each degree $2nk - (n-1) + (n-3) + d$, where $d = 0, 2, \dots, 2(n-1)$. If we write h_k for rank $HC_k^0(S^3, \xi_{std})$, we have

$$\begin{array}{c|cccccccccccc}
k & 2n-2 & 2n-1 & 2n & \cdots & 4n-4 & 4n-3 & 4n-2 & 4n-1 & 4n & \cdots & 6n-4 & \cdots \\
\hline
h_k & \underbrace{1 \quad 0 \quad 1 \quad \cdots \quad 1}_{H_*(\mathbb{C}P^{n-1},\mathbb{Q})} & & & & & 0 & \underbrace{1 \quad 0 \quad 1 \quad \cdots \quad 1}_{H_*(\mathbb{C}P^{n-1},\mathbb{Q})} & & & & & \cdots
\end{array}$$

4.2 Tight contact structures on T^3

Let us consider again the contact structures ξ_n, $n \geq 1$ on T^3, defined as $\xi_n = \ker \alpha_n$, where

$$\alpha_n = \cos(n\theta)dx + \sin(n\theta)dy.$$

4.2.1 Contact homology over $\mathbb{Q}$

The corresponding Reeb vector field is given by

$$R_{\alpha_n} = \cos(n\theta)\frac{\partial}{\partial x} + \sin(n\theta)\frac{\partial}{\partial y}.$$

Hence, the orbits are straight lines foliating the 2-tori $\{\theta = \text{ const}\}$. Pick any nontrivial homotopy class $\bar{a}$ in this family of 2-tori. There are n values of θ corresponding to the slope of $\bar{a}$, so we obtain n circles of closed Reeb orbits in class $\bar{a}$.

We choose the basis $e_1 = \frac{\partial}{\partial\theta}, e_2 = -\sin(n\theta)\frac{\partial}{\partial x} + \cos(n\theta)\frac{\partial}{\partial y}$ for ξ. In this basis, the linearized Reeb flow is given by the path of matrices

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

Hence, the Maslov index for a closed Reeb orbit γ is $\mu(\gamma) = \frac{1}{2}$. Note that it is independent of the choice of $\bar{a}$.

The symplectization $(\mathbb{R} \times T^3, d(e^t\alpha_n))$ does not contain J-holomorphic cylinders other than the vertical ones. Indeed, for any J-holomorphic cylinder C, $\int_C d\gamma \geq 0$, since $d\alpha$ is nonnegative on complex lines. Moreover, this integral vanishes if and only if C is vertical. On the other hand, by Stokes theorem, $\int_C d\gamma = \mathcal{A}(\gamma^+) - \mathcal{A}(\gamma^-)$, if C is asymptotic to the closed orbit γ^+ for $t \to +\infty$ and to γ^- for $t \to -\infty$. But all the closed Reeb orbits in class $\bar{a}$ have the same action, so the conclusion follows.

Hence, the differential d coincides with the Morse-Witten differential of S^1, and the contact homology $HC_*^{\bar{a}}(T^3, \xi_n)$ will contain a copy of $H_*(S^1, \mathbb{Q})$ for each orbit space. Taking the grading into account, we obtain

$$HC_k^{\bar{a}}(T^3, \xi_n) = \oplus_{j=1}^n H_{k+1}(S^1, \mathbb{Q}) \qquad \text{for all } k \in \mathbb{Z}.$$

In particular, we see that contact homology distinguishes the contact structures ξ_n, $n = 1, 2, \ldots$

4.2.2 Contact homology over $\mathbb{Q}[H_2(T^3, \mathbb{Z})]$

If we choose instead the coefficient ring $\mathbb{Q}[H_2(T^3, \mathbb{Z})]$, we must pay attention to the homology classes of the generalized holomorphic cylinders. On each orbit space S^1, there are 2 such cylinders lying over the 2 gradient trajectories l_1 and l_2 joining the maximum M and the minimum m of a perfect Morse function.

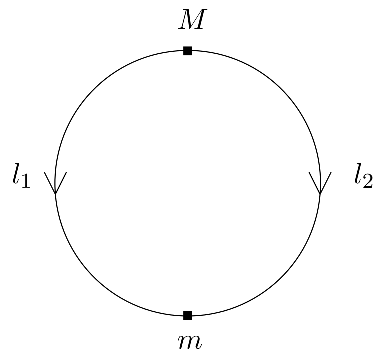

Figure 4.1: Morse function on the orbit space S^1.

In order to associate a homology class to a J-holomorphic cylinder, we need to make some choices. We choose the representative for class $\bar{a}$ to be the closed Reeb orbit at m. Then, the spanning surface from this representative to m can be chosen to be trivial. Finally, we choose the spanning surface from the representative to M to be the cylinder lying above the gradient trajectory l_1.

Then, the generalized holomorphic cylinder lying above l_1 has a zero homology class, and the generalized holomorphic cylinder lying above l_2 has the homology class of the 2-torus $T_{x,y}$ spanning the x and y directions. Hence, the differential d of contact homology is given by

$$\begin{aligned} dM &= (1 - e^{T_{x,y}})m, \\ dm &= 0, \end{aligned}$$

on each orbit space S^1. Therefore,

$$\begin{aligned} HC_k^{\bar{a}}(T^3, \xi_n) &= 0 \text{ if } k \neq -1, \\ HC_{-1}^{\bar{a}}(T^3, \xi_n) &= \oplus_{j=1}^{n} \mathbb{Q}[H_2(T^3, \mathbb{Z})/\langle T_{x,y}\rangle] \end{aligned}$$

over the coefficient ring $\mathbb{Q}[H_2(T^3, \mathbb{Z})]$.

This example shows that contact homology strongly depends on the choice of the coefficient ring. It is therefore important to specify the coefficient ring that was used to carry out the computation.

4.3 Applications of contact homology

The first application of contact homology is of course to distinguish contact structures on a given manifold M.

Theorem 1. *There are infinitely many pairwise non-isomorphic contact structures, with the same classical invariants, on the following manifolds :*

(i) S^{4k+1}, *for* $k \geq 1$ *(Ustilovsky [27]) ,*

(ii) T^5, $T^2 \times S^{2k+1}$, *for* $k \geq 1$ *(Bourgeois [1]).*

We now turn to other applications of contact homology. We first need a few definitions; let (V, J) be a complex manifold.

Definition 13. *A function* $f : V \to \mathbb{R}$ *is strictly plurisubharmonic if the* 2*-form* $\omega_f = dd^c f$*, where* $d^c f = df \circ J$*, satisfies* $\omega_f(v, Jv) > 0$ *for all* $v \neq 0$.

Definition 14. *A complex manifold* (V, J) *is Stein if it admits a proper, positive, stricly plurisubharmonic function* f *such that* ∂V *is a regular level set of* f.

The boundary $M = \partial V$ of a Stein manifold admits a contact structure ξ, which is the maximal complex subspace of TM. Moreover, a Stein manifold of dimension $2n$ has the homotopy type of a CW complex of dimension n.

Definition 15. *A Stein manifold of dimension n is called subcritical if it has the homotopy type of a CW complex of dimension* $n - 1$.

If these conditions are satisfied, we say that $M = \partial V$ with ξ as above, is a subcritical Stein-fillable contact manifold. Contact homology was computed for some manifolds of this type [28].

Theorem 2. (M.L. Yau) *Let* (M, ξ) *be a* $(2n-1)$*-dimensional subritical Stein-fillable contact manifold with* $n > 2$, $c_1(\xi) = 0$, *and* (V, J) *a subcritical Stein domain such that* $\partial V = M$ *and* ξ *is the maximal complex subbundle of* TM. *Let* $b^i = \mathrm{rk}(\mathrm{H^i(V,M)})$ *and denote by* h_i *the rank of the ith contact homology group* $HC_i^0(M, \xi)$ *of* (M, ξ). *Then* $h_i = 0$ *for* $i < n-1$, $h_{i-2} - h_{i-4} = b^i$ *for* $i \in \mathbb{Z}$.

Contact homology can also be used to study contact embeddings. The following result [18] is similar to the non-squeezing theorem in symplectic geometry.

Theorem 3. (S.S. Kim) *If* $r < R$, *there is no contact embedding* $\Psi \in \mathrm{Diff}_\xi^0(\mathbb{R}^{2n} \times S^1)$ *such that* $\Psi(B_R \times S^1) \subset B_r \times S^1$.

Contact homology can also be used to study the space $\Xi(M)$ of contact structures on M. By Gray stability, it is natural to first look at the connected components of $\Xi(M)$, i.e. $\pi_0(\Xi(M))$. But it is also interesting to consider the topology of the connected components, and in particular $\pi_k(\Xi(M), \xi_0)$, for $k > 0$. The first result in this direction was obtained by Geiges and Gonzalo [11], and can be reproved using contact homology [3].

Proposition 4. (Geiges-Gonzalo, Bourgeois) *$\pi_1(\Xi(T^3), \xi_n)$ contains an infinite cyclic subgroup.*

Sketch of proof. We want to show that the loop of contact structures $\xi_{n,s} = \ker \alpha_{n,s}$, for $s \in \mathbb{R}/\mathbb{Z}$, where

$$\alpha_{n,s} = \cos(n\theta + 2\pi s)dx + \sin(n\theta + 2\pi s)dy,$$

generates an infinite cyclic subroup of $\pi_1(\Xi(T^3), \xi_n)$.
Using this loop, we can construct a symplectic cobordism $(\mathbb{R} \times T^3, d(e^t \alpha_{n,f(t)}))$, where $f : \mathbb{R} \to [0, 1]$ is a smooth increasing function, such that $f(t) = 0$ for t small and $f(t) = 1$ for t large.
Counting J-holomorphic curves in this symplectic cobordism, we obtain a chain map $\Phi : (C_*^{\bar{a}}, d) \to (C_*^{\bar{a}}, d)$ inducing an automorphism of $HC_*^{\bar{a}}(T^3, \xi_n)$.
This chain map Φ is easily computed, because it can be shown that the J-holomorphic cylinders have to closely follow the closed Reeb orbits in their motion along the θ coordinate as $s \in \mathbb{R}/\mathbb{Z}$. Hence, $\Phi\gamma = \gamma e^{T_{\theta,x}}$ for every closed Reeb orbit γ, where $T_{\theta,x}$ is the homology class of the 2-torus spanned by the coordinates θ and x. This chain map induces multiplication by $e^{T_{\theta,x}}$ on $HC_*^{\bar{a}}(T^3, \xi_n)$. Looking back on our computation with coefficient ring $\mathbb{Q}[H_2(T^3, \mathbb{Z})]$, this automorphism is distinct from the identity.
Assume by contradiction that the loop $\xi_{n,s}$, $s \in \mathbb{R}/\mathbb{Z}$, is contractible. Using a contraction, we can construct a 1-parameter family of symplectic cobordisms, interpolating between the original cobordism $(\mathbb{R} \times T^3, d(e^t \alpha_{n,f(t)}))$ and the symplectization $(\mathbb{R} \times T^3, d(e^t \alpha_n))$. According to the proof of invariance for contact homology, the automorphism of contact homology obtained from the original cobordism coincides with the automorphism induced by the symplectization, i.e. the identity. But this contradicts the above computation of Φ.
Similarly, iterating the loop $\xi_{n,s}$, $s \in \mathbb{R}/\mathbb{Z}$, we obtain various powers of the automorphism Φ, and these are also distinct from the identity. □

Using similar techniques, we can obtain the following results [3].

Proposition 5. *There are infinitely many contact structures ξ on T^5 such that $\pi_1(\Xi(T^5), \xi)$ contains the subgroup $\mathbb{Z}^3$.*

The contact structures on T^5 in this proposition are the ones constructed in Theorem 1.

Proposition 6. *Let ξ_{std} be the canonical contact structure on the unit cotangent bundle of T^4. Then $\pi_3(\Xi(T^4 \times S^3), \xi_{std})$ contains an infinite cyclic subgroup.*

The 3-sphere of contact structures is obtained by acting with S^3 on itself in $T^4 \times S^3$.

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