Holomorphic curves and the moment map

Five Lectures by Dietmar Salamon

Introduction

These lectures were delivered by Dietmar Salamon at the Summer School in Symplectic Geometry, Paris, July 12-19th and they are based on joint work of Kai Cieliebak, Rita Gaio, Ignasi Mundet and Dietmar Salamon. Their main point of concern is the study of holomorphic curves in symplectic quotients.

Lecture 1

1.1 Group actions on symplectic manifolds

Let (M, ω) be a symplectic manifold and G a compact Lie group with Lie algebra $\mathfrak{g}$. An action of G on M induces an *infinitesimal action* of $\mathfrak{g}$ on the tangent space at any point $x \in M$ as a linear map

$$\begin{aligned} L_x : \mathfrak{g} &\longrightarrow T_x M \\ \eta &\longmapsto X_\eta(x) = \frac{d}{dt}_{|t=0} \exp(t\eta)x \end{aligned}$$

As an example, in the case of a circle action on M there is, basically, only one interesting element η as above, namely a generator of the Lie algebra $\mathfrak{g} = i\mathbb{R}$. One calls the action *hamiltonian* if there is a function $H : M \longrightarrow \mathbb{R}$ which is a primitive of the 1-form $\iota(X_\eta)\omega$ and, in this case, X_η will be called a *hamiltonian vector field.*

In the higher dimensional case one calls the action of G *hamiltonian* if there is a *moment map*

$$\mu : M \longrightarrow \mathfrak{g}^*$$

such that

$$\text{(generalized Hamilton equations)} \quad \forall \eta \in \mathfrak{g},\ \iota(X_\eta)\omega = d\langle \mu, \eta \rangle$$

$$\text{(G-equivariance)} \quad \forall x \in M,\ \mu(g \cdot x) = g \cdot \mu(x)$$

Roughly speaking, that means that not only all embedded 1-dimensional actions are to be hamiltonian, but they are required to fit together in a pointwise linear and globally equivariant way.

Notes by A. Oancea

In what follows we shall suppose that G is endowed with a bi-invariant metric $\langle\cdot,\cdot\rangle$ which will allow the identification of $\mathfrak{g}^*$ and $\mathfrak{g}$; the moment map will take values in $\mathfrak{g}$ and we will ask that

$$\forall\eta\in\mathfrak{g},\quad \iota(X_\eta)\omega = d\langle\mu,\eta\rangle$$

$$\forall x\in M,\quad \mu(g\cdot x) = g\cdot\mu(x)$$

The equivariance of μ ensures that the level sets which are invariant under the action of G correspond precisely to the central elements $\tau\in Z(\mathfrak{g})$. Indeed, the condition

$$x\in\mu^{-1}(\tau)\Rightarrow g\cdot x\in\mu^{-1}(\tau)$$

is equivalent with $\tau\in\mathfrak{g}$ being invariant under the adjoint action of G on $\mathfrak{g}$, whose linearization at $e\in G$ is given by the Lie bracket, hence $\tau\in Z(\mathfrak{g})$. The converse follows in a similar way.

A similar argument shows that, if the moment map exists, then it is unique up to adding a central element in $\mathfrak{g}$: the Hamiltonian condition forces the difference of any two moment maps to be a constant, which moreover has to be central by G-equivariance.

We shall therefore study in the sequel hamiltonian actions of compact Lie groups G on symplectic manifolds (M,ω), carrying a moment map denoted by μ, subject to the following hypothesis:

(H1) *μ is proper, $\tau\in Z(\mathfrak{g})$ is a regular value of μ* and *G acts freely on $\mu^{-1}(\tau)$*

These three conditions will ensure respectively the existence of the compact smooth quotient

$$\overline{M} = \mu^{-1}(\tau)/G$$

carrying a symplectic structure induced by ω. The quotient will sometimes be denoted

$$\overline{M} = M/\!\!/G(\tau)$$

1.1.1 Examples

1. Complex projective spaces. Let $G=\mathbb{S}^1$ act on $(M,\omega)=(\mathbb{C}^n,\omega_0)$ by complex multiplication (to the left) on each coordinate. Here $\omega_0=\sum_j dx_j\wedge dy_j=\frac{i}{2}\sum_j dz_j\wedge d\bar z_j$ is the standard symplectic form. The action is hamiltonian with moment map $\mu:\mathbb{C}^n\longrightarrow i\mathbb{R}$ given by

$$\mu(z) = -\frac{i}{2}|z|^2 \tag{1.1}$$

To see this, identify $\mathfrak{g} = i\mathbb{R}$ with $\mathbb{R}$ in the obvious way and the infinitesimal action will be $X_\eta(z) = \eta X_1(z)$, with $X_1(z) = iz = -i(\overline{z}\frac{\partial}{\partial \overline{z}} - z\frac{\partial}{\partial z})$. Thus

$$\iota(iz)\omega_0 = d(-\frac{1}{2}|z|^2)$$

Now, any element in the Lie algebra is central. For $\tau = -\frac{i}{2}$, one gets

$$\mu^{-1}(-\frac{i}{2}) = \mathbb{S}^{2n-1}$$

$$\mathbb{C}^n /\!\!/ \mathbb{S}^1(-\frac{i}{2}) = \mathbb{C}P^{n-1}$$

2. Complex grassmannians. One can identify the space $\mathcal{M}_{k,n}(\mathbb{C})$ of complex matrices having k lines and n columns with $(M,\omega) = (\mathbb{C}^{n\times k}, \omega_0)$ by successively writing the lines of a matrix in order to form a vector in $\mathbb{C}^{n\times k}$. The natural action of the unitary group $G = U(k)$ on $\mathcal{M}_{k,n}(\mathbb{C})$ by left multiplication induces an action of G on M. Let us fix on $\mathfrak{u}(k)$ the scalar product

$$\langle \Xi, \Upsilon \rangle = tr(\Xi^* \Upsilon)$$

We claim that the map

$$\begin{aligned} \mu : M &\longrightarrow \mathfrak{g} = \mathfrak{u}(k) \\ \phi &\longmapsto -\frac{i}{2}\phi\phi^* \end{aligned} \tag{1.2}$$

is a moment map for this action. First notice that the symplectic form ω_0 can be read directly on $\mathcal{M}_{k,n}(\mathbb{C})$ as

$$\omega_0(A,B) = \mathrm{Im}(tr(AB^*)) = -\frac{i}{2}tr(AB^* - BA^*)$$

The infinitesimal action is

$$\begin{aligned} \mathfrak{g} &\longrightarrow T_M\mathcal{M}_{k,n}(\mathbb{C}) \cong \mathcal{M}_{k,n}(\mathbb{C}) \\ \Xi &\longmapsto X_\Xi(M) = \Xi \cdot M \end{aligned}$$

and we successively have

$$\begin{aligned} (\iota(\Xi\cdot M)\omega_0)(B) &= -\frac{i}{2}tr(\Xi M B^* - BM^*\Xi^*) \\ &= -\frac{i}{2}tr(\Xi M B^* + BM^*\Xi) \\ &= -\frac{i}{2}tr\left((MB^* + BM^*)^*\Xi\right) \\ &= d\langle \mu, \Xi\rangle_{|M}(B) \end{aligned}$$

The central elements in $\mathfrak{u}(k)$ are represented by diagonal matrices and $\mu^{-1}(-\frac{i}{2}\,\mathrm{Id})$ consists of all unitary k-frames in $\mathbb{C}^n$. Thus

$$\mathbb{C}^{n\times k}/\!\!/U(k)(-\frac{i}{2}\,\mathrm{Id}) = G(k,n)$$

is the grassmannian of k-planes in $\mathbb{C}^n$.

3. Complex flag manifolds. Let $n > k_1 > k_2 > \cdots > k_l > 0$ be a strictly decreasing sequence of positive integers. We call

$$\begin{aligned} F(k_1,k_2,\ldots,k_l;n) \;=\; \{ & \, V_l \subset V_{l-1} \subset \ldots V_2 \subset V_1 \subset \mathbb{C}^n \quad : \\ & V_i \text{ is a complex vector subspace of } \mathbb{C}^n \text{ of dimension } k_i \} \end{aligned}$$

the *manifold of flags of type* $(k_1,k_2,\ldots,k_l)$ *in* $\mathbb{C}^n$. Notice that $F(k_1;n) = G(k_1,n)$ and flag manifolds can be thought of as generalizations of the grassmannians.

The product group $G = U(k_l)\times U(k_{l-1})\times\cdots\times U(k_1)$ acts on the product manifold $M = \mathcal{M}_{k_l,k_{l-1}}(\mathbb{C})\times\mathcal{M}_{k_{l-1},k_{l-2}}(\mathbb{C})\times\cdots\times\mathcal{M}_{k_1,n}(\mathbb{C})$ by left and right multiplication:

$$(g_l,g_{l-1},\ldots,g_1)\cdot(\phi_l,\phi_{l-1},\ldots,\phi_1) = (g_l\phi_l g_{l-1}^*, g_{l-1}\phi_{l-1}g_{l-2}^*,\ldots,g_2\phi_2 g_1^*, g_1\phi_1)$$

The standard symplectic form on $M \cong \mathbb{C}^{k_{l-1}\times k_l}\times\mathbb{C}^{k_{l-2}\times k_{l-1}}\times\cdots\times\mathbb{C}^{n\times k_1}$ is

$$\omega_0\left((A_l,A_{l-1},\ldots,A_1),(B_l,B_{l-1},\ldots,B_1)\right) = -\frac{i}{2}\sum_{i=1}^{l} tr(A_iB_i^* - B_iA_i^*)$$

while the scalar product on the Lie algebra $\mathfrak{g} = \mathfrak{u}(k_l)\times\mathfrak{u}(k_{l-1})\times\cdots\times\mathfrak{u}(k_1)$ is given by

$$\langle\, (\Xi_l,\Xi_{l-1},\ldots,\Xi_1),(\Upsilon_l,\Upsilon_{l-1},\ldots,\Upsilon_1)\,\rangle = \sum_{i=1}^{l} tr(\Xi_i^*\Upsilon_i)$$

We claim that the map

$$\mu : M \longrightarrow \mathfrak{g} = \prod_{i=1}^{l}\mathfrak{u}(k_i)$$

$$\phi = (\phi_l,\phi_{l-1},\ldots,\phi_1) \longmapsto -\frac{i}{2}(\phi_l\phi_l^*, \phi_{l-1}\phi_{l-1}^* - \phi_l^*\phi_l,\ldots,\phi_1\phi_1^* - \phi_2^*\phi_2)$$

is a moment map for the above action. The G-equivariance is obvious and one only has to check the Hamilton equation. The infinitesimal action of $\mathfrak{g}$ is

$$\mathfrak{g} \longrightarrow T_\phi M$$

$$\Xi = (\Xi_l, \Xi_{l-1}, \dots, \Xi_1) \longmapsto X_\Xi(\phi) = (\Xi_l\phi_l + \phi_l\Xi_{l-1}^*, \Xi_{l-1}\phi_{l-1} + \phi_{l-1}\Xi_{l-2}^*, \dots, \Xi_1\phi_1)$$

and thus

$$\begin{aligned}
\big(\iota(X_\Xi(\phi))\omega_0\big)(B) &= -\frac{i}{2}\sum_{i=1}^{l} tr(X_\Xi(\phi)_i B_i^* - B_i X_\Xi(\phi)_i^*) \\
&= -\frac{i}{2}\sum_{i=2}^{l} tr\big((\Xi_i\phi_i - \phi_i\Xi_{i-1})B_i^* - B_i(\Xi_i\phi_i - \phi_i\Xi_{i-1})^*\big) \\
&\quad -\frac{i}{2} tr(\Xi_1\phi_1 B_1^* - B_1\phi_1^*\Xi_1^*) \\
&= -\frac{i}{2}\sum_{i=1}^{l} tr\big((\phi_i B_i^* + B_i\phi_i^*)\Xi_i\big) - \frac{i}{2}\sum_{i=2}^{l} tr\big((\phi_i^* B_i + B_i^*\phi_i)\Xi_{i-1}\big) \\
&= d\langle\mu, \Xi\rangle_{|_\phi}(B)
\end{aligned}$$

The central elements of $\mathfrak{g}$ are $\Xi = (\Xi_l, \Xi_{l-1}, \dots, \Xi_1)$ such that each Ξ_i is central in $\mathfrak{u}(k_i)$, i.e. each Ξ_i is a diagonal matrix. We claim that

$$\mu^{-1}\big(-\frac{i}{2}(\mathrm{Id}, \mathrm{Id}, \dots, \mathrm{Id})\big)/G = F(k_1, k_2, \dots, k_l; n)$$

In order to see this, use the following simple observation: if $k < p$ and $A \in \mathcal{M}_{k,p}(\mathbb{C})$ verifies $AA^* = D$, with $D \in \mathcal{M}_k(\mathbb{C})$ a diagonal matrix having positive real entries, then

$$A^*A = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}_p(\mathbb{C})$$

Proof: If $AA^* = D$, there is a $g \in U(k)$ such that $A = g \cdot M$, with $M = \begin{pmatrix} D^{1/2} & | & 0 \end{pmatrix}$. Then $A^*A = M^*g^*gM = M^*M$, i.e.

$$A^*A = \begin{pmatrix} {D^{1/2}}^* D^{1/2} & 0 \\ 0 & 0 \end{pmatrix}$$

□

Thus $\mu^{-1}\big(-\frac{i}{2}(\mathrm{Id}, \mathrm{Id}, \dots, \mathrm{Id})\big)$ is consituted of all elements $\phi = (\phi_l, \dots, \phi_1)$ satisfying

$$\phi_l\phi_l^* = \mathrm{Id},\ \phi_{l-1}\phi_{l-1}^* = \begin{pmatrix} 2\cdot\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix}, \dots, \phi_1\phi_1^* = \begin{pmatrix} l\cdot\mathrm{Id} & & & 0 \\ & (l-1)\cdot\mathrm{Id} & & \\ & & \ddots & \\ 0 & & & \mathrm{Id} \end{pmatrix}$$

These are to be interpreted as follows: *up to normalization*, ϕ_1 represents ordered unitary k_1-frames in $\mathbb{C}^n$, ϕ_2 represents ordered unitary k_2-frames inside $\mathrm{Span}(\phi_1)$

and so on, up to ϕ_l which represents ordered unitary k_l-frames inside $\mathrm{Span}(\phi_{l-1})$. The quotient will therefore be the desired flag manifold.

Basically, the action of G is designed so that the multiplication to the left identifies frames spanning the same vector space, while the multiplication to the right keeps track of the identification of frames at the upper level.

4. A configuration space. Consider the diagonal action of $G = SO(3)$ on $M = \underbrace{\mathbb{S}^2 \times \ldots \times \mathbb{S}^2}_{2k+1}$, where the product is taken an odd number of times. The Lie algebra consists of antisymmetric matrices

$$\mathfrak{g} = \mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -p & n \\ p & 0 & -m \\ -n & m & 0 \end{pmatrix} \quad : \quad m, n, p \in \mathbb{R} \right\}$$

We shall identify $\mathfrak{g}$ with $\mathbb{R}^3$ by associating to a matrix of the above form the vector $(m, n, p) \in \mathbb{R}^3$. The following hold:

(i) the Lie bracket in $\mathfrak{so}(3)$ corresponds to the vector product in $\mathbb{R}^3$;

(ii) the adjoint action of $SO(3)$ on $\mathfrak{so}(3)$ corresponds to the natural left action of $SO(3)$ on $\mathbb{R}^3$

(iii) the infinitesimal action of $SO(3)$ on $\mathbb{R}^3$ corresponds to performing the vector product in $\mathbb{R}^3$.

Define now

$$\mu : \mathbb{S}^2 \times \ldots \times \mathbb{S}^2 \longrightarrow \mathfrak{g} \cong \mathbb{R}^3$$
$$(x_1, \ldots, x_{2k+1}) \longmapsto \sum_{i=1}^{2k+1} x_i$$

We claim that μ is a moment map for the action of G on M, having 0 as a regular value. The equivariance is obvious by taking into account (ii).

Let us now focus on the Hamiltonian condition. Keeping in mind (iii), the infinitesimal action of G on M at a point $x = (x_1, \ldots, x_{2k+1})$ is given by

$$\mathbb{R}^3 \longrightarrow T_x M \cong x_1^{\perp} \times \ldots \times x_{2k+1}^{\perp}$$
$$v \longmapsto X_v(x) = (v \times x_1, \ldots, v \times x_{2k+1})$$

The symplectic form on M is

$$\omega = \sigma_1 \oplus \ldots \oplus \sigma_{2k+1}$$

where σ_i is the standard area form on the i-th factor. Pick any $Y \in T_xM$ and get

$$\begin{aligned} \big[\iota(X_v(x))\omega\big](Y) &= \sum_{i=1}^{2k+1} \sigma_i(v \times x_i, Y_i) \\ &= \sum_{i=1}^{2k+1} (Y_i, v) \\ &= d\langle \mu, v\rangle_{|x}(Y) \end{aligned}$$

which shows that μ also verifies Hamilton's equations.

As for seeing that 0 is a regular value for μ, it is enough to write

$$d\mu_{|x}(X_1, \ldots, X_{2k+1}) = X_1 + \ldots + X_{2k+1}, \qquad X_i \in x_i^{\perp}$$

which shows that x is a critical point iff all the x_i's lie on the same line (i.e. all the $x_i^{\perp}$ coincide). The set $\mu^{-1}(0)$ consists of all configurations with center of mass equal to 0, and in the case of an *odd* number of points it is impossible to have them all lying on the same line. Thus 0 is a regular value and $M/\!\!/SO(3)(0)$ describes up to a rotation all configurations having center of mass equal to 0.

5. Toric varieties. Let T be a finite dimensional torus and

$$\rho = (\rho_1, \ldots, \rho_n) : T \longrightarrow (\mathbb{S}^1)^n$$

be a homomorphism of Lie groups. We can make T act on $\mathbb{C}^n$ via ρ and the diagonal action of $(\mathbb{S}^1)^n$.

Denote the Lie algebra of T by $\mathfrak{t}$. We exhibit two remarkable lattices, called "integral" by analogy with the case $T = (\mathbb{S}^1)^n$. They are defined by

$$\Lambda := \big\{\tau \in \mathfrak{t} \ : \ exp(\tau) = 1\big\} \subset \mathfrak{t}$$

$$\Lambda^* := \big\{w \in \mathfrak{t}^* \ : \ \langle w, \tau\rangle \in 2\pi\mathbb{Z}, \quad \forall\, \tau \in \Lambda\big\} \subset \mathfrak{t}^*$$

The action of T is in fact determined by its *weights* $w_j \in \Lambda^*, \quad 1 \leq j \leq n$, which are defined by the identity

$$\rho_j(exp(\tau)) = exp(i\langle w_j, \tau\rangle), \quad \forall\, \tau \in \mathfrak{t}$$

Remark: Identify naturally $\mathrm{Lie}(\mathbb{S}^1)^n$ with $\mathbb{R}^n$, and let $(e_1,\dots,e_n)$ be the standard basis. Then the weights of the diagonal action on $\mathbb{C}^n$ are precisely the elements of the dual basis $(e_1^*,\dots,e_n^*)$ and it is easy to verify that

$$w_j = \rho^*(e_j^*), \quad 1 \le j \le n$$

We claim that the action of T on $\mathbb{C}^n$ is hamiltonian with moment map

$$\mu : \mathbb{C}^n \longrightarrow \mathfrak{t}^*$$

$$\mu(z) = \mu_\rho(z) = -\frac{1}{2}\sum_{j=1}^n |z_j|^2 w_j$$

It is a general fact that, given a morphism of Lie groups $\rho : H \longrightarrow G$ and a hamiltonian action of G on M with moment map $\mu_G : M \longrightarrow \mathfrak{g}^*$, the induced action of H on M is hamiltonian with moment map $\mu_H = \rho^* \circ \mu_G : M \longrightarrow \mathfrak{h}^*$.

In our case, taking into acount the preceding remark, it is enough to see that the diagonal action of $(\mathbb{S}^1)^n$ on $\mathbb{C}^n$ is hamiltonian with moment map

$$\mu : \mathbb{C}^n \longrightarrow \mathrm{Lie}(\mathbb{S}^1)^n \cong \mathbb{R}^n$$

$$\mu(z) = -\frac{1}{2}\sum_{j=1}^n |z_j|^2 e_j^*$$

The infinitesimal action at a point $z \in \mathbb{C}^n$ is given by

$$X_v(z) = (iv_1 z_1, \dots, iv_n z_n), \quad v \in \mathbb{R}^n \cong \mathrm{Lie}(\mathbb{S}^1)^n$$

and after a computation similar to that of Example 1 we get

$$\begin{aligned} \iota\big(X_v(z)\big)\omega_0 &= \frac{1}{2}\sum_{j=1}^n v_j\, \iota\big(\overline{z}_j \frac{\partial}{\partial \overline{z}_j} - z_j \frac{\partial}{\partial z_j}\big) dz_j \wedge d\overline{z}_j \\ &= \sum_{j=1}^n -\frac{1}{2} d\big(|z_j|^2\big) v_j \\ &= d\langle \mu, v\rangle_{|z} \end{aligned}$$

As to the properness of the action, we quote the following

Lemma 1. *[G] The action of T on $\mathbb{C}^n$ is proper if and only if one of the following four equivalent conditions is fulfilled:*

(i) $\mu_\rho^{-1}(0) = 0$*;*

(ii) $\sum_j s_j\, w_j = 0,\ \ s_j \geq 0 \Rightarrow s_1 = \ldots = s_n = 0$*;*

(iii) $\mu_\rho(\mathbb{C}^n) \setminus \{0\}$ *is contained in a positive open halfspace;*

(iv) there is a vector $v \in \mathfrak{t}$ *such that* $w_j(v) > 0$ *for all* j*.*

The quotients of the type $\mathbb{C}^n /\!\!/ T(\tau)$ with τ a generic element of $\mathfrak{t}^*$ are called *toric varieties.*

6. The Atiyah-Bott construction for the moduli space of flat connections ([AB]; see also [CGS], §2.2 and [CdS], §25 for an introduction). Let S be a compact Riemann surface on which we fix a Riemannian metric, G a compact or semi-simple Lie group and $P \xrightarrow{\pi} S$ a principal G-bundle over S, on which G acts freely *on the left.* Let us look at a construction which generalizes the already presented finite dimensional framework: an infinite dimensional Lie group (the gauge group) will act on an infinite dimensional symplectic manifold (the space of connections) in a Hamiltonian way, with moment map given by the curvature and such that the quotient space will be the moduli space of flat connections (modulo gauge equivalence).

Definition 1. *A connection on* P *is a Lie algebra valued* 1*-form* $A \in \Omega^1(P, \mathfrak{g})$ *that is*

- *G-equivariant :* $A_{gp}(gv) = gA_p(v)g^{-1},\ v \in T_pP,\ g \in G$*;*
- *vertical:* $A(\xi_P) = \xi,\ \xi \in \mathfrak{g}$ *with* ξ_P *being the infinitesimal generator of the flow* $exp(t\xi) : P \longrightarrow P$*.*

We shall denote the space of connections by

$$M = \mathcal{A}(P) \tag{1.3}$$

Any connection A as above determines a splitting $TP = V \oplus H$, where $V = \ker\ \pi_*$ is the vertical subbundle of TP and $H = \ker A$. This in turn determines splittings

$$\Omega^1(P) = \Omega^1_{\mathrm{Vert}}(P) \oplus \Omega^1_{\mathrm{Horiz}}(P)$$

$$\Omega^2(P) = \Omega^2_{\mathrm{Vert}}(P) \oplus \Omega^2_{\mathrm{Mixed}} \oplus \Omega^2_{\mathrm{Horiz}}(P)$$

where, for example, $\Omega^1_{\mathrm{Horiz}}(P) = \{\omega \in \Omega^1(P)\ :\ \omega_{|_V} = 0\}$ and the other spaces have similar obvious meanings. By definition, $A \in \Omega^1_{\mathrm{Vert}}(P, \mathfrak{g})$ if the splitting is determined by A itself.

Let A be a connexion 1-form on P. Then $A + a$ is still a connection 1-form if a is a $\mathfrak{g}$-valued horizontal and equivariant 1-form. Conversely, the difference of any two connection 1-forms is an element of $\big(\Omega^1_{\text{Horiz}}(P,\mathfrak{g})\big)^G$ i.e. a horizontal and equivariant 1-form. The space $\mathcal{A}(P)$ is thus an affine space modeled on $\big(\Omega^1_{\text{Horiz}}(P,\mathfrak{g})\big)^G$. Moreover, it can be endowed with a *symplectic structure* defined as

$$\omega(\alpha,\beta)=\int_S\langle\alpha\wedge\beta\rangle,\quad \alpha,\beta\in\big(\Omega^1_{\text{Horiz}}(P,\mathfrak{g})\big)^G\cong T_AM \tag{1.4}$$

where $\langle\cdot,\cdot\rangle$ is an *invariant* scalar product on $\mathfrak{g}$ (obtained either by averaging if G is compact or by using the Killing form if G is semi-simple).

In order to be able to perform the wedge product of two Lie algebra valued 1-forms, one needs a supplementary bilinear operation (call it $\mathcal{B}(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\longrightarrow\mathfrak{g}$): the wedge product of two 1-forms $\alpha=\sum a_i\otimes X_i$ and $\beta=\sum b_j\otimes Y_j$ in $\Omega^1(P)\otimes\mathfrak{g}$ will be

$$\alpha\wedge\beta\overset{\text{Not.}}{=}\mathcal{B}(\alpha\wedge\beta)=\sum a_i\wedge b_j\otimes\mathcal{B}(X_i,Y_j)\in\Omega^2(P)\otimes\mathfrak{g}$$

If $\mathcal{B}$ is real-valued, then $\mathcal{B}(\alpha\wedge\beta)\in\Omega^2(P)$. Moreover, if α and β are equivariant while $\mathcal{B}$ is invariant with respect to the adjoint action on $\mathfrak{g}$, then $\mathcal{B}(\alpha\wedge\beta)$ is *invariant* with respect to the left action on P and thus descends to a 2-form on the base. This is how (1.4) is to be understood.

Consider now the *"gauge group"*

$$\mathcal{G}=\mathcal{G}(P)=\big(\text{Map}(P,G)\big)^G$$

of maps $g:P\longrightarrow G$ that are equivariant with respect to the left adjoint action of G on itself, i.e. $g(hx)=hg(x)h^{-1},\ h\in G,\ x\in P$. The group $\mathcal{G}$ acts on the left on $\mathcal{A}(P)$ by

$$g\cdot A=-g^{-1}dg+g^{-1}Ag$$

One can define the gauge group in an equivalent way as being the group of all fibre preserving diffeomorphisms $f:P\longrightarrow P$ that are G-equivariant. The correspondence between the two definitions is given by $f(p)=g(p)p,\ p\in P$. The action on $\mathcal{A}(P)$ is given by defining $g\cdot A$ to be the push-forward of A through the diffeomorphism defined by g. The above formula should be read as

$$(g\cdot A)_{|p}=-g^{-1}(p)dg_{|p}+g^{-1}(p)A_{|p}g(p)$$

The Lie algebra of $\mathcal{G}$ is obviously

$$\text{Lie}(\mathcal{G})=\big(\Omega^0(P,\mathfrak{g})\big)^G$$

The pairing

$$\big(\Omega^2_{\text{Horiz}}(P,\mathfrak{g})\big)^G\times\big(\Omega^0(P,\mathfrak{g})\big)^G\longrightarrow\mathbb{R}$$
$$(\eta,\omega)\longmapsto\int_S\langle\eta\wedge\omega\rangle$$

is perfect and exhibits $\left(\Omega^2_{\mathrm{Horiz}}(P,\mathfrak{g})\right)^G$ as $\mathrm{Lie}(\mathcal{G})^*$.

Atiyah and Bott [AB] have remarked that the action of $\mathcal{G}$ on M is Hamiltonian with *moment map* given by the curvature $F_A = dA + [A,A]$, $A \in \mathcal{A}(P)$:

$$\mu : \mathcal{A}(P) \longrightarrow \left(\Omega^2_{\mathrm{Horiz}}(P,\mathfrak{g})\right)^G = \mathrm{Lie}(\mathcal{G})^*$$

$$A \longmapsto F_A$$

The infinitesimal action at $A \in \mathcal{A}(P)$ is

$$\mathrm{Lie}(\mathcal{G}) = \left(\Omega^0(P,\mathfrak{g})\right)^G \longrightarrow \left(\Omega^1_{\mathrm{Horiz}}(P,\mathfrak{g})\right)^G = T_A\mathcal{A}(P)$$

$$\alpha \longmapsto -d_A\alpha \overset{\text{not.}}{=} X_\alpha(A)$$

where d_A denotes the covariant derivative induced by A. It is defined as the horizontal part of the exterior derivative and on $\mathfrak{g}$-valued functions is computed to be

$$d_A\alpha_{|_p}(v) = d\alpha_{|_p}(v_{\mathrm{horiz}}) = d\alpha_{|_p}(v) + [A_p(v), \alpha(p)]$$

Take now $\alpha \in \left(\Omega^0(P,\mathfrak{g})\right)^G$ and $\beta \in \left(\Omega^1_{\mathrm{Horiz}}(P,\mathfrak{g})\right)^G$. One has

$$\begin{aligned} \big(\iota(X_\alpha(A))\omega\big)(\beta) &= -\int_S \langle d_A\alpha \wedge \beta\rangle \\ &\overset{\text{Stokes}}{=} \int_S \langle \alpha \wedge d_A\beta\rangle = \big(d\langle\mu,\alpha\rangle\big)_{|_A}(\beta) \end{aligned}$$

The last equality is obtained as follows

$$\begin{aligned} \big(d\langle\mu,\alpha\rangle\big)_{|_A}(\beta) &= \frac{d}{dt}_{|t=0} \int_S \langle \alpha \wedge \big(d(A+t\beta) + [A+t\beta, A+t\beta]\big)\rangle \\ &= \int_S \langle \alpha \wedge \big(d\beta + 2[A,\beta]\big)_{\mathrm{Horiz}}\rangle \\ &= \int_S \langle \alpha \wedge d_A\beta\rangle \end{aligned}$$

One has $[A,\beta] = 0$ since A is vertical while β is horizontal.

This completes the proof of the fact that the action is hamiltonian with moment map F_A. The symplectic quotient $\mathcal{A}(P)/\!\!/\mathcal{G}(0)$ is the *moduli space of flat connections modulo gauge equivalence*. It turns out that this space is a finite dimensional symplectic orbifold.

1.2 The Cauchy-Riemann equation for pseudo-holomorphic curves in symplectic quotients

We shall assume throughout the rest of the lectures that the condtition $(H1)$ is fulfilled. Let us first derive a useful description of the tangent space

$$T_{[x]}\overline{M} = \frac{T_x\mu^{-1}(\tau)}{T_x\, Gx} = \frac{\ker d\mu_x}{\operatorname{im} L_x}$$

One can think of $T_{[x]}\overline{M}$ as being the middle homology group of the complex

$$0 \longrightarrow \mathfrak{g} \xrightarrow{L_x} T_xM \xrightarrow{d\mu_x} \mathfrak{g} \longrightarrow 0$$

where $d\mu_x \circ L_x(\cdot) = [\cdot, \mu(x)] = 0$ if $\mu(x) = \tau \in Z(\mathfrak{g})$. In the presence of scalar products one has a baby Hodge decomposition which is readily verified

$$T_xM = \operatorname{im} L_x \oplus \operatorname{im} d\mu_x^* \oplus \big(\ker d\mu_x \cap \ker L_x^*\big) \tag{1.5}$$

The term $\ker d\mu_x \cap \ker L_x^*$ corresponds to harmonic elements for the formal laplacian $L_xL_x^* + d\mu_x^* d\mu_x$ and is isomorphic to the middle homology group of the above complex. Hence one can identify

$$T_{[x]}\overline{M} \cong \ker d\mu(x) \cap \ker L_x^* \tag{1.6}$$

As usual in symplectic geometry, scalar products are to be built from compatible almost complex structures and we are thus led to considering such J's satisfying

$$\begin{cases} \omega(\cdot, J\cdot) > 0 \text{ is a a positive definite riemannian metric} \\ J \text{ is } G \text{ - invariant i.e. } g^*J = J \end{cases}$$

Notation. Call $\mathcal{J}_G(M,\omega)$ the set of almost complex structures satisfying the above two conditions. It is a non-empty contractible set (homotopically equivalent with the set of G-invariant riemannian metrics on M).

Notation. Call $\langle \cdot, \cdot \rangle_J$ the scalar product $\omega(\cdot, J\cdot)$.

If $J \in \mathcal{J}_G(M,\omega)$, then by equivariance it descends to an almost complex structure $\overline{J}$ on $\overline{M}$. Indeed, (1.6) and the identity

$$d\mu_x J = L_x^* \tag{1.7}$$

imply that the tangent space $T_{[x]}\overline{M}$ is stable under J. In order to prove (1.7), choose arbitrary $\eta \in \mathfrak{g}$, $\xi \in T_xM$ and write

$$\langle \eta, L_x^*\xi \rangle_{\mathfrak{g}} = \langle L_x\eta, \xi \rangle_J = \omega(L_x\eta, J\xi) = \omega(X_\eta(x), J\xi) = \langle \eta, d\mu_x J\xi \rangle_{\mathfrak{g}}$$

Let us now describe holomorphic curves $\overline{u} : \mathbb{C} \longrightarrow \overline{M}$. Each such curve lifts to a smooth $u : \mathbb{C} \longrightarrow M$ with $\mu(u) = \tau$. Conversely, a map $u : \mathbb{C} \longrightarrow M$ descends to a $\overline{J}$-holomorphic map into $\overline{M}$ if and only if there exist maps $\Phi, \Psi : \mathbb{C} \longrightarrow \mathfrak{g}$ such that

$$\begin{cases} \partial_s u + L_u \Phi + J(\partial_t u + L_u \Psi) = 0 \\ \mu(u) = \tau \end{cases} \tag{1.8}$$

In fact, in view of (1.5) and (1.6), the maps Φ and Ψ are uniquely determined by the conditions

$$\begin{cases} L_u^*\big(\partial_s u + L_u \Phi\big) = 0 \\ L_u^*\big(\partial_t u + L_u \Psi\big) = 0 \end{cases} \tag{1.9}$$

which mean that $\partial_s u + L_u \Phi$ and $\partial_t u + L_u \Psi$ are the harmonic representatives of $\partial_s u$ and $\partial_t u$ respectively.

There is a natural gauge group of maps $g : \mathbb{C} \longrightarrow G$ acting on triples (u, Φ, Ψ) as

$$g^*(u, \Phi, \Psi) = (g^{-1}u, g^{-1}\partial_s g + g^{-1}\Phi g, g^{-1}\partial_t g + g^{-1}\Psi g)$$

Two triples (u, Φ, Ψ) and (u', Φ', Ψ') induce the same (holomorphic) map $\overline{u} : \mathbb{C} \longrightarrow \overline{M}$ if and only if there is a $g : \mathbb{C} \longrightarrow G$ such that $(u', \Phi', \Psi') = g^*(u, \Phi, \Psi)$.

We focus now on the global version of (1.8). Let Σ be a compact oriented Riemann surface with a fixed almost complex structure J_Σ. A map $\overline{u} : \Sigma \longrightarrow \overline{M}$ need not lift to a map $u : \Sigma \longrightarrow M$. However, if one considers $\mu^{-1}(\tau)$ as a principal bundle over $\overline{M}$, then $\overline{u}$ obviously lifts as an equivariant map from the pull-back bundle to $\mu^{-1}(\tau)$ and hence to M.

It will be made precise in lecture 2 that a principal bundle P as above is determined up to isomorphism by some equivariant homology class. In view of the fact that we shall construct the moduli space of maps u representing such an equivariant homology class, it is non-restrictive to fix from the very beginning a principal bundle $\pi : P \to \Sigma$ and investigate under what conditions an equivariant map $u : P \to M$ descends to a $(J_\Sigma, \overline{J})$-holomorphic curve.

We therefore study pairs (u, A) consisting of

- a G-equivariant map $u : P \longrightarrow M$ satisfying $\mu(u) \equiv \tau$,
- a connection $A \in \mathcal{A}(P)$,

where $\pi : P \to \Sigma$ is a fixed principal G-bundle. The covariant derivative of the map $u : P \to M$ with respect to the connection A is given by $d_A u : TP \to u^*TM$ defined as

$$d_A u(p)v \overset{\text{def}}{=} d_A u(p)v_{\text{horiz}} = du(p)v + X_{A_p(v)}(u(p)) \in T_{u(p)}M$$

The 1-form $d_A u \in \Omega^1(P, u^*TM)$ is equivariant (as are the map u and the horizontal distribution) and horizontal (vanishing by definition on the vertical part of TP). As a consequence, d_A descends on Σ as an element of $\Omega^1(\Sigma, u^*TM/G)$. Notice that u^*TM/G is a complex vector bundle.

Definition 2. *The twisted Cauchy-Riemann operator $\overline{\partial}_{J,A}$ associated to the connection A is the $(0,1)$ part of d_A i.e.*

$$\overline{\partial}_{J,A}(u) = \frac{1}{2}\big(d_A u + J \circ d_A u \circ J_\Sigma\big) \tag{1.10}$$

Even if J_Σ does not act on T_pP, the composition $d_A u \circ J_\Sigma$ is to be understood as follows : take a vector in T_pP, project it to $T_{\pi(p)}\Sigma$, apply J_Σ, lift to T_pP and finally apply $d_A u$. As $d_A u$ vanishes on vertical vectors, the result is well defined and independent of the lift.

$$\begin{tikzcd}
T_pP \arrow[d, "\pi"'] & T_pP \arrow[d, dashed, "\text{lift}"] \arrow[dr, "d_A u"] & \\
T_{\pi(p)}\Sigma \arrow[r, "J_\Sigma"] & T_{\pi(p)}\Sigma & T_{u(p)}M
\end{tikzcd}$$

One can verify that $\overline{\partial}_{J,A}(u) \in \Omega^{0,1}(\Sigma, u^*TM/G)$ coincides in local coordinates with the first term in (1.8).

Here is now an equivalent way of looking at solutions of $\overline{\partial}_{J,A}(u) = 0$. Consider the associated bundle $\widetilde{M} = P \times_G M$ with fibres diffeomorphic to M. The connection A induces a horizontal distribution on $P \times_G M$ and one can define therefore an almost complex structure $\widetilde{J}$ induced by J_Σ and J. An equivariant map $u : P \to M$ gives rise to a section $\widetilde{u} : \Sigma \to \widetilde{M}$ which is $\widetilde{J}$-holomorphic if and only if $\overline{\partial}_{J,A}(u) = 0$.

We shall consider in lecture 2 the moduli space of solutions of the equation

$$\begin{cases} \overline{\partial}_{J,A}(u) = 0 \\ *F_A + \mu(u) = \tau \end{cases} \tag{1.11}$$

This is a deformed version of (1.8). It is inspired by the Atiyah-Bott construction and has the advantage (over (1.10) combined with $\mu(u) = \tau$ alone) of uncoupling A and u by introducing the curvature term. Its similarities with both the Yang-Mills equation and the Gromov-Witten construction of moduli spaces make one hope that it can provide an alternative way of identifying Gromov and Seiberg-Witten invariants.

One essential feature of (1.11) is that, unlike the classical Gromov-Witten theory, it puts into play the metric on Σ via the Hodge $*$-operator. We shall fix throughout the rest of the lectures the volume form dvol_Σ and hence the metric on Σ. It will play an essential role in the results to follow.

Lecture 2

In this lecture, we study the solutions of the equation

$$\begin{cases} \partial_{J,A}(u) = 0 \\ *F_A + \mu(u) = \tau \end{cases} \tag{2.1}$$

2.1 Gauge invariance

Definition 3. *The group of gauge transformations is*

$$\mathcal{G}(P) = \{g \ : P \longrightarrow G / g(ph) = h^{-1}g(p)h\}.$$

This group acts on $\mathcal{C}^\infty_G(P, M) \times \mathcal{A}(P)$ by :

$$g^*(u, A) = (g^{-1}u, g^{-1}dg + g^{-1}Ag).$$

Thus, if (u, A) is solution of (2.1), then $g^*(u, A)$ is solution of (2.1).

2.2 Equivariant homology

Let EG be an infinite dimensional contractible space on which the group G acts freely. And let $\mathrm{BG} = \mathrm{EG}/\mathrm{G}$.

Example : If we assume that $\mathrm{G} \subset \mathrm{U(k)}$, then we can define explicitly :

$$\mathrm{EG^n} = \{\theta \in \mathbb{C}^{\mathrm{n\times k}} \,/\, \theta^*\theta = 1\}$$

Thus $\mathrm{EG^n} \subset \mathrm{EG^{n+1}}$ (by embedding $\mathbb{C}^n$ in $\mathbb{C}^{n+1}$), and we define :

$$\mathrm{EG} = \cup_{\mathrm{n}} \mathrm{EG^{n+1}}.$$

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Definition 4. *Let* $\mathrm{M_G} = \mathrm{M} \times_{\mathrm{G}} \mathrm{EG}$*. The equivariant homology of M is defined as the homology of* $\mathrm{M_G}$*.*

There exists an equivariant map from P to EG, obtained from a map $\bar{\theta}\ : \Sigma \longrightarrow \mathrm{BG}$, which is the classifying map for P :

$$\begin{array}{ccc} P & \xrightarrow{\theta} & \mathrm{EG} \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\bar{\theta}} & \mathrm{BG} \end{array}$$

$\bar{\theta}$ lifts by pull-back to θ.

Then the map $u \times \theta\ : P \longrightarrow M \times \mathrm{EG}$ is equivariant and so descends to a function u_G :

$$\begin{array}{ccc} P & \xrightarrow{u\times\theta} & \mathrm{M} \times \mathrm{EG} \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{u_G} & \mathrm{M_G} \end{array}$$

We define :

Definition 5. $[u] = (u_G)_*[\Sigma]\ \in H_2(M_G, \mathbb{Z})$

There is a natural projection $\pi\ : M_G \longrightarrow \mathrm{BG}$ (induced by the projection $\mathrm{EG} \longrightarrow \mathrm{BG}$), and so the homology class $[u]$ determines a homology class in $H_2(\mathrm{BG})$: $(\pi)_*([u]) = (\pi \circ u_G)_*([\Sigma])$. If G is connected, then this class determines the bundle P up to isomorphism (if $(\pi \circ u_G)_*([\Sigma]) = (\pi \circ u'_G)_*([\Sigma])$, then P and P' are isomorphic).

We fix a class $B \in H_2(M_G, \mathbb{Z})$. We consider the moduli space of all solutions which represent that homology class :

$$\widetilde{\mathcal{M}}_{B,\Sigma}(\tau, J)\ := \{(u, A) \in \mathcal{C}^\infty_G(P, M) \times \mathcal{A}(P)\,/\,(u, A)\ \textit{is solution of}\ (2.1)\ \textit{and}\ [u] = B\}.$$

Let :

$$\mathcal{M}_{B,\Sigma}(\tau, J)\ := \widetilde{\mathcal{M}}_{B,\Sigma}(\tau, J)/\mathcal{G}(P).$$

2.3 Energy

Definition 6. *We define the energy of an element of* $\mathcal{C}_G^\infty(P,M)\times\mathcal{A}(P)$ *:*

$$E(u,A) \;:= \frac{1}{2}\int_\Sigma (|d_A u|^2 + |F_A|^2 + |\mu(u)-\tau|^2)dvol_\Sigma. \tag{2.2}$$

Lemma 2.

$$E(u,A) = \int_\Sigma (|\bar\partial_{J,A}(u)|^2 + \frac{1}{2}|*F_A + \mu(u)-\tau|^2)dvol_\Sigma + \int_\Sigma (u^*\omega - d\langle\mu(u)-\tau, A\rangle). \tag{2.3}$$

The last term is a topological invariant of the class B (Σ is closed, but the integral is different of 0 because $\langle\mu(u)-\tau,A\rangle$ is only defined on P and does not descend on Σ). Thus, if (u,A) is solution of (2.1), then it is a minimum of E in its homology class.

proof :

We choose a holomorphic coordinates chart $\eta \;: U \longrightarrow \Sigma$, where $U(\ni s+it)$ is an open set of $\mathbb{C}$, and $\tilde\eta \;: U \longrightarrow P$ a lift of η. Then we consider u in local coordinates : $u = u^{loc}(= u\circ\tilde\eta) \;: U \longrightarrow M$. Also in local coordinates we have $A = A^{loc} = \Phi ds + \Psi dt$, where $\Phi,\Psi \;: U \longrightarrow \mathfrak{g}$, and $dvol_\Sigma = \lambda^2 ds\wedge dt$, where $\lambda \;: U \longrightarrow (0,\infty)$.
Thus,

$$F_A = (\partial_s\Psi - \partial_t\Phi + [\Phi,\Psi])ds\wedge dt,$$

$$d_A u (:= du + L_u A) = (\partial_s u + L_u\Phi)ds + (\partial_t u + L_u\Psi)dt,$$

$$\bar\partial_{J,A}(u) = \frac{1}{2}((\partial_s u + L_u\Phi) + J(\partial_t u + L_u\Psi)ds - J((\partial_s u + L_u\Phi) + J(\partial_t u + L_u\Psi))dt).$$

Then, the equation (2.1) locally writes :

$$\begin{cases}(\partial_s u + L_u\Phi) + J(\partial_t u + L_u\Psi) = 0\\ \partial_s\Psi - \partial_t\Phi + [\Phi,\Psi] + \lambda^2(\mu(u)-\tau) = 0.\end{cases} \tag{2.4}$$

Let $K \;:= \partial_s\Psi - \partial_t\Phi + [\Phi,\Psi]$, $v_s \;:= \partial_s u + L_u\Phi$ and $v_t \;:= \partial_t u + L_u\Psi$. Then the local energy $E^{loc} = \int e\, dsdt$, where :

$$e \;:= \frac{1}{2}|v_s + Jv_t|^2 + \frac{\lambda^2}{2}|\lambda^{-2}K + \mu(u) - \tau|^2 + R,$$

with R a mixed term :

$$R = \omega(v_s, v_t) - \langle K, \mu(u) - \tau \rangle.$$

By direct calculations, we have :

$$R = \omega(\partial_s u, \partial_t u) - \partial_s \langle \mu(u) - \tau, \Psi \rangle + \partial_t \langle \mu(u) - \tau, \Phi \rangle = u^* \omega - d\langle \mu(u) - \tau, A \rangle,$$

and this ends the proof.

2.4 Unique continuation

Definition 7. *A solution (u, A) of (2.1) is called horizontal if $d_A u \equiv 0$, $F_A \equiv 0$ and $\mu(u) \equiv \tau$.*

We see that a solution (u, A) is horizontal $\Leftrightarrow$ $[u]$ is torsion. Indeed, if $[u]$ is torsion, then $E = 0$ (according to the lemma). Conversely, if (u, A) is horizontal, then every equivariant cohomology class vanishes when evaluated on $[u]$.

We can prove :

Lemma 3. *If (u, A) is a solution of (2.1) and $d_A u = 0$, $\mu(u) = \tau$ on some open set, then (u, A) is horizontal.*

So, we have :

Corollary 1. *Assume $U \subset \Sigma$ is an open set, $d_A u = 0$ on U, and $L_{u(p)} : \mathfrak{g} \longrightarrow T_{u(p)}M$ is injective $\forall\, p \in U$. Then (u, A) is horizontal.*

proof of the corollary : According to the lemma, we just have to show that $\mu(u) = \tau$.
We have $0 = d_A u \in \Omega^1(\Sigma, u^*TM/G)$. We use the identity : $0 = \nabla_A d_A u = L_u F_A$.
As L_u is injective, we deduce $F_A = 0$ in U. So, $\mu(u) = \tau$ in U (because (u, A) is solution of (2.1)).

This lemma and the corollary are results of unique continuation.

2.5 Compactness

In general, the moduli space $\mathcal{M}_{B,\Sigma}(\tau, J)$ will not be compact. (As usual for holomorphic curves, we do not have compactness if there exist holomorphic spheres. So, there will be a problem if M is compact and $\mathbb{S}^1 \subset G$.)

Thus, we look at no compact M.
In order to have compactness, we have to make some hypothesis of convexity :

Definition 8. *A convex structure for* (M,ω,μ) *is a pair* (f,J) *where* $f\ : M \longrightarrow [0,\infty)$, $J \in \mathcal{J}_G(M,\omega)$, *such that :*

(*i*) f *is* G*-equivariant and proper,*

(*ii*) $f(x) \geq c_0 \Rightarrow \langle \nabla_\xi \nabla f(x), \xi\rangle \geq 0,\ \forall \xi\ \in T_xM,$

(*iii*) $f(x) \geq c_0(\tau) \Rightarrow df(x)J_xL_x(\mu(x)-\tau) \geq 0.$

example : If $M = \mathbb{C}^n$ and $G \subset U(n)$, then we can choose $f(x) = |x|^2$ and $J = i$.

The existence of such a pair guarantees that the solutions of (2.1) stay in compact set :

Lemma 4. *If (f,J) is a convex structure, then, for all solutions* (u,A) *of (2.1),*

$$\max_P f \circ u \leq max(c_0, c_0(\tau)).$$

proof : Computations show that $\Delta(f\circ u) \geq 0$ whenever $f \circ u \geq max(c_0, c_0(\tau))$.

Let (H2) be :

(H2) there exists such a convex structure (f,J) and $\langle [\omega], \pi_2(M)\rangle = 0$.

Theorem 1. *Assume (H1) and (H2). Then* $\forall\, B\ \in H_2(M_G,\mathbb{Z})$, $\mathcal{M}_{B,\Sigma}(\tau,J)$ *is compact,* i.e. *for all sequences* $(u_\nu, A_\nu)\ \in \widetilde{\mathcal{M}}_{B,\Sigma}(\tau,J)$, $\exists g_\nu\ \in \mathcal{G}(P)$ *such that* $g_\nu^*(u_\nu,A_\nu)$ *has a* $\mathcal{C}^\infty$*-convergent subsequence.*

sketch of proof :

- All solutions of the equation (2.1) stay in some given compact subset K of M (*i.e.* $u(P) \subset K$).
- As u verifies the equation $*F_A + \mu(u) = \tau$, $\exists c$ such that $\forall (u,A) \in \widetilde{\mathcal{M}}_{B,\Sigma}(\tau,J)$, $\parallel F_A \parallel_{L^\infty} \leq c$.
- For any p large enough ($p > 2$ for example), we can assume $\parallel A_\nu \parallel_{W^{1,p}} \leq C$ (thanks to Uhlenbeck theorem).
- Thus, the only obstruction to compactness will be "bubbling", if

$$c_\nu\ := \parallel d_{A_\nu} u_\nu \parallel_{L_\infty} \longrightarrow \infty. \tag{2.5}$$

In local coordinates, the following equation holds true :

$$(\partial_s u_\nu + L_{u_\nu}\Phi_\nu) + J(\partial_t u_\nu + L_{u_\nu}\Psi_\nu) = 0,$$

and (2.5) writes : there exist (s_ν, t_ν) such that :

$$|\partial_s u_\nu(s_\nu, t_\nu) + L_{u_\nu(s_\nu,t_\nu)}\Phi_\nu| = c_\nu \longrightarrow \infty. \tag{2.6}$$

Let $\epsilon_\nu := \frac{1}{c_\nu}$.
Let $v_\nu(s,t) := u_\nu(s_\nu + \epsilon_\nu s, t_\nu + \epsilon_\nu t)$. Then $|\partial_s v_s(0)| \sim 1$ (because $L_{u_\nu}\phi_\nu$ is bounded, so $\epsilon_\nu L_{u_\nu}\phi_\nu \longrightarrow 0$).
Moreover, $\partial_s v_\nu + J\partial_t v_\nu = -\epsilon_\nu w_\nu$, with $w_\nu = L_{u_\nu}\phi_\nu + L_{u_\nu}\psi_\nu$ bounded in L^∞. So, $\partial_s v_\nu + J\partial_t v_\nu \longrightarrow 0$.
Thus, by elliptic bootstrapping, we obtain a subsequence of (v_ν) which converges to $v : \mathbb{C} \longrightarrow M$, with $\partial_s v + J\partial_t v = 0$ and $|\partial_s v(0)| = 1$. But, by hypothesis, there is no sphere with positive ω. So, we have a contradiction.

Lecture 3

In this lecture, we will complete the construction of the moduli space $\mathcal{M}_{B,\Sigma}$ of solutions of

$$\left\{ \begin{array}{rcl} \overline{\partial}_{J,A} u & = & 0 \\ *F_A + \mu(u) & = & \tau \end{array} \right. \tag{3.1}$$

We will use it to define an invariant Φ, and compute it in a simple example.

3.1 Fredholm theory

The tangent space to the configuration space $\mathcal{B} = C^\infty_G(P, M) \times \mathcal{A}(P)$ is given by

$$T_{(u,A)}\mathcal{B} = \Omega^0(\Sigma, u^*TM/G) \times \Omega^1(\Sigma, \mathfrak{g}_P)$$

where $\mathfrak{g}_P = P \times_G \mathfrak{g}$ is the bundle associated to P for the adjoint representation of G on $\mathfrak{g}$.

After linearizing equations (3.1) at $(u, A) \in \mathcal{B}$, we obtain the operator

$$\mathcal{D}_{u,A} : \Omega^0(\Sigma, u^*TM/G) \oplus \Omega^1(\Sigma, \mathfrak{g}_P) \to \Omega^{0,1}(\Sigma, u^*TM/G) \oplus \Omega^0(\Sigma, \mathfrak{g}_P) \oplus \Omega^0(\Sigma, \mathfrak{g}_P)$$

that is given by

$$\mathcal{D}_{u,A} \begin{pmatrix} \xi \\ \alpha \end{pmatrix} = \begin{pmatrix} D_{u,A}\xi + (L_u\alpha)^{0,1} \\ d\mu(u)\xi + *d_A\alpha \\ L_u^*\xi - d_A^*\alpha \end{pmatrix}$$

where $D_{u,A}$ is the linearized Cauchy-Riemann operator at u.

The first two components correspond to the linearization of the 2 equations (3.1). On the

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other hand, we are only interested in solutions of (3.1) modulo gauge equivalence. That is why we have a third component, corresponding to the condition

$$d_A^*\alpha = L_u^*\xi$$

This is the equation of a local slice for the gauge group, that is a subspace of $T_{(u,A)}\mathcal{B}$ that is supplementary to the tangent space of the gauge orbit at (u, A).

In order to prove the Fredholm property, we can concentrate on the highest order terms only for each component. These are respectively $D_{u,A}\xi$, $*d_A\alpha$ and $d_A^*\alpha$.
Of course, in order for the operator $\mathcal{D}_{u,A}$ to be Fredholm, we have to extend its domain and target spaces to some Sobolev spaces.

Lemma 5. *$\mathcal{D}_{u,A}$ is Fredholm and index$(\mathcal{D}_{u,A}) = (\frac{1}{2}\dim M - \dim G)\chi(\Sigma) + 2\langle c_1, B\rangle$, where $B = [u] \in H_2(M_G, \mathbb{Z})$ and $c_1 = c_1(TM \times_G EG) \in H^2(M_G, \mathbb{Z})$.*

Proof. By the Riemann-Roch theorem, the operator

$$D_{u,A} : \Omega^0(\Sigma, u^*TM/G) \to \Omega^{0,1}(\Sigma, u^*TM/G)$$

is Fredholm with index $\frac{1}{2}\dim M\chi(\Sigma) + 2c_1(u^*TM/G)$.
On the other hand, consider the operator $d_A \oplus d_A^* : \Omega^1(\Sigma, \mathfrak{g}_P) \to \Omega^2(\Sigma, \mathfrak{g}_P) \oplus \Omega^0(\Sigma, \mathfrak{g}_P)$. Its kernel is $H^1(\Sigma, \mathfrak{g}_P)$ and its cokernel is $H^2(\Sigma, \mathfrak{g}_P) \oplus H^0(\Sigma, \mathfrak{g}_P)$. Therefore, adding the index of both summands, we obtain the expected formula. $\square$

In order to construct the moduli space of solutions, we need the operator $\mathcal{D}_{u,A}$ to be surjective.

Lemma 6. *If $\mathcal{D}_{u,A}$ is onto for all $(u, A) \in \widetilde{\mathcal{M}}_{B,\Sigma}(J)$ and if $\mathcal{G}$ acts freely on $\widetilde{\mathcal{M}}_{B,\Sigma}(J)$, then $\mathcal{M}_{B,\Sigma} = \widetilde{\mathcal{M}}_{B,\Sigma}/\mathcal{G}$ is a smooth manifold of dimension $2m =$ index$(\mathcal{D}_{u,A})$.*

This lemma can be proved using the implicit function theorem.

Lemma 6 shows that it is important to know whether $\mathcal{G}$ acts freely on $\widetilde{\mathcal{M}}_{B,\Sigma}$.

Definition 9. *A solution (u, A) of (3.1) is called irreducible if there exists $p \in P$ such that*

(i) $G_{u(p)} = \{1\}$

(ii) $\mathrm{Im}\ L_{u(p)} \cap \mathrm{Im}\ JL_{u(p)} = \{0\}$

Lemma 7. *Assuming (H1) and Vol(Σ) >> 1, every solution of (3.1) is irreducible.*

Proof. Using the energy functional, we have

$$\int_\Sigma |\mu(u) - \tau|^2 dvol_\Sigma \leq E(u, A)$$

Therefore, for a large volume of Σ,

$$\min_P |\mu(u) - \tau|^2 \leq \frac{E(u, A)}{\mathrm{Vol}(\Sigma)} \leq \delta$$

for some small $\delta > 0$. This holds for every solution (u, A) because the energy is a topological invariant.
Therefore, we can choose p so that $u(p)$ is arbitrarily close to $\mu^{-1}(\tau)$. Assumption (H1) then guarantees that p satisfies the 2 properties of definition 9. □

Given a convex structure (f, J_0) on M, choose a compact subset $K \subset M$ so that f satisfies (H2) for all $x \in M \setminus K$.

Let $\mathcal{J} = \{J : \Sigma \to \mathcal{J}_G(M, \omega) : z \to J_z \text{ such that } J_z = J_0 \text{ in } M \setminus K\}$.

Of course, all solutions of (3.1) are contained in K. Let $\mathcal{J}_{reg}(B) = \{J \in \mathcal{J} :$ all $(u, A) \in \widetilde{\mathcal{M}}_{B,\Sigma}(J)$ is irreducible and $\mathcal{D}_{u,A}$ is onto$\}$.

Theorem 2. *If B is not a torsion class, then $\mathcal{J}_{reg}$ is a countable intersection of open dense subsets of $\mathcal{J}$.*

Remark. We need to assume that B is not torsion so that $d_A u \neq 0$ somewhere; this is necessary to find generic J.

The proof of this theorem uses Sard-Smale theorem and the analysis of the linearized operator when J varies. Details can be found in chapter 3 of [McS].

Corollary 2. *Assume (H1), (H2), B is not torsion, Vol(Σ) >> 1 and $J \in \mathcal{J}_{reg}(B)$. Then $\mathcal{M}_{B,\Sigma}(\tau, J)$ is a compact oriented smooth manifold of dimension $2m$.*

Proof. We combine theorems 1 and 2, and lemmata 6 and 7. □

3.2 The invariant Φ

Let us construct an evaluation map $\mathcal{M}_{B,\Sigma} \to M_G$. Fix a point $p_0 \in P$. Consider the bundle $\mathcal{P}_{B,\Sigma} = \widetilde{\mathcal{M}}_{B,\Sigma}/\mathcal{G}_0 \to \mathcal{M}_{B,\Sigma}$, where $\mathcal{G}_0 = \{g \in \mathcal{G} \mid g(p_0) = 1\}$. To this principal

G-bundle corresponds a unique classifying map $\theta_0 : \mathcal{M}_{B,\Sigma} \to BG$ up to homotopy, so that the diagram

$$\begin{array}{ccc} \mathcal{P}_{B,\Sigma} & \xrightarrow{\bar{\theta}_0} & EG \\ \downarrow & & \downarrow \\ \mathcal{M}_{B,\Sigma} & \xrightarrow{\theta_0} & BG \end{array}$$

is commutative. The map $\bar{\theta}_0$ is equivariant : $\bar{\theta}_0(g^{-1}u, g^*A) = g(p_0)\bar{\theta}_0(u, A)$.

We can then define

$$\begin{array}{rcl} ev_G : \mathcal{M}_{B,\Sigma} & \to & M \times_G EG \\ [u, A] & \to & [u(p_0), \bar{\theta}_0(u, A)] \end{array}$$

Note that the G-equivalence class in M_G is well-defined, because of the behavior of $\bar{\theta}_0$.

Using ev_G, we can construct a map $\Phi_{B,\Sigma}^{M,\mu,\tau} : H^*(M_G) \to \mathbb{Z}$ defined by

$$\Phi_{B,\Sigma}^{M,\mu,\tau}(\alpha) = \int_{\mathcal{M}_{B,\Sigma}(\tau,J)} ev_G^*\alpha$$

It is independent of the choice of J, but depends on τ via some wall crossing formula.

Remark. Using the projection map $\mathcal{M}_{B,\Sigma} \to \mathcal{A}(P)$, we can also pull-back a cohomology class from the space of connections. We can also let the point p_0 vary.

3.3 Example

Let $M = \mathbb{C}^n$, $G = S^1$ acting on $\mathbb{C}^n$ by multiplication of each component by a phase. Then, $H^*(M_G) = H^*(BS^1)$ since $\mathbb{C}^n$ is contractible. A circle bundle P over Σ is classified by $\deg(P) = d$ (corresponding to the data of B); let g be the genus of Σ.

Proposition 1.

$$\Phi_{d,g}^{\mathbb{C}^n, S^1}(c^m) = n^g$$

where $m = n(d + 1 - g) + g - 1$ and c is the generator of $H^*(BS^1)$.

This formula was first obtained via the vortex equation by Weitsmann et al.

Proof. Fix $A_0 \in \mathcal{A}(P)$. Let

$$\mathcal{A}_0 = \{A \in \mathcal{A}(P) \mid d^*(A - A_0) = 0 \text{ and } *F_A = -\frac{2\pi i d}{\mathrm{Vol}(\Sigma)}\}$$

and

$$\mathcal{G}_0 = \{g \in \mathcal{G}(P) \mid d^*(g^{-1}dg) = 0\}$$

Intuitively, $\mathcal{A}_0$ is an analog of the set of flat connections, and $\mathcal{G}_0$ consists of gauge transformations preserving $\mathcal{A}_0$.

We have $\mathcal{A}_0/\mathcal{G}_0 \cong T^{2g}$. To see this, pick a basis $\gamma_1, \dots, \gamma_{2g}$ of $H_1(\Sigma, \mathbb{Z})$ consisting of simple curves, and such that $\gamma_i \cdot \gamma_{i+g} = 1$, all other intersection numbers being zero. Let α_i be harmonic 1-forms representing the cohomology classes dual to γ_i.
Let $A_t = A_0 + \sum_{j=1}^{2g} t_j 2\pi i \alpha_j$, where $t_j \in \mathbb{R}$.

Claim. $A_{t+k} = g_k^* A_t$, where $k = (k_1, \dots, k_{2g})$, $k_j \in \mathbb{Z}$ and $g_k \in \mathcal{G}_0$ satisfy

$$\frac{1}{2\pi i} \int_{\gamma_j} g_k^{-1} dg_k = k_j$$

Hence, after dividing out by gauge equivalence, we obtain $t \in T^{2g}$.

Let $E = P \times_{S^1} \mathbb{C}$ be the complex line bundle over Σ associated to P. Let $\mathbb{E} = E \times T^{2g}$ over $\Sigma \times T^{2g}$. We will denote $\mathbb{E}|_{\Sigma \times \{t\}}$ by $\mathbb{E}_t$. It will be equipped with connection A_t.

Consider now the family of operators

$$\begin{tikzcd} \Omega^0(\mathbb{E}_t) \arrow[rr, "\overline{\partial}_{A_t}"] \arrow[dr] & & \Omega^{0,1}(\mathbb{E}_t) \arrow[dl] \\ & T^{2g} & \end{tikzcd}$$

where $\overline{\partial}_{A_t}$ is the $(0,1)$ part of the covariant derivative d_{A_t}.

The kernel manifold $\mathcal{M}_0$ is defined to be

$$\mathcal{M}_0 = \{(t, s) \mid t \in T^{2g}, s \in \Omega^0(\Sigma, \mathbb{E}_t^{\oplus n}), \overline{\partial}_{A_t} s = 0, \|s\|_{L^2} = 1\}/S^1$$

Intuitively, we expect this manifold to be closely related to the moduli space $\mathcal{M}_{d,g}^{\mathbb{C}^n, S^1}$. Indeed, sections of $E^{\oplus n}$ are equivariant maps $P \to \mathbb{C}^n$, and the condition $\|s\|_{L^2} = 1$ is somewhat analogous to $\mu(u) = -\frac{i}{2}$.

Fact 1. $\mathcal{M}_0$ is cobordant to $\mathcal{M}_{d,g}^{\mathbb{C}^n, S^1}$.

Fact 2.

$$\int_{\mathcal{M}_{d,g}^{\mathbb{C}^n, S^1}} c^m = \int_{T^{2g}} c_g(-IND(\overline{\partial}_t^{\oplus n}))$$

where $IND(\overline{\partial}_t) = \ker(\overline{\partial}_t) - \mathrm{coker}(\overline{\partial}_t)$ is the index bundle of $\overline{\partial}_t$.

By the Atiyah-Singer index theorem, we have

$$\mathrm{ch}(IND^{\oplus n}) = n \int_\Sigma \mathrm{td}(T\Sigma)\mathrm{ch}(\mathbb{E}) \in H^*(T^{2g})$$

The characteristic classes are given by

$$\begin{aligned} \mathrm{td}(T\Sigma) &= 1 + (1-g)\sigma \\ \mathrm{ch}(\mathbb{E}) &= 1 + c_1(\mathbb{E}) + \frac{1}{2}c_1(\mathbb{E})^2 \\ c_1(\mathbb{E}) &= d\sigma + \sum_{j=1}^{2g} \alpha_j \wedge dt_j \end{aligned}$$

where σ is the positive generator of $H^2(\Sigma, \mathbb{Z})$.
Hence

$$\begin{aligned} c_1(\mathbb{E})^2 &= -\sigma \sum_{j=1}^{2g} dt_j \wedge dt_{j+g} \\ &= -\sigma\Omega \end{aligned}$$

where Ω is the intersection 2-form on $H^1(\Sigma, \mathbb{R})$.
Therefore,

$$\begin{aligned} \mathrm{td}(T\Sigma)\mathrm{ch}(\mathbb{E}) &= (1 + (1-g)\sigma)(1 + d\sigma - \sigma\Omega + \dots) \\ &= 1 + (d + 1 - g)\sigma - \sigma\Omega + \dots \end{aligned}$$

The omitted terms have a zero contribution after integration on Σ. We obtain

$$\mathrm{ch}(IND) = d + 1 - g - \Omega$$

so that

$$\begin{aligned} c_g(-IND^{\oplus n}) &= \frac{1}{g!}c_1(-IND^{\oplus n})^g \\ &= n^g \frac{\Omega^g}{g!} \end{aligned}$$

After integrating on T^{2g}, we obtain n^g as predicted. $\square$

Lecture 4

In this lecture, we will establish the relationship between the invariant Φ and the Gromov-Witten invariants of the symplectic reduction $\overline{M} = M/\!\!/G$.

$$\begin{array}{ccc} \text{On } M: & & \text{On } \overline{M}: \\ \left\{ \begin{array}{rcl} \overline{\partial}_{J,A} u &=& 0 \\ *F_A + \mu(u) &=& 0 \end{array} \right. & \longleftrightarrow & \left\{ \begin{array}{rcl} \overline{\partial}_{J,A} u &=& 0 \\ \mu(u) &=& 0 \end{array} \right. \end{array}$$

The correspondance will be established using an adiabatic limit. We replace $\mathrm{Vol}(\Sigma)$ with $\mathrm{Vol}(\Sigma)\ \epsilon^{-2}$. The equations become

$$\left\{ \begin{array}{rcl} \overline{\partial}_{J,A} u &=& 0 \\ *F_A + \epsilon^{-2}\mu(u) &=& 0 \end{array} \right. \tag{4.1}$$

For $\epsilon = 1$, we obtain the equations on M, but if we let $\epsilon \to 0$, we obtain the equations on $\overline{M}$.

4.1 Correspondance theorem

Let us define

$$\mathcal{V} = \{v : D^2 \to M \mid v(\partial D) \subset \mu^{-1}(0), v(e^{i\theta}) = g(\theta)x_0 \text{ where } x_0 \in \mu^{-1}(0), g : \mathbb{R}/2\pi\mathbb{Z} \to G\}$$

Let $m : \mathcal{V} \to \mathbb{Z}$ be the Maslov index. Given a map $v : D^2 \to M$, we can trivialize the tangent bundle of M over D^2, so that the differentials $g(\theta)_*$ give a loop of symplectic matrices based at $g(0)_* = \mathrm{Id}$. Then $m(v)$ is the Maslov index of this loop.

Remark. $m(v)$ is nothing but $2c_1$ evaluated on the equivariant homology class of v.

Notes by F. Bourgeois

We now introduce a monotonicity assumption on $\overline{M}$.

$$\text{(H3)} \qquad \exists \lambda > 0 \text{ such that } \forall v \in \mathcal{V}, \int_{D^2} v^* \omega = \lambda m(v)$$

We define the minimal Maslov number of $\overline{M}$ by

$$N = \min_{v \in \mathcal{V}, m(v) > 0} m(v)$$

so that $m(\mathcal{V}) = N\mathbb{Z}$.

Note that, since

$$\overline{M} \sim \mu^{-1}(0) \times_G EG \hookrightarrow M \times_G EG = M_G$$

we have maps $H_*(\overline{M}) \to H_*(M_G)$ and $H^*(M_G) \to H^*(\overline{M})$. We will denote both maps by κ.

Theorem 3. (R. Gaio, D. Salamon [GS]) *Assuming (H1), (H2) and (H3), if* $B = \kappa(\bar{B})$*,* $\bar{\alpha}_i = \kappa(\alpha_i)$*,* $\sum \deg(\alpha_i) = \dim \mathcal{M}_{B,\Sigma}$ *and* $\deg(\alpha_i) < 2N$*, then*

$$GW^{\overline{M}}_{\bar{B},\Sigma}(\bar{\alpha}_1, \dots, \bar{\alpha}_k) = \Phi^{M,\mu}_{B,\Sigma}(\alpha_1 \cup \ldots \cup \alpha_k)$$

Remark 1. The Gromov-Witten invariant GW is defined here for fixed (Σ, j_Σ) and with fixed marked points $z_i \in \Sigma$:

$$GW^{\overline{M}}_{\bar{B},\Sigma}(\bar{\alpha}_1, \dots, \bar{\alpha}_k) = \#\{\bar{u} : \Sigma \to \overline{M}(j_\Sigma, J) - \text{holomorphic} \mid [\bar{u}] = \bar{B}, \bar{u}(z_i) \in Y_i\}$$

where Y_i are generic cycles in $\overline{M}$ representing homology classes dual to $\bar{\alpha}_i$.

Remark 2. We cannot allow $\deg(\alpha_i) \geq 2N$. Consider indeed the example from lecture 3 : $M = \mathbb{C}^n$, S^1 acts on $\mathbb{C}^n$ by multiplication of each component by a phase, so that $\overline{M} = \mathbb{C}P^{n-1}$. In that case, $N = n$ and

$$\Phi^{\mathbb{C}^n, S^1}_{d,g}(c^m) = n^g$$

where $m = n(d + 1 - g) + g - 1$, but

$$GW^{\mathbb{C}P^{n-1}}_{d,g}(c^{m_1}, \dots, c^{m_k}) = 0$$

if $m_i \geq n$ for some i.

4.2 Extension to quantum cohomology

The condition on the degree of α_i is rather restrictive. It is interesting to reformulate theorem 3 on the quantum cohomology of $\overline{M}$ rather than its usual cohomology.

Recall that the quantum cohomology $QH^*(\overline{M})$ of $\overline{M}$ consists of elements of the form

$$\bar{\alpha} = \sum_{\bar{B}} \bar{\alpha}_{\bar{B}} e^{\bar{B}}$$

where $\bar{\alpha}_{\bar{B}} \in H^*(\overline{M})$, $\bar{B} \in H_2(\overline{M})$ and $\deg(\bar{\alpha}_{\bar{B}}) + 2c_1(\bar{B}) = \deg(\bar{\alpha})$, with the condition

$$\#\{\bar{B} \,|\, \bar{\alpha}_{\bar{B}} \neq 0, \int_{\bar{B}} \omega \leq C\} < \infty$$

for all $C \in \mathbb{R}$.

a) Product structure on $QH^*(\overline{M})$: pick a basis $\bar{e}_i$ of $H^*(\overline{M})$; let $\bar{e}_j^*$ be the dual basis, i.e. $\int_{\overline{M}} \bar{e}_i \cup \bar{e}_j^* = \delta_{ij}$. Then the quantum product is given by

$$\bar{\alpha}_1 * \bar{\alpha}_2 = \sum_{\bar{B}_1, \bar{B}_2, \bar{B}} \sum_i GW_{\bar{B}-\bar{B}_1-\bar{B}_2}(\bar{\alpha}_{1,\bar{B}_1}, \bar{\alpha}_{2,\bar{B}_2}, \bar{e}_i^*)\ \bar{e}_i\ e^{\bar{B}}$$

b) Multilinear map $GW_{\bar{B}} : QH^*(\overline{M}) \otimes \ldots \otimes QH^*(\overline{M}) \to \mathbb{Z}$ given by

$$GW_{\bar{B}}(\bar{\alpha}_1, \ldots, \bar{\alpha}_k) = \sum_{\bar{B}_i} GW_{\bar{B}-\bar{B}_1-\ldots-\bar{B}_k}(\bar{\alpha}_{1,\bar{B}_1}, \ldots, \bar{\alpha}_{k,\bar{B}_k})$$

c) Gluing theorem for GW :

$$GW_{\bar{B}}(\bar{\alpha}_1, \ldots, \bar{\alpha}_k) = GW_{\bar{B}}(\bar{\alpha}_1 * \ldots * \bar{\alpha}_k)$$

Hence, we can view the Gromov-Witten invariant as a map $GW_{\bar{B}} : QH^*(\overline{M}) \to \mathbb{Z}$.

We are now in position to reformulate theorem 3.

Corollary 3. *Assume (H1), (H2) and (H3). If $H^*(M_G)$ is generated by classes of degree $< 2N$, then there exists a unique ring homomorphism $\varphi : H^*(M_G) \to QH^*(\overline{M})$ such that if $\deg \alpha < 2N$ then $\varphi(\alpha) = \kappa(\alpha)$.*
Moreover,

a) φ is onto.

b) The diagram

$$\begin{tikzcd} H^*(M_G) \arrow[rr, "\varphi"] \arrow[dr, "\Phi_{B,\Sigma}"'] & & QH^*(\overline{M}) \arrow[dl, "GW^{\overline{M}}_{B,\Sigma}"] \\ & \mathbb{Z} & \end{tikzcd}$$

is commutative.

Proof. If $\alpha = \sum_i \alpha_{i_1} \cup \ldots \cup \alpha_{i_m}$, then we define $\varphi(\alpha) = \sum_i \kappa(\alpha_{i_1}) \cup \ldots \cup \kappa(\alpha_{i_m})$. This is well-defined, because of theorem 3, i.e. $\varphi(\alpha) \neq 0 \implies \alpha \neq 0$. $\square$

4.3 Proof of the theorem

Let $\mathcal{M}^{\epsilon}_{B,\Sigma}$ be the moduli space of solutions of (4.1). We wish to establish a relation between $\mathcal{M}^{\epsilon}_{B,\Sigma}$ for $\epsilon > 0$ and $\mathcal{M}^{0}_{B,\Sigma}$.

For $C_0 > 0$, let $\mathcal{M}^{0}_{B,\Sigma}(C_0) = \{(u, A) \in \mathcal{M}^{0}_{B,\Sigma} \mid \|d_A u\|_{L^\infty} \leq C_0, \|F_A\|_{L^\infty} \leq C_0\}$.

Theorem 4. *For all $C_0 > 0$ and $B \in H_2(M_G)$, there exists $\epsilon_0 > 0$ such that $\forall \epsilon \in (0, \epsilon_0)$, there is an embedding*

$$\mathcal{T}^{\epsilon} : \mathcal{M}^{0}_{B,\Sigma}(C_0) \to \mathcal{M}^{\epsilon}_{B,\Sigma}$$

Both moduli spaces have the same dimension, so we can think of this embedding as the elementary embedding $f : \mathbb{R} \to S^1$ such that $f(\mathbb{R}) = (\delta, 2\pi - \delta)$ for some $\delta > 0$. Of course, such a map is not surjective, but we can control the image of $\mathcal{T}^{\epsilon}$ using the following result.

Theorem 5. *Fix $\delta > 0$. For all $C > 0$, there are $\epsilon_0, C_0 > 0$ such that if*

$$(u, A) \in \mathcal{M}^{\epsilon}_{B,\Sigma} \qquad \|d_A u\|_{L^\infty} \leq C \qquad \|\mu(u)\|_{L^\infty} \leq \delta$$

then $[u, A] \in$ Im $\mathcal{T}^{\epsilon}$.

Roughly speaking, the discrepancy between GW and Φ will correspond to solutions of (4.1) that are never contained in the image of the embedding $\mathcal{T}^{\epsilon}$, for every $\epsilon > 0$.
If $\epsilon_i \to 0$ and $(u_i, A_i) \in \mathcal{M}^{\epsilon_i}_{B,\Sigma} \setminus \text{ Im } \mathcal{T}^{\epsilon_i}$, then $|d_{A_i} u_i(w_i)| \to \infty$, for some $w_i \in \Sigma$ and $w_i \to w$. Therefore, the sequence (u_i, A_i) will converge to a holomorphic map in $\overline{M}$, with a bubble at w.

Case 1 : $w \neq z_i$
The bubble takes some energy, hence the dimension of the moduli space for the remaining part decreases, by the monotonicity assumption (H3). So, there will be no holomorphic curves with the prescribed conditions at points z_i.

Case 2 : $w = z_j$

There will be no condition at z_j for the remaining part, but the dimension of the moduli space will drop even more, so the conclusion is the same.

Intuitively, the conclusion of this discussion is that the solutions outside $\mathrm{Im}\mathcal{T}^\epsilon$ do not contribute to GW, so the invariants must coincide.

Remark. When bubbling occurs, we are in one of the following cases :

(i) $\|d_{A_i} u_i\|_{L^\infty}\epsilon_i \to \infty$

This corresponds to the case of a bubble in M. But these do not exist, by assumption (H2).

(ii) $\|d_{A_i} u_i\|_{L^\infty}\epsilon_i \to 0$

This corresponds to the case of a bubble in $\overline{M}$, and it will contribute to $GW^{\overline{M}}$.

(iii) $\|d_{A_i} u_i\|_{L^\infty}\epsilon_i \to 1$

After rescaling, we lose the factor ϵ in equation (4.1). We obtain the bubbling of a vortex over $\mathbb{C}$, satisfying :

$$\begin{cases} \partial_s u + L_u\Phi + J(\partial_t u + L_u\Psi) & = & 0 \\ \partial_s\Psi - \partial_t\Phi + [\Phi, \Psi] + \mu(u) & = & 0 \end{cases}$$

with finite energy.

Let $\mathcal{M}_B^0$ be the moduli space of finite energy vortices over $\mathbb{C}$. In radial coordinates, we have

$$u(re^{i\theta}) \to g(\theta)x_0 \quad \text{when} \quad r \to \infty$$

with $\mu(x_0) = 0$.

Hence, we can define an evaluation map $ev_\infty : \mathcal{M}_B^0 \to \overline{M}$ by $ev_\infty(u, \Phi, \Psi) = [x_0]$.

On the other hand, we have an evaluation map at point $0 \in \mathbb{C}$: $ev_0 : \mathcal{M}_B^0 \to M_G$.

Using these maps, we can construct explicitly the homomorphism φ of corollary 3 :

$$\varphi(\alpha) = \sum_i \sum_{\bar{B}} \int_{\mathcal{M}^0_{\kappa(\bar{B})}} (ev_0^*\alpha \cup ev_\infty^*\bar{e}_i^*)\ \bar{e}_i\ e^{\bar{B}}$$

Lecture 5

This lecture treats explicit examples and gives some hints on possible links with other classical constructions of symplectic or gauge theoretical invariants.

Example 1: Vortex equations. Consider the standard action of $G = \mathbb{S}^1$ on $M = \mathbb{C}$, with moment map $\mu(z) = -\frac{i}{2}|z|^2$ (see also (1.1)). The quotient at a nonzero element of $\mathrm{Lie}(\mathbb{S}^1) = i\mathbb{R}$ is a point, but the solutions of (1.11) are still interesting to look at.

Consider $P \xrightarrow{\pi} \Sigma$ a circle bundle of degree d. The complex line bundle associated to P is precisely $E = P \times_{\mathbb{S}^1} \mathbb{C}$ and the degree of P is $\langle c_1(E), [\Sigma]\rangle$. An equivariant map $u : P \to \mathbb{C}$ is identified with a section $\Theta : \Sigma \to E$, while a connection A on P gives rise to a hermitian connection on E - still denoted by A . One has $H_*^{\mathbb{S}^1}(\mathbb{C}) = H_*(\mathbb{C}P^\infty)$ and any map u as above represents the class $d \cdot [\mathbb{C}P^1]$.

The operator $\overline{\partial}_{J_{\mathbb{C}},A} : \big(\mathrm{Map}(P, \mathbb{C})\big)^G \longrightarrow \Omega^{0,1}(\Sigma, TM/\mathbb{S}^1)$ translates precisely to the Cauchy-Riemann operator

$$\overline{\partial}_A : \Gamma(\Sigma, E) \longrightarrow \Omega^{0,1}(\Sigma, E)$$

and the equations (1.11) become the "vortex equations"

$$\begin{cases} \overline{\partial}_A \Theta = 0 \\ *iF_A + \frac{|\Theta|^2}{2} = \tau, \qquad \tau \in \mathbb{R} \end{cases} \tag{5.1}$$

The necessary condition of existence of solutions is easily obtained by integrating over Σ the second equation in (5.1) and writes

$$\tau > \frac{2\pi d}{\mathrm{Vol}(\Sigma)}$$

Notes by A. Oancea

The moduli space

$$\mathcal{M}_d(\Sigma) = \frac{\{(A,\Theta) : (5.1)\}}{\mathrm{Map}(\Sigma, \mathbb{S}^1)}$$

can be identified via the zeroes of Θ with the set of effective divisors of degree d, hence with

$$\mathrm{Sym}^d(\Sigma) = \frac{\Sigma \times \ldots \times \Sigma}{\mathfrak{S}_d}$$

Example 2: Bradlow pairs. Let $G = U(2)$ act on $M = \mathbb{C}^2$ by multiplication on the left, with moment map $\mu(z) = -\frac{i}{2}zz^*$ (see also (1.2)). The quotient is the empty set for any nonzero central element of $\mathfrak{u}(2)$, but the space of solutions of (1.11) is again interesting.

Let $P \xrightarrow{\pi} \Sigma$ be a $U(2)$-bundle of degree d and $E = P \times_{U(2)} \mathbb{C}^2$ the associated rank 2 vector bundle. As in the preceding example, the equations (1.11) become

$$\begin{cases} \overline{\partial}_A \Theta = 0 \\ *iF_A + \frac{1}{2}\Theta\Theta^* = \tau \mathrm{Id} \end{cases} \tag{5.2}$$

with $A \in \mathcal{A}(E)$ and $\Theta \in \Gamma(\Sigma, E)$. The term $\Theta\Theta^*$ is well defined as an element of $\mathrm{End}(E)$ because it is invariant under the action of $U(2)$.

A first necessary condition of existence of solutions is obtained as before by integrating over Σ the trace of the second equation in (5.2) and writes $\tau > \frac{\pi d}{\mathrm{Vol}\ \Sigma}$. It turns out that the complete necessary and sufficient condition for the existence of solutions is

$$\frac{\pi d}{\mathrm{Vol}\ \Sigma} < \tau \leq \frac{2\pi d}{\mathrm{Vol}\ \Sigma}$$

Thaddeus [Th] studied in detail the behaviour of the moduli space

$$\mathcal{M}_\tau = \frac{\{(A,\Theta)\ : (5.2)\}}{\mathcal{G}(E)}$$

as τ varies. The critical parameters are

$$\tau_k = \frac{2\pi k}{\mathrm{Vol}\ \Sigma}, \qquad \frac{d}{2} < k \leq d, \qquad k \in \mathbb{Z}$$

For large d, the picture is the following :

- if $\tau_{d-1} < \tau < \tau_d$, then $\mathcal{M}_\tau \sim \mathbb{C}P^{3g-3+d}$, with g the genus of Σ
- $\mathcal{M}_\tau$ is smooth for $\tau \neq \tau_k$ and the singular part of $\mathcal{M}_{\tau_k}$ can be identified with $\mathrm{Sym}^{d-k}(\Sigma)$

- if $\frac{\pi d}{\text{Vol } \Sigma} < \tau < \tau_{d/2+1}$ then $\mathcal{M}_\tau$ is a bundle on the moduli space of flat $U(2)$ connections on E with projective spaces as fibres.

By studying the change of topology in $\mathcal{M}_\tau$ when τ crosses the critical values, one can obtain information on the topology of the moduli space of flat $U(2)$-connections.

In [BDW] the authors construct a "master space " incorporating all the $\mathcal{M}_\tau$'s . Fix a point z_0 on Σ and put $\mathcal{G}_0 = \{g \in \mathcal{G}(E) \ : \det(g(z_0)) = 1\}$. Then define

$$\mathcal{M}^{\text{Bradlow}} = \frac{\{(A,\Theta) \ : (5.2) \text{ for some } \tau\}}{\mathcal{G}_0}$$

The group $\mathbb{S}^1$ acts on $\mathcal{M}^{\text{Bradlow}}$ via gauge transformations and this action is hamiltonian with moment map

$$\begin{aligned} \mathcal{M}^{\text{Bradlow}} &\longrightarrow i\mathbb{R} \\ (A,\Theta) &\longmapsto -\frac{i}{2}\int_\Sigma |\Theta|^2 dvol_\Sigma \end{aligned}$$

Then

$$\mathcal{M}_\tau = \mathcal{M}^{\text{Bradlow}} /\!\!/ \mathbb{S}^1\big(i(2\pi d - 2\tau \text{Vol } \Sigma)\big)$$

Example 3: Anti-self-dual Yang-Mills equations. In what follows, the setting of (1.11) is enlarged to the infinite dimensional symplectic manifold of connections on a principal bundle (see also (1.3)). Let S be a compact oriented 2-dimensional riemannian manifold and $Q \xrightarrow{\pi} S$ a non-trivial $SO(3)$-bundle. For simplicity, we shall write G for $SO(3)$ and $\mathfrak{g}$ for $\mathfrak{so}(3)$. Recall from lecture 1 that the space of connections $M = \mathcal{A}(Q)$ carries a symplectic structure and that the action of the gauge group $\mathcal{G} = \mathcal{G}(Q)$ of equivariant fibre preserving diffeomorphisms is hamiltonian with moment map given by the curvature

$$\begin{aligned} \mathcal{A}(Q) &\longrightarrow \big(\Omega^2_{\text{Horiz}}(P,\mathfrak{g})\big)^G = \text{Lie}(\mathcal{G})^* \\ A &\longmapsto F_A \end{aligned}$$

The symplectic quotient at 0 was seen to be the moduli space of flat connections on Q. Note that the Hodge $*$-operator on S induces a Hodge operator on $\big(\Omega^*_{\text{Horiz}}(P,\mathfrak{g})\big)^G$ as follows : if $\alpha \in \Omega^k_{\text{Horiz}}(P,\mathfrak{g})\big)^G$, there is a unique form $\beta \in \Omega^{2-k}_{\text{Horiz}}(P,\mathfrak{g})\big)^G$ denoted by $*\alpha$ such that

$$*\langle \eta \wedge \alpha \rangle = \langle \eta \wedge \beta \rangle, \qquad \forall\, \eta \in \big(\Omega^0(P,\mathfrak{g})\big)^G$$

when seen as forms on S. For $k=1$, this gives a *complex structure* on

$$T_A\mathcal{A}(Q) = \big(\Omega^1_{\text{Horiz}}(P,\mathfrak{g})\big)^G$$

Recall from lecture 1 that the *infinitesimal action* is given by the covariant derivative

$$\big(\Omega^0(P,\mathfrak{g})\big)^G \longrightarrow \big(\Omega^1_{\text{Horiz}}(P,\mathfrak{g})\big)^G$$

$$\eta \longmapsto -d_A\eta$$

Hence, the local equations (1.8) write

$$\begin{cases} \partial_s A - d_A\Phi + *(\partial_t A - d_A\Psi) = 0 \\ \partial_s\Psi - \partial_t\Phi + [\Phi,\Psi] + \epsilon^{-2} * F_A = 0 \end{cases} \tag{5.3}$$

Here s and t are local coordinates on the Riemann surface Σ, $A + \Phi ds + \Psi dt$ is a connexion on $\mathbb{C}\times S$ realized as the local expression of a connexion on $\Sigma\times S$ while the local volume on Σ writes $\epsilon^{-2}ds\wedge dt$. These are precisely the equations of anti-self-dual instantons on $\Sigma\times S$.

When letting the volume of Σ tend to infinity (as in lecture 4) the last equation becomes

$$*F_A = 0$$

and one obtains the equations of holomorphic curves in the moduli space of flat connections. The adiabatic limit argument in the main theorem of lecture 4 establishes a correspondence between anti-self-dual instantons on $\Sigma\times S$ and holomorphic curves in the moduli space of flat connections. This was the main idea in the proof of the Atiyah-Floer conjecture [DS].

Example 4: Seiberg-Witten equations. We consider another infinite dimensional version of our construction. Let $E \xrightarrow{\pi} S$ be a degree d line bundle over a compact oriented Riemann surface. Consider the space

$$M = \big\{(A,\Theta)\in\mathcal{A}(E)\times\Gamma(S,E)\ :\ \overline{\partial}_A\Theta = 0\big\}$$

The usual symplectic structure on $\mathcal{A}(E)\times\Gamma(S,E)$ induces a symplectic structure on M. The gauge group $\mathcal{G} = \mathrm{Map}(S,\mathbb{S}^1)$ acts on M in a hamiltonian way with moment map given by

$$M \longrightarrow \Omega^0(S) = \mathrm{Lie}(\mathcal{G})$$

$$(A,\Theta) \longmapsto *F_A - \frac{i}{2}|\Theta|^2$$

The symplectic quotient $M/\!\!/\mathcal{G}(-i\tau)$ is precisely the moduli space $\mathcal{M}_d(S)$ of solutions of the vortex equations (5.1). This was identified previously with $\mathrm{Sym}^d(S)$.

On the other hand, the equations (1.8) now write

$$\begin{cases} \overline{\partial}_A \Theta = 0 \\ \partial_s \Theta + \Phi\Theta + i(\partial_t \Theta + \Psi\Theta) = 0 \\ \partial_s A - d\Phi + *(\partial_t A - d\Psi) = 0 \\ \partial_s \Psi - \partial_t \Phi + \epsilon^{-2}\big(* F_A - \frac{i}{2}|\Theta|^2 + i\tau\big) = 0 \end{cases} \tag{5.4}$$

where $s+it$ are local coordinates on Σ, $A(s,t) \in \mathcal{A}(E)$, $\Theta(s,t) \in \Gamma(S,E)$, $\Phi(s,t)$, $\Psi(s,t) \in \Omega^0(S, i\mathbb{R})$ and $dvol_\Sigma = \epsilon^{-2} ds \wedge dt$. These are precisely the Seiberg-Witten equations on the product $\Sigma \times S$ in the *integrable case* i.e. when the complex structure on S is independent of s and t.

Again, as ϵ goes to zero, the last equation becomes

$$*F_A - \frac{i}{2}|\Theta|^2 + i\tau = 0$$

and thus (5.4) describes precisely holomorphic curves in $\mathcal{M}_d(S)$ (see (1.8)). The adiabatic limit argument works again [S] and provides a correspondence between the Seiberg-Witten equations on $\Sigma \times S$ and holomorphic curves in $\mathrm{Sym}^d(S)$.

Conclusions. Let X be a 4-dimensional symplectic manifold. Donaldson proved that X admits the structure of a Lefschetz fibration over the sphere $\mathbb{S}^2$ with generic fibre S. By cutting out the singular fibres and replacing the generic ones by $\mathrm{Sym}^d(S)$ or $\mathcal{M}^{\mathrm{flat}}(S)$ (the moduli space of flat $SO(3)$-connections on S) one gets symplectic manifolds denoted by $X^{[d]}$ and X^{flat} respectively. Make the following "notations":

$SW(X)$ = Seiberg-Witten invariants of X;

$\mathcal{D}(X)$ = Donaldson invariants of X;

$\mathrm{Gr}(X^{\mathrm{flat}})$ = Gromov invariants of X^{flat};

$\mathrm{Gr}(X^{[d]})$ = Gromov invariants of $X^{[d]}$;

$\mathrm{Gr}(X)$ = Gromov invariants of X.

The diagram below expresses known (continous arrows) or conjectured (dotted arrows) relations between the quantities above.

$$\begin{tikzcd}
SW(X) \arrow[r, leftrightarrow, "{[PT]}"] \arrow[d, leftrightarrow, "\text{adiabatic limit}"'] \arrow[dd, leftrightarrow, bend right=60, "{[T]}"'] & \mathcal{D}(X) \arrow[d, leftrightarrow, "\text{Atiyah-Floer conjecture [DS]}"] \\
\mathrm{Gr}(X^{[d]}) \arrow[r, leftrightarrow, dotted] \arrow[d, leftrightarrow, dotted] & \mathrm{Gr}(X^{\mathrm{flat}}) \\
\mathrm{Gr}(X)
\end{tikzcd}$$

The link between $\mathrm{Gr}(X^{[d]})$ and $\mathrm{Gr}(X^{\mathrm{flat}})$ should go through a study related to the work of Thaddeus [Th] (cf. Example 2 above). The link between $\mathrm{Gr}(X^{[d]})$ and $\mathrm{Gr}(X)$ is the object of a current research project of Simon Donaldson and Ivan Smith. When completed, it should provide together with the adiabatic limit technique an alternative connection between $SW(X)$ and $\mathrm{Gr}(X)$, besides the classical one of Taubes [T].

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