A MORSE-BOTT APPROACH TO CONTACT HOMOLOGY

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Abstract

Contact homology was introduced by Eliashberg, Givental and Hofer. In this theory, we count holomorphic curves in the symplectization of a contact manifold, which are asymptotic to periodic Reeb orbits. These closed orbits are assumed to be nondegenerate and, in particular, isolated. This assumption makes practical computations of contact homology very difficult.

In this thesis, we develop computational methods for contact homology in Morse-Bott situations, in which closed Reeb orbits form submanifolds of the contact manifold. We require some Morse-Bott type assumptions on the contact form, a positivity property for the Maslov index, mild requirements on the Reeb flow, and $c_1(\xi) = 0$.

We then use these methods to compute contact homology for several examples, in order to illustrate their efficiency. As an application of these contact invariants, we show that T^5 and $T^2 \times S^3$ carry infinitely many pairwise non-isomorphic contact structures in the trivial formal homotopy class.

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Contents

Chapter 1

Introduction

1.1 Contact geometry

The main object of contact geometry is the study of contact structures on differentiable manifolds. These naturally arise in different branches of physics, such as classical mechanics and geometric optics.

Definition 1.1. *A 1-form α on a $(2n-1)$-dimensional manifold M is called contact if $\alpha \wedge (d\alpha)^{n-1}$ is a volume form on M.*

Equivalently, the 2-form $d\alpha$ defines a symplectic form on the hyperplane distribution $\xi = \ker \alpha$.

Definition 1.2. *A hyperplane distribution $\xi \subset TM$ is called a contact structure if it is defined locally by a contact form. We say that (M, ξ) is a contact manifold.*

Note that, if the hyperplane distribution is co-orientable, then it admits a global defining 1-form α. In what follows, we will always assume that ξ is co-orientable.
Any other choice of a contact form for ξ is given by $f\alpha$, where f is a nonvanishing function on M. The conformal class of the symplectic form $d\alpha$ on ξ is independent of f, since $d(f\alpha)$ restricts to $fd\alpha$ on ξ.

Contact manifolds, like symplectic manifolds, have no local invariants. This is a consequence of Darboux' theorem.

Theorem 1.3. *Let (M, ξ) be a contact manifold with contact form α. For every $p \in M$, there exist an open neighborhood U of p and a diffeomorphism $\varphi : U \to \mathbb{R}^{2n-1}$ with coordinates $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}, \vartheta$ such that $\varphi^*\alpha = d\vartheta + x \cdot dy = d\vartheta + \sum_{i=1}^{n-1} x_i dy_i$.*

On the other hand, contact structures are stable under isotopy, as shown by Gray's theorem.

Theorem 1.4. *Let $\{\xi_t\}_{t\in[0,1]}$ be a smooth family of contact structures on a closed manifold M. Then there exists a smooth path $\{\varphi_t\}_{t\in[0,1]}$ of diffeomorphisms of M such that $\varphi_0 = Id$ and $\varphi_t^*\xi_t = \xi_0$ for all $t \in [0, 1]$.*

This is a fundamental result in contact topology, since it shows that there is a hope of classifying contact structures on a given closed manifold M. When $\dim M = 3$, many such classification results have been obtained already (see for example Eliashberg [2] [3], Giroux [9] [11], Honda [17] [18]) using purely 3-dimensional techniques. The starting point of this intense research is the result of Martinet [23] that every closed orientable 3-manifold admits a contact structure.

When $\dim M > 3$, those techniques cannot be used and the only general way that was available until the mid 90s in order to distinguish contact structures was the computation of their formal homotopy class.

Definition 1.5. *An almost contact structure is the data of a hyperplane distribution ξ equipped with a complex structure J. Equivalently, it is a reduction of the structure group of TM to $U(n-1)$.*

A contact structure ξ defines an almost contact structure, since the set of complex structures J on ξ that are compatible with $d\alpha$ is contractible and independent of the choice of α.

Definition 1.6. *The formal homotopy class of a contact structure ξ is the homotopy class of the corresponding almost contact structure.*

But this "classical invariant" can sometimes be very difficult to compute. Moreover, several examples of distinct contact structures in the same formal homotopy class are known.

1.2 Contact homology

Contact homology [4] was introduced in the mid 90s by Eliashberg and Hofer, and it was soon followed by Symplectic Field Theory [5]. These theories are based on the introduction of pseudo-holomorphic curves in symplectic geometry by Gromov [12]. We count pseudo-holomorphic curves in the symplectization $(\mathbb{R}\times M, d(e^t\alpha))$ of (M,α) that converge, for $t \to \pm\infty$, to closed orbits of the Reeb vector field R_α associated to α by the conditions $\imath(R_\alpha)d\alpha = 0$ and $\alpha(R_\alpha) = 1$. Note that the dynamic properties of R_α strongly depend on the choice of a contact form α for ξ. We construct a chain complex generated by the closed Reeb orbits and whose differential counts the pseudo-holomorphic curves mentioned above. Contact homology is the homology of that chain complex and turns out to be a contact invariant, i.e. is independent of the choice of α.

The usefulness of contact homology has already been demonstrated by several computations for certain contact manifolds. Unfortunately, these computations are limited and uneasy, because of an important assumption in the theory : the closed Reeb orbits must be nondegenerate (and, in particular, isolated). Therefore, when the contact manifold admits a natural and very symmetric contact form, this contact form has to be perturbed before starting the computation. As a consequence of this, the Reeb flow becomes rather chaotic and hard to study. But the worst part comes from the Cauchy-Riemann equation, which becomes perturbed as well. It is then nearly impossible to compute the moduli spaces of holomorphic curves. To avoid these difficulties, one would like to extend the theory to a larger set of admissible contact forms. That is the goal of this thesis.

Contact homology can be thought of as some sort of Morse theory for the action functional for loops γ in M : $\mathcal{A}(\gamma) = \int_\gamma \alpha$. The critical points of $\mathcal{A}$ are the closed orbits under the Reeb flow φ_t and the corresponding critical values are the periods of these orbits. The set of critical values of $\mathcal{A}$ is called the action spectrum and will be denoted by $\sigma(\alpha)$. If the contact form is very symmetric, the closed Reeb orbits will not be isolated, so we have to think of $\mathcal{A}$ as a Morse-Bott functional. These considerations motivate the following definition.

Definition 1.7. *A contact form α on M is said to be of Morse-Bott type if the action spectrum $\sigma(\alpha)$ of α is discrete and if, for every $T \in \sigma(\alpha)$, $N_T = \{p \in M \mid \varphi_T(p) = p\}$ is a closed, smooth submanifold of M, such that rank $d\alpha|_{N_T}$ is locally constant and $T_pN_T = \ker(\varphi_{T*} - I)_p$.*

The last condition is exactly the Morse-Bott condition for the action functional $\mathcal{A}$, since the linearized Reeb flow corresponds to the Hessian of $\mathcal{A}$.
We define S_T to be the quotient of N_T under the circle action induced by the Reeb flow.

1.3 Main theorems

We will construct a Morse-Bott chain complex (C_*, d) involving the Reeb dynamics and holomorphic curves for a contact form α of Morse-Bott type.

Theorem 1.8. *Let α be a contact form of Morse-Bott type for a contact structure ξ on M satisfying $c_1(\xi) = 0$.
Assume that, for all $T \in \sigma(\alpha)$, N_T and S_T are orientable, $\pi_1(S_T)$ has no disorienting loop, and all Reeb orbits in S_T are good. Assume that the almost complex structure J is invariant under the Reeb flow on all submanifolds N_T. Assume that the orbit spaces S_T have index positivity and that the Reeb field R_α has bounded return time.
Then the homology $H_*(C_*, d)$ of the Morse-Bott chain complex (C_*, d) of (M, α) is isomorphic to the contact homology $HC_*(M, \xi)$ of $(M, \xi = \ker \alpha)$ with coefficients in the Novikov ring of $H_2(M, \mathbb{Z})/\mathcal{R}$.*

It is sometimes better to consider instead cylindrical homology, for which we count cylindrical curves only. In that case, we construct a Morse-Bott chain complex $(C_*^{\bar{a}}, d)$ for each homotopy class $\bar{a}$ of periodic Reeb orbits.

Theorem 1.9. *Let α be a contact form of Morse-Bott type for a contact structure ξ on M satisfying $c_1(\xi) = 0$.
Assume that, for all $T \in \sigma(\alpha)$, N_T and S_T are orientable, $\pi_1(S_T)$ has no disorienting*

loop, and all Reeb orbits in S_T are good. Assume that the almost complex structure J is invariant under the Reeb flow on all submanifolds N_T. Assume that cylindrical homology is well defined : $C_k^0 = 0$ for $k = -1, 0, +1$, that all orbit spaces S_T of contractible periodic orbits have index positivity, and that the Reeb field R_α has bounded return time.
Then the homology $H_*(C_*^{\bar{a}}, d)$ of the Morse-Bott chain complex $(C_*^{\bar{a}}, d)$ of (M, α) is isomorphic to the cylindrical homology $HF_*^{\bar{a}}(M, \xi)$ of $(M, \xi = \ker \alpha)$ with coefficients in the Novikov ring of $H_2(M, \mathbb{Z})/\mathcal{R}$.

The assumptions and the notation in these statements will be explained later in this thesis. The strategy of the proof is to rewrite the chain complex for contact homology using the Morse-Bott data. Therefore, the homology of the Morse-Bott chain complex is isomorphic to contact homology, and it is a contact invariant as a consequence of its original definition [5].
We then apply these Morse-Bott techniques to compute contact homology for certain families of contact structures. As a result, we obtain some information on the contact topology of some manifolds.

Corollary 1.10. *There are infinitely many pairwise non-isomorphic contact structures on T^5 and on $T^2 \times S^3$ in the trivial formal homotopy class.*

This thesis is organized as follows.
In chapter 2, we introduce holomorphic curves in the symplectization of a contact manifold, with a contact form of Morse-Bott type. We then explain how to perturb this setup to obtain non-degenerate closed Reeb orbits.
In chapter 3, we study the asymptotic behavior of holomorphic curves with finite Hofer energy, in the neighborhood of a puncture. We show that the map converges exponentially fast to a closed Reeb orbit within an orbit space.
In chapter 4, we prove a compactness theorem for holomorphic curves in a symplectization. We also generalize this result to the case of a contact form of Morse-Bott type with a vanishing perturbation.
In chapter 5, we derive the Fredholm theory for holomorphic curves in a symplectization. We also work out estimates for gluing holomorphic curves and fragments of

gradient trajectories in the orbit spaces.

In chapter 6, we explain how to achieve transversality for the moduli spaces of (generalized) holomorphic curves, using virtual cycle techniques.

In chapter 7, we generalize the construction of coherent orientations for the moduli spaces of holomorphic curves with a contact form of Morse-Bott type.

In chapter 8, we use the preceding results to construct the moduli spaces of holomorphic curves in different setups and prove the main theorems.

In chapter 9, we use the main theorems to compute contact homology in various classes of examples, illustrating the efficiency of these Morse-Bott techniques.

In chapter 10, we apply theorem 1.9 to the study of certain families of contact structures and deduce some information about the contact topology of certain manifolds.

Chapter 2

Morse-Bott setup

2.1 Holomorphic curves with degenerate asymptotics

Let (M, α) be a compact, $(2n-1)$-dimensional contact manifold. We denote the Reeb vector field associated to α by R_α. We are interested in the periodic orbits γ of R_α, i.e. curves $\gamma : [0,T] \to M$ such that $\frac{d\gamma}{dt} = R_\alpha$ and $\gamma(0) = \gamma(T)$. The period T of γ is also called action and can be computed using the action functional $\int_\gamma \alpha$.

If α is not a generic contact form for $\xi = \ker \alpha$ but has some symmetries, then the closed Reeb orbits are not isolated but come in families. Let $N_T = \{p \in M \mid \varphi_T(p) = p\}$, where φ_t is the flow of R_α. We assume that α is of Morse-Bott type in the sense of definition 1.7, so that N_T is a smooth submanifold of M. The Reeb flow on M induces an S^1 action on N_T. Denote the quotient N_T/S^1 by S_T; this is an orbifold with singularity groups $\mathbb{Z}_k$. The singularities correspond to Reeb orbits with period T/k, covered k times. Since $\sigma(\alpha)$ is discrete, there will be countably many such orbit spaces S_T. We will denote by S_i the connected components of the orbit spaces $(i = 1, 2, \dots)$.

The contact distribution ξ is equipped with a symplectic form $d\alpha$. Let $\mathcal{J}$ be the set of complex structures on ξ, compatible with $d\alpha$. This set is nonempty and contractible.

Note that $\mathcal{J}$ is independent of the choice of contact form α for ξ (for a given coorientation of ξ), because the conformal class of $d\alpha$ is fixed. Let $\tilde{J} \in \mathcal{J}$; we can extend $\tilde{J}$ to an almost complex structure J on the symplectization $(\mathbb{R} \times M, d(e^t\alpha))$, where t denotes the coordinate of $\mathbb{R}$, by $J|_\xi = \tilde{J}$ and $J\frac{\partial}{\partial t} = R_\alpha$. Note that if we replace α with $f\alpha$, where f is a positive function on M, we can keep the same $\tilde{J}$, but the extension to $\mathbb{R} \times M$ is modified.

Let (Σ, j) be a compact Riemann surface, let $x_1^0, \dots, x_{s^0}^0$ be marked points on Σ, and let $x_1^+, \dots, x_{s^+}^+, x_1^-, \dots, x_{s^-}^- \in \Sigma$ be a set of punctures. We are interested in J-holomorphic curves

$$\tilde{u} = (a, u) : (\Sigma \setminus \{x_1^+, \dots, x_{s^+}^+, x_1^0, \dots, x_{s^0}^0, x_1^-, \dots, x_{s^-}^-\}, j) \to (\mathbb{R} \times M, J)$$

which have the following behavior near the punctures : $\lim_{p \to x_i^\pm} a(p) = \pm\infty$ and the map u converges, near a puncture, to a closed Reeb orbit. We say that x_i^+ $(i = 1, \dots, s^+)$ are positive punctures and x_j^- $(j = 1, \dots, s^-)$ are negative punctures. We will show in chapter 3 that such an asymptotic behavior for the J-holomorphic maps is guaranteed if the Hofer energy is finite.

Definition 2.1. *Let $\mathcal{C} = \{\phi \in C^0(\mathbb{R}, [0,1]) \mid \phi' \geq 0\}$; then the Hofer energy is defined by*

$$E(\tilde{u}) = \sup_{\phi \in \mathcal{C}} \int_\Sigma \tilde{u}^* d(\phi\alpha).$$

We now show that punctures at finite distance can be eliminated if the Hofer energy is finite.

Lemma 2.2. *Let $\tilde{u} : (D^2 \setminus \{0\}, j) \to (\mathbb{R} \times M, J)$ be a holomorphic map with $E(\tilde{u}) < \infty$. Suppose that a is bounded in a neighborhood of the origin. Then $\tilde{u}$ extends continuously to a holomorphic map $\tilde{v} : (D^2, j) \to (\mathbb{R} \times M, J)$ with $E(\tilde{v}) = E(\tilde{u}) < \infty$.*

Proof. Let K be a compact subset of $\mathbb{R} \times M$ containing $\tilde{u}(D^2 \setminus \{0\})$. Choose $\phi \in \mathcal{C}$ such that $\phi'(t) > 0$ for all $(t, x) \in K$. Then, $d(\phi\alpha)$ is nondegenerate on K and the energy of $\tilde{u}$ is its area with respect to the symplectic form $d(\phi\alpha)$. Hence, we can apply the usual removable singularity lemma for a compact symplectic manifold [28]. $\square$

As a corollary of this lemma, we can always assume that the domain of a holomorphic map in $\mathbb{R} \times M$ with finite energy has no punctures that are mapped at finite distance.

We want to associate a homology class to a holomorphic map. In order to do this, we need to fix some additional data. Choose a base point in each orbit space S_T and, for the corresponding Reeb orbit, choose a capping disk in M (if the Reeb orbit is not contractible, we choose instead a representative for its homotopy class, and a homotopy from the orbit to the representative). Then, given a holomorphic map with asymptotic Reeb orbits $\gamma_1^+, \ldots, \gamma_{s^+}^+, \gamma_1^-, \ldots, \gamma_{s^-}^-$, we join each asymptotic Reeb orbit $\gamma_i^\pm$ to the base point of the corresponding orbit space. Gluing the holomorphic curve, the cylinders lying above the paths and the capping disks, we obtain a homology class in $H_2(M, \mathbb{Z})$.
However, the result depends on the homotopy class of the chosen path in S_T. Clearly, the homology class is well-defined modulo $\mathcal{R} = \text{Image } (i_T \circ \pi_T^{-1} : H_1(S_T, \mathbb{Z}) \to H_2(M, \mathbb{Z}))$, where $i_T : N_T \to M$ is the embedding of N_T into M and $\pi_T : N_T \to S_T$ is the quotient under the Reeb flow. The elements of $\mathcal{R}$ are analogous to the rim tori of Ionel and Parker [19]. Note that $c_1(\xi)$ vanishes on $\mathcal{R}$, because ξ restricted to a torus lying above a loop in S_T is the pullback of a vector bundle over that loop. Hence, the quotient of the Novikov ring of $H_2(M, \mathbb{Z})$ by $\mathcal{R}$ is well-defined and we can choose to work with these somewhat less precise coefficients.
Note that it would be possible to recover more information on the homology class, using a topological construction as in [19], but this would be very impractical for computations. Therefore, we prefer to content ourselves with $H_2(M, \mathbb{Z})/\mathcal{R}$.

The moduli spaces of such J-holomorphic curves are defined under the following equivalence relation :

$$\begin{aligned}&(\Sigma \setminus \{x_1^+, \ldots, x_{s^+}^+, x_1^0, \ldots, x_{s^0}^0, x_1^-, \ldots, x_{s^-}^-\}, j, \tilde{u})\\ &\qquad \sim (\Sigma' \setminus \{x_1'^+, \ldots, x_{s^+}'^+, x_1'^0, \ldots, x_{s^0}'^0, x_1'^-, \ldots, x_{s^-}'^-\}, j', \tilde{u'})\end{aligned}$$

if there exists a biholomorphism

$$h : (\Sigma \setminus \{x_1^+, \dots, x_{s^+}^+, x_1^-, \dots, x_{s^-}^-\}, j) \to (\Sigma' \setminus \{x'^+_1, \dots, x'^+_{s^+}, x'^-_1, \dots, x'^-_{s^-}\}, j')$$

such that $h(x_i^\pm) = x'^\pm_i$ for $i = 1, \dots, s^\pm$, $h(x_i^0) = x'^0_i$ for $i = 1, \dots, s^0$ and $\tilde{u} = \tilde{u'} \circ h$.

We will denote the moduli space of J-holomorphic maps of genus g, of homology class $A \in H_2(M)/\mathcal{R}$, with s^0 marked points, s^+ positive punctures and asymptotic Reeb orbits in $S_1^+, \dots, S_{S^+}^+$, with s^- negative punctures and asymptotic Reeb orbits in $S_1^-, \dots, S_{s^-}^-$ by

$$\mathcal{M}^A_{g,s^+,s^-,s^0}(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-).$$

In addition to the usual evaluation maps $ev_i^0 : \mathcal{M} \to M$ $(i = 1, \dots, s^0)$ at the marked points, this moduli space $\mathcal{M}$ will be equipped with evaluation maps $ev_i^+ : \mathcal{M} \to S_i^+$ $(i = 1, \dots, s^+)$ and $ev_j^- : \mathcal{M} \to S_j^-$ $(j = 1, \dots, s^-)$ at the positive and negative punctures.

Let $\theta_1, \dots, \theta_m$ be a set of representatives for a basis of $H_*(M)$. We introduce variables $t_1, \dots, t_m$ associated to these cycles, with grading $|t_i| = \dim \theta_i - 2$. We formally define $\bar{\theta} = \sum_{i=1}^m t_i \theta_i$ and express every possible condition at the marked points using the fibered product

$$\mathcal{M} \times_M \bar{\theta} \dots \times_M \bar{\theta}$$

that is defined using the evaluation maps ev_i^0, $i = 1, \dots, s^0$.

In order to construct contact homology, we just consider moduli spaces with genus $g = 0$ and one positive puncture : $s^+ = 1$. However, we will construct these moduli spaces in full generality, since that does not really require more work.

2.2 Perturbation of the contact form

Let us construct a function $\bar{f}_T$ with support in a small neighborhood of $\cup_{T' \leq T} N_{T'}$ and such that $d\bar{f}_T(R_\alpha) = 0$ on $N_{T'}$. In particular, $\bar{f}_T$ will descend to a differentiable function f_T on the orbifold S_T. We will choose $\bar{f}_T$ in such a way that it induces a

Morse function f_T on S_T.

We proceed by induction on T. For the smallest $T \in \sigma(\alpha)$, the orbit space S_T is a smooth manifold. Pick any Morse function f_T on it.
For larger values of $T \in \sigma(\alpha)$, S_T will be an orbifold having as singularities the orbit spaces $S_{T'}$ such that T' divides T. We extend the functions $f_{T'}$ to a function f_T on S_T, so that the Hessian of f_T restricted to the normal bundle to $S_{T'}$ is positive definite.
Finally, we extend $\bar{f}_T$ to a tubular neighborhood of N_T so that it is constant on the fibers of the normal bundle of N_T (for some metric invariant under the Reeb flow). We then use cut off depending on the distance from N_T. We can choose the radial size of the tubular neighborhood of N_T to be very small, so that R_α is C^1-close to its value on N_T in the tubular neighborhood.

Consider the perturbed contact form $\alpha_\lambda = (1 + \lambda \bar{f}_T)\alpha$, where λ is a small positive constant.

Lemma 2.3. *For all T, we can choose $\lambda > 0$ small enough so that the periodic orbits of R_{α_λ} in M of action $T' \leq T$ are nondegenerate and correspond to the critical points of $f_{T'}$.*

Proof. The new Reeb vector field $R_{\alpha_\lambda} = R_\alpha + X$ where X is characterized by

$$\begin{aligned} i(X)d\alpha &= \lambda \frac{d\bar{f}_T}{(1+\lambda \bar{f}_T)^2} \qquad \text{on } \xi, \\ \alpha(X) &= -\lambda \frac{\bar{f}_T}{1+\lambda \bar{f}_T}. \end{aligned}$$

The first equation describes the transversal deformations of the Reeb orbits. These vanish when $df_T = 0$, that is at critical points of f_T. On the other hand, if λ is small enough, the perturbation cannot create new periodic orbits, for a fixed action range, because we have an upper bound on the deformation of the flow for the corresponding range of time. The surviving periodic orbits are nondegenerate, because the Hessian at a critical point is nondegenerate. This corresponds to first order variations of X, that is of the linearized Reeb flow. ∎

Let $p \in S_{T'}$ be a simple Reeb orbit that is a critical point of $f_{T'}$. Then we will denote the closed orbit corresponding to $p \in S_{kT'}$ by $\gamma^p_{kT'}$ $(k = 1, 2, \ldots)$.

We can compute the Conley-Zehnder index of these closed Reeb orbits for a small value of λ.

Lemma 2.4. *If λ is as in lemma 2.3 and $kT' \leq T$, then*

$$\mu_{CZ}(\gamma^p_{kT'}) = \mu(S_{kT'}) - \frac{1}{2}\dim S_{kT'} + \mathrm{index}_p(f_{kT'}).$$

Proof. Let H be the Hessian of f_T at critical point p. Then, the ξ-component of X is given by $-\lambda JHx$, where x is a local coordinate in a uniformization chart near p. The linearized Reeb flow now has a new crossing at $t = 0$, with crossing form $-\lambda H$. Its signature is $\sigma(0) = \mathrm{index}_p(f_{kT}) - (\dim S_{kT} - \mathrm{index}_p(f_{kT}))$. Half of this must be added to $\mu(S_{kT})$ to obtain the Conley-Zehnder index of the nondegenerate orbit. □

On the other hand, we have to make sure that all Reeb orbits with action greater than T do not interfere with the closed orbits characterized above. For this, we need to make an assumption on the behavior of the Reeb field R_α.

Definition 2.5. *We say that the Reeb field R_α has bounded return time if there exists $\Delta T < \infty$ such that, for every Reeb trajectory leaving a small tubular neighborhood U_T of N_T at p, we have $\varphi_t(p) \in U_T$ for some $0 < t < \Delta T$.*

This assumption is automatically satisfied in a number of cases, for example if $N_T = M$ for some $T \in \sigma(\alpha)$, or in the more general examples discussed in chapters 9 and 10. We will also need the following definition adapted from [34].

Definition 2.6. *We say that the orbit spaces S_T have index positivity if there exist constants $c > 0, c'$ such that $\mu(S_T) > cT + c'$ for all $T \in \sigma(\alpha)$.*

Then, we have the following result.

Lemma 2.7. *Assume that the orbit spaces S_T have index positivity, that $c_1(\xi) = 0$ and that the Reeb field has bounded return time. Then there exists $\lambda_0 > 0$ such that, if $0 < \lambda < \lambda_0$, all period orbits γ_λ of R_{α_λ} of action greater than T satisfy $\mu_{CZ}(\gamma_\lambda) > \frac{c}{2}T$.*

In order to prove this lemma, we will need the following result from [34].

Lemma 2.8. (Catenation Lemma) *Let Ψ_1, Ψ_2 be paths of symplectic matrices satisfying $\Psi_i(0) = I$. Then*

$$|\mu(\Psi_1 * \Psi_2) - \mu(\Psi_1) - \mu(\Psi_2)| \leq 2n.$$

The catenation $\Psi_1 * \Psi_2$ of the paths $\Psi_1, \Psi_2 : [0,1] \to Sp(2n)$ is defined as follows :

$$\Psi_1 * \Psi_2(t) = \begin{cases} \Psi_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \Psi_1(1)\Psi_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Proof of lemma 2.7. First choose a trivialization of TM along the closed orbit γ_λ of R_{α_λ}. If γ_λ is contractible, just take the trivialization induced by any capping disk. The Conley-Zehnder index is independent of the choice of the capping disk since $c_1(\xi) = 0$. If γ_λ is not contractible, choose a homotopy to a standard representative of its homotopy class, with a prescribed trivialization. Again, the index will not depend on the choice of the homotopy.

Note that all closed orbits of R_{α_λ} must intersect the neighborhood of N_T where α is perturbed. By the bounded return time assumption, it is clear that closed Reeb orbits spend most of their time in a small neighborhood U_T of N_T, when $\lambda > 0$ is sufficiently small. Because of this, we can just concentrate on the contributions to μ_{CZ} due to the portions of trajectories in U_T. Indeed, for λ sufficiently small, and for $\delta > 0$ independent of λ, every fragment γ^i_λ of γ_λ in U_T with length $L = \frac{\delta}{\lambda}$ is C^1-close to a multiple γ^i of a non-perturbed orbit in S_T. In particular, we have $|\mu_{CZ}(\gamma^i_\lambda) - \mu_{CZ}(\gamma^i)| \leq 2n$. But by assumption, $\mu_{CZ}(\gamma^i) > cL + c'$. Hence, $\mu_{CZ}(\gamma^i_\lambda) > cL + c' - 2n$.

Next, we catenate the fragments γ^i_λ to rebuild the closed orbit γ_λ. By a repeated use of the catenation lemma, we obtain $\mu_{CZ}(\gamma_\lambda) > (cL + c' - 4n)\frac{T}{L} = cT + (c' - 4n)\lambda\frac{T}{\delta}$. Hence, if λ is sufficiently small, we can make sure that $\mu_{CZ}(\gamma_\lambda) > \frac{c}{2}T$. □

2.3 Morse theory on the orbit spaces

In this section, we indicate how to generalize the results of Morse theory to certain orbifolds, in the specific case of the orbit spaces S_T with the functions f_T.

Definition 2.9. *A Morse function f_T on the orbit space S_T is called admissible if, for every critical point γ of f_T with minimal period T/k, the unstable manifold of γ is contained in $S_{T/k}$.*

By construction, the Morse functions f_T introduced in the last section are admissible. This assumption is enough to extend Morse theory to our orbifolds.

Proposition 2.10. *If the Morse function f_T is admissible on the orbit space S_T, then the Morse-Witten complex of f_T is well-defined and its homology is isomorphic to the singular homology of S_T.*

Proof. Let $\gamma \in S_T$ be a critical point of f_T with minimal period T/k. Pick a uniformization chart U for S_T, centered at γ. Since f_T is admissible, the unstable sphere of γ is fixed by the group acting on U. Hence, its image in S_T is topologically a sphere, and $W^u(\gamma)$ is a smooth disk embedded in S_T.
Moreover, if δ is another critical point of f_T with minimal period dividing T/k, then its stable manifold intersects $W^u(\gamma)$ at smooth points only. Therefore, the moduli spaces of gradient flow trajectories can be defined as in the smooth case. The stable and unstable manifolds will intersect transversely after a small perturbation of f_T in the smooth part of S_T/k.
Next, we want to show that $d^2 = 0$ and homology is isomorphic to $H_*(S_T)$. But we have already seen that the unstable manifolds provide a cell decomposition of S_T, so that the Morse-Witten complex of f_T coincide with the complex of that cell decomposition. $\square$

Note that the assumption that the unstable manifolds are fixed by the uniformizing groups is essential. It is easy to construct an example where neither the unstable nor the stable manifolds of a critical point are fixed, and $d^2 \neq 0$.

Chapter 3

Asymptotic behavior

In this chapter, we generalize to any dimension $2n-1$ for M the estimates of Hofer, Wysocki and Zehnder [16] for the asymptotic behavior of holomorphic maps in a symplectization when $\dim M = 3$.

3.1 Local coordinates

In order to study the asymptotic behavior of J-holomorphic curves near a submanifold N_T of closed Reeb orbits, we need to use appropriate coordinate charts.

Lemma 3.1. *Let (M, α) be a $(2n-1)$-dimensional manifold with contact form α of Morse-Bott type. Let γ be a closed Reeb orbit in $N_i \subset N_T$, with minimal period T/m. Then, there exists a tubular neighborhood V of $P = \gamma(\mathbb{R})$ and a neighborhood $U \subset S^1 \times \mathbb{R}^{2n-2}$ of $S^1 \times \{0\}$ and a covering map $\phi : U \to V$ such that*

(i) $V \cap N_i$ is invariant under Reeb flow,

(ii) $\phi|_{S^1\times\{0\}}$ covers P exactly m times,

(iii) the preimage under ϕ of any periodic orbit γ' in $V \cap N_i$ consists of one or more $S^1 \times \{p\}$, where $p \in \mathbb{R}^k \subset \mathbb{R}^{2n-2}$,

(iv) $\phi^*\alpha = f\alpha_0$ where $f|_{\mathbb{R}^k} = T$ and $df|_{\mathbb{R}^k} = 0$.

Proof. Let $\pi : \tilde{V} \to V$ be a m-fold covering of V so that all Reeb trajectories in $\tilde{V}$ are closed with period T. Let $\tilde{N}$ be the preimage of N_T and $\tilde{\gamma}$ be a lift of γ. Let Δ_1 be the distribution in N corresponding to the kernel of $d\alpha$ on $\xi \cap TN_T$. It is easy to check that this distribution is integrable. Δ_1 is invariant under the Reeb flow, so we can find local coordinates in $\tilde{N}$ so that the first coordinate, $\vartheta \in \mathbb{R}/\mathbb{Z}$, is a coordinate along Reeb orbits, and the integral leaves of Δ_1 are flat coordinate planes. Note that the supplementary coordinate planes to the integral leaves are contact manifolds with 1-form α, since the kernel of $d\alpha$ is spanned by just R_α. Therefore, we can apply a parametric version of Lutz-Martinet theorem and obtain coordinates in $\tilde{N}$ so that α is standard. Finally, we have to extend these coordinates to $\tilde{V}$. Choose vector fields spanning the normal bundle to $\tilde{N}$ so that α and $d\alpha$ are standard in $T\tilde{V}$ along $\tilde{N}$. Then we obtain coordinates in $\tilde{V}$ by exponentiating these vector fields using some metric. Using some Moser-type argument, we change the coordinates away from $\tilde{N}$ so that $\alpha = f\alpha_0$. By construction, $f = T$ and $df = 0$ on $\tilde{N}$. □

In the tubular neighborhood $U \subset S^1 \times \mathbb{R}^{2n-2}$, the coordinate for the S^1 factor will be denoted by ϑ and the coordinates for the $\mathbb{R}^{2n-2}$ factor will be denoted by $z = (x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1})$. In these coordinates, $\alpha_0 = d\vartheta + x \cdot dy$ and $\phi^{-1}(V \cap N_i)$ is a linear subspace generated by ϑ and $k-1$ components of z. We will denote by

$$z = \begin{bmatrix} z_{in} \\ z_{out} \end{bmatrix}$$

the splitting between the coordinates z_{in} tangential to N_i and the rest z_{out} of the components of z.

Fix $\sigma > 0$. Let $I_l = [a_l, b_l] \subset \mathbb{R}$ be a sequence of intervals, and let $\tilde{u}_l = (a_l, u_l) : S^1 \times I_l \to \mathbb{R} \times M$ be a solution of the Cauchy-Riemann equations with finite energy, such that $u_l(s,t) \in S^1 \times [-\sigma, +\sigma]^{2n-2}$ for all $s \in I_l$, $t \in S^1$.

Assume that, for every multi-index α,

$$\begin{aligned}
\sup_{(s,t)\in I_l\times S^1} |\partial^\alpha z_{out}(s,t)| &\to 0,\\
\sup_{(s,t)\in I_l\times S^1} |\partial^\alpha (z_{in}(s,t) - z_{in}(s,0))| &\to 0,\\
\sup_{(s,t)\in I_l\times S^1} |\partial^\alpha (\vartheta(s,t) - \vartheta(s,0) - t)| &\to 0,
\end{aligned}$$

and for every multi-index α with $|\alpha| \geq 1$,

$$\begin{aligned}
\sup_{(s,t)\in I_l\times S^1} |\partial^\alpha z_{in}(s,t)| &\to 0,\\
\sup_{(s,t)\in I_l\times S^1} |\partial^\alpha (\vartheta(s,t) - t)| &\to 0,
\end{aligned}$$

when $l \to \infty$.

In the local coordinates of lemma 3.1, the Cauchy-Riemann equations read :

$$\begin{aligned}
z_s + Mz_t + (a_t I - a_s M)\tilde{S} z_{out} &= 0,\\
a_s - (\vartheta_t + x \cdot y_t) f &= 0,\\
a_t + (\vartheta_s + x \cdot y_s) f &= 0,
\end{aligned}$$

where M is the matrix of the almost complex structure on ξ with respect to basis $(\partial_{x_1}, \dots \partial_{x_{n-1}}, -x_1\partial_\vartheta + \partial_{y_1}, \dots - x_{n-1}\partial_\vartheta + \partial_{y_{n-1}})$, and $\tilde{S}$ is a $(2n-2)\times(2n-k-1)$ matrix given by

$$\tilde{S} = -\frac{1}{T^2}\begin{bmatrix} 0 \\ \nabla^2_{N_i^\perp} f \end{bmatrix}.$$

Define $A(s) : L^2_1(S^1, \mathbb{R}^{2n-2}) \subset L^2(S^1, \mathbb{R}^{2n-2}) \to L^2(S^1, \mathbb{R}^{2n-2})$ by

$$\begin{aligned}
A(s)\gamma &= -M\gamma_t - (a_t I - a_s M)\tilde{S}\gamma_{out}\\
&= -M\gamma_t - S\gamma_{out}.
\end{aligned}$$

When $\vartheta = \vartheta(a_l, 0) + t$, $a = Ts$, $z_{out} = 0$ and $z_{in} = z_{in}(a_l, 0)$, we will denote this operator by A_l. These operators are self-adjoint with respect to the inner product

$$\begin{aligned} \langle u, v \rangle_l &= \int_0^1 d\alpha_0(u, Mv)dt \\ &= \int_0^1 \langle u, -J_0 M_l v \rangle dt \end{aligned}$$

where J_0 is the standard complex structure on $\mathbb{R}^{2n-2}$ and $\langle \cdot, \cdot \rangle = d\alpha_0(\cdot, J_0 \cdot)$. The kernel of A_l is independent of l and is generated by constant loops with values in the tangent space to N_i. Let P_l be the orthonormal projection to $\ker A_l$ with respect to $\langle \cdot, \cdot \rangle_l$, and $Q_l = I - P_l$. The operator Q_l clearly has the following properties : $(Q_l z)_t = z_t$, $(Q_l z)_s = Q_l z_s$, $(Q_l z)_{out} = z_{out}$ and $Q_l A_l = A_l Q_l$.

3.2 Convergence to a Reeb orbit

We will first obtain some estimates for the decaying rate of z_{out}. For $s \in I_l$, let $g_l(s) = \frac{1}{2}\|Q_l z(s)\|_l^2$ and let $\beta_l(s) = (\vartheta(a_l, 0) - \vartheta(s, 0), z_{in}(a_l, 0) - z_{in}(s, 0))$.

Lemma 3.2. *There exist $l_0 > 0$ and $\bar{\beta} > 0$ such that for $l \geq l_0$ and $s \in I_l$ satisfying $|\beta_l(s)| \leq \bar{\beta}$, we have*

$$g_l''(s) \geq c_1^2 g_l(s)$$

where $c_1 > 0$ is a constant independent of l.

Proof. We have

$$g_l''(s) \geq \langle Q_l z_{ss}, Q_l z \rangle_l.$$

Let us compute the right hand side. First

$$\begin{aligned} Q_l z_s &= Q_l A(s) z(s) \\ &= Q_l A_l z + Q_l [A(s) - A_l] z \\ &= Q_l A_l z + Q_l [\Delta_l z_t + \hat{\Delta}_l z_{out} + (\tilde{\Delta}_l \beta_l) z_t + (\bar{\Delta}_l \beta_l) z_{out}] \\ &= A_l Q_l z + Q_l \Delta_l (Q_l z)_t + Q_l \hat{\Delta}_l (Q_l z)_{out} + Q_l (\tilde{\Delta}_l \beta_l)(Q_l z)_t + Q_l (\bar{\Delta}_l \beta_l)(Q_l z)_{out}, \end{aligned}$$

where

$$\begin{aligned}
\Delta_l &= M(z_{in}(s,0),0,\vartheta(s,0)+t)-M,\\
\hat{\Delta}_l &= S(z_{in}(s,0),0,\vartheta(s,0)+t)-S,\\
\tilde{\Delta}_l\beta_l &= M_l-M(z_{in}(s,0),0,\vartheta(s,0)+t),\\
\bar{\Delta}_l\beta_l &= S_l-S(z_{in}(s,0),0,\vartheta(s,0)+t).
\end{aligned}$$

The expressions Δ_l and $\hat{\Delta}_l$ contain the dependence in z_{out} so that

$$\begin{aligned}
\sup_{(s,t)\in\theta_l}|\partial^\alpha\Delta_l(s,t)| &\to 0,\\
\sup_{(s,t)\in\theta_l}|\partial^\alpha\hat{\Delta}_l(s,t)| &\to 0.
\end{aligned}$$

On the other hand, the expressions $\tilde{\Delta}_l$ and $\bar{\Delta}_l$ contain the dependence in z_{in}, therefore

$$\begin{aligned}
\sup_{(s,t)\in\theta_l}|\partial^\alpha\tilde{\Delta}_l(s,t)| &\leq c_\alpha,\\
\sup_{(s,t)\in\theta_l}|\partial^\alpha\bar{\Delta}_l(s,t)| &\leq c_\alpha.
\end{aligned}$$

Taking the derivative once more, we obtain

$$\begin{aligned}
Q_l z_{ss} &= A_l(Q_l z)_s + Q_l(\frac{\partial}{\partial s}\Delta_l)(Q_l z)_t + Q_l\Delta_l(Q_l z_s)_t\\
&\quad +Q_l(\frac{\partial}{\partial s}\hat{\Delta}_l)(Q_l z)_{out} + Q_l\hat{\Delta}((Q_l z)_{out})_s\\
&\quad +Q_l(\frac{\partial}{\partial s}\tilde{\Delta}_l)\beta_l(Q_l z)_t + Q_l(\tilde{\Delta}_l\frac{d}{ds}\beta_l)(Q_l z)_t + Q_l(\tilde{\Delta}_l\beta_l)(Q_l z_s)_t\\
&\quad +Q_l(\frac{\partial}{\partial s}\bar{\Delta}_l)\beta_l(Q_l z)_{out} + Q_l(\bar{\Delta}_l\frac{d}{ds}\beta_l)(Q_l z)_{out} + Q_l(\bar{\Delta}_l\beta_l)(Q_l z_s)_{out}.
\end{aligned}$$

Taking the inner product with $Q_l z$, we obtain

$$\begin{aligned}
\langle Q_l z_{ss}, Q_l z\rangle_l &= \langle Q_l z_s, A_l(Q_l z)\rangle_l + \langle(\frac{\partial}{\partial s}\Delta_l)(Q_l z)_t, Q_l z\rangle_l + \langle \Delta_l(Q_l z_s)_t, Q_l z\rangle_l \\
&\quad + \langle(\frac{\partial}{\partial s}\hat{\Delta}_l)(Q_l z)_{out}, Q_l z\rangle_l + \langle \hat{\Delta}((Q_l z)_{out})_s, Q_l z\rangle_l \\
&\quad + \langle(\frac{\partial}{\partial s}\tilde{\Delta}_l)\beta_l(Q_l z)_t, Q_l z\rangle_l + \langle(\tilde{\Delta}_l \frac{d}{ds}\beta_l)(Q_l z)_t, Q_l z\rangle_l \\
&\quad + \langle(\tilde{\Delta}_l\beta_l)(Q_l z_s)_t, Q_l z\rangle_l \\
&\quad + \langle(\frac{\partial}{\partial s}\bar{\Delta}_l)\beta_l(Q_l z)_{out}, Q_l z\rangle_l + \langle(\bar{\Delta}_l \frac{d}{ds}\beta_l)(Q_l z)_{out}, Q_l z\rangle_l \\
&\quad + \langle(\bar{\Delta}_l\beta_l)(Q_l z_s)_{out}, Q_l z\rangle_l .
\end{aligned}$$

Let us denote the 11 terms of the right hand side by $T_1, \ldots, T_{11}$.

Substitute $Q_l z_s$ by its value in T_1 :

$$\begin{aligned}
T_1 &= \|A_l Q_l z\|_l^2 + \langle Q_l \Delta_l Q_l z_t, A_l Q_l z\rangle_l + \langle Q_l \hat{\Delta}_l (Q_l z)_{out}, A_l Q_l z\rangle_l \\
&\quad + \langle Q_l(\tilde{\Delta}_l\beta_l) Q_l z_t, A_l Q_l z\rangle_l + \langle Q_l(\bar{\Delta}_l\beta_l)(Q_l z)_{out}, A_l Q_l z\rangle_l .
\end{aligned}$$

Use integration by parts in T_3 and T_8 :

$$\begin{aligned}
T_3 &= \langle \Delta_l (Q_l z_s)_t, Q_l z\rangle_l \\
&= \int_0^1 \langle (Q_l z_s)_t, -\Delta_l^* J_0 M Q_l z\rangle dt \\
&= -\int_0^1 \langle Q_l z_s, (-\frac{\partial}{\partial t}\Delta_l^* J_0 M) Q_l z\rangle dt - \int_0^1 \langle Q_l z_s, -\Delta_l^* J_0 M Q_l z_t\rangle dt
\end{aligned}$$

and similarly

$$T_8 = -\int_0^1 \langle Q_l z_s, (-\frac{\partial}{\partial t}(\tilde{\Delta}_l\beta_l) J_0 M) Q_l z\rangle dt - \int_0^1 \langle Q_l z_s, -(\tilde{\Delta}_l\beta_l) J_0 M Q_l z_t\rangle dt.$$

Applying Cauchy-Schwarz inequality to all terms T_i and taking into account the asymptotic behavior of Δ_l, $\hat{\Delta}_l$ and of $\tilde{\Delta}_l$, $\bar{\Delta}_l$, we obtain

$$\begin{aligned}
T_1 &\geq \|A_l Q_l z\|_l^2 - \epsilon(l)\|Q_l z_t\|_l \|A_l Q_l z\|_l - \epsilon(l)\|Q_l z\|_l \|A_l Q_l z\|_l \\
&\quad - c|\beta_l|\|Q_l z_t\|_l \|A_l Q_l z\|_l - c|\beta_l|\|Q_l z\|_l \|A_l Q_l z\|_l, \\
T_2 &\geq -\epsilon(l)\|Q_l z_t\|_l \|Q_l z\|_l, \\
T_3 &\geq -\epsilon(l)\|Q_l z_s\|_l \|Q_l z\|_l - \epsilon(l)\|Q_l z_s\|_l \|Q_l z_t\|_l, \\
T_4 &\geq -\epsilon(l)\|Q_l z\|_l^2, \\
T_5 &\geq -\epsilon(l)\|Q_l z_s\|_l \|Q_l z\|_l, \\
T_6 &\geq -c|\beta_l|\|Q_l z_t\|_l \|Q_l z\|_l, \\
T_7 &\geq -\epsilon(l)\|Q_l z_t\|_l \|Q_l z\|_l, \\
T_8 &\geq -c|\beta_l|\|Q_l z_s\|_l \|Q_l z\|_l - c|\beta_l|\|Q_l z_s\|_l \|Q_l z_t\|_l, \\
T_9 &\geq -c|\beta_l|\|Q_l z\|_l^2, \\
T_{10} &\geq -\epsilon(l)\|Q_l z\|_l^2, \\
T_{11} &\geq -c|\beta_l|\|Q_l z_s\|_l \|Q_l z\|_l,
\end{aligned}$$

where $\epsilon(l)$ denotes a positive constant converging to zero as $l \to \infty$.

Using the expression for $Q_l z_s$ we obtain

$$\|Q_l z_s\|_l \leq \|A_l Q_l z\|_l + \epsilon(l)\|Q_l z_t\|_l + \epsilon(l)\|Q_l z\|_l + c|\beta_l|\|Q_l z_t\|_l + c|\beta_l|\|Q_l z\|_l.$$

On the other hand, it is clear from the definition of Q_l that

$$\|A_l Q_l z\|_l \geq c_1(\|(Q_l z)_t\|_l^2 + \|Q_l z\|_l^2)^{\frac{1}{2}}.$$

Using these 2 inequalities to eliminate $Q_l z$, $Q_l z_s$ and $Q_l z_t$ from the estimates for the T_i, we end up with

$$\langle Q_l z_{ss}, Q_l z \rangle_l \geq (1 - \epsilon(l) - c|\beta_l|)\|A_l Q_l z\|_l^2.$$

Therefore, if we choose l_0 sufficiently large and $\bar{\beta}$ sufficiently small, then for $l > l_0$ and $|\beta_l(s)| < \bar{\beta}$ we will have

$$\langle Q_l z''(s), Q_l z(s)\rangle_l \geq \frac{1}{2}\|A_l Q_l z(s)\|_l^2.$$

From this we deduce that

$$\begin{aligned} g_l''(s) &\geq \langle Q_l z''(s), Q_l z(s)\rangle_l \\ &\geq \frac{1}{2}\|A_l Q_l z(s)\|_l^2 \\ &\geq \frac{c_1^2}{2}\|Q_l z\|_l^2 = c_1^2 g_l(s). \end{aligned}$$

□

Define $s_l = \sup\{s \in I_l \,|\, |\beta_l(s')| \leq \bar{\beta} \text{ for all } s' \in [a_l, s]\}$. Then from lemma 3.2, we deduce that

$$g_l(s) \leq \max(g_l(a_l), g_l(s_l))\frac{\cosh(c_1(s - \frac{a_l+s_l}{2}))}{\cosh(c_1\frac{a_l-s_l}{2})}$$

for $s \in [a_l, s_l]$.

Now, let us derive some estimates for z_{in}. Let e be a unit vector in $\mathbb{R}^{2n-2}$ with $e_{out} = 0$.

Lemma 3.3. *For $l \geq l_0$ and $s \in [a_l, s_l]$, we have*

$$|\langle z(s), e\rangle_l - \langle z(a_l), e\rangle_l| \leq \frac{4d}{c_1}\max(\|Q_l z(a_l)\|_l, \|Q_l z(s_l)\|_l).$$

Proof. The inner product of the Cauchy-Riemann equation with e gives

$$\frac{d}{ds}\langle z, e\rangle_l + \langle M z_t, e\rangle_l + \langle S z_{out}, e\rangle_l = 0.$$

But we have

$$\begin{aligned}\langle Mz_t, e\rangle_l &= \int_0^1 \langle M(Q_l z)_t, -J_0 M_{c,d} e\rangle dt \\ &= \int_0^1 \langle Q_l z, \frac{d}{dt}(M^* J_0 M_{c,d})e\rangle dt\end{aligned}$$

so that

$$|\langle Mz_t, e\rangle_l| \leq d_1 \|Q_l z\|_l$$

and similarly

$$\langle S(Q_l z)_{out}, e\rangle_l = \int_0^1 \langle (Q_l z)_{out}, S^*(-J_0) M_{c,d} e\rangle dt$$

so that

$$|\langle Sz_{out}, e\rangle_l| \leq d_2 \|Q_l z\|_l.$$

Therefore

$$\langle z(s), e\rangle_l - \langle z(a_l), e\rangle_l \leq d \int_{R_l}^s \|Q_l z(\sigma)\|_l d\sigma$$

for $s \in I_l$. By lemma 3.2, we have

$$\|Q_l z(\sigma)\|_l \leq \max(Q_l z(a_l), Q_l z(s_l)) \sqrt{\frac{\cosh(c_1(s - \frac{a_l + s_l}{2}))}{\cosh(c_1 \frac{a_l - s_l}{2})}}$$

for $\sigma \in [a_l, s_l]$. Hence, using the fact that $\sqrt{\cosh u} < \sqrt{2}\cosh\frac{u}{2}$ and that $\sqrt{\cosh u} > \sqrt{2}\sinh\frac{u}{2}$, we obtain

$$\begin{aligned}|\langle z(s), e\rangle_l - \langle z(a_l), e\rangle_l| &\leq \frac{4d}{c_1}\max(\|Q_l z(a_l)\|_l, \|Q_l z(s_l)\|_l)\sqrt{2}\frac{\sinh(c_1\frac{a_l - s_l}{4})}{\sqrt{\cosh(c_1\frac{a_l - s_l}{2})}} \\ &\leq \frac{4d}{c_1}\max(\|Q_l z(a_l)\|_l, \|Q_l z(s_l)\|_l).\end{aligned}$$

□

Combining our estimates, we can now determine the behavior of a holomorphic map near puncture at infinity.

Proposition 3.4. *Let $\tilde{u} = (a, u) : \mathbb{R}^+ \times S^1 \to (\mathbb{R} \times M, J)$ be a holomorphic map with $E(\tilde{u}) < \infty$. Suppose that the image of $\tilde{u}$ is unbounded in $\mathbb{R} \times M$. Then there exists a periodic Reeb orbit γ in M such that $\lim_{s\to\infty} u(s,t) = \gamma(Tt)$ and $\lim_{s\to\infty} a(s,t) = \pm\infty$ in $C^\infty(S^1)$.*

Proof. By standard results [13], we can find a sequence $R_l \to \infty$ such that

$$\begin{aligned} \lim_{l\to\infty} u(R_l, t) &= \gamma(Tt), \\ \lim_{l\to\infty} a(R_l, t) &= \pm\infty \end{aligned}$$

for some periodic Reeb orbit γ of period T.
Let $\delta_l > 0$ be the largest number such that $u(s,t) \in S^1 \times [-\sigma, +\sigma]^{2n-2}$ for all $s \in [R_l, R_l + \delta_l]$, $t \in S^1$. Then, all the above results clearly apply to $I_l = [R_l, R_l + \delta_l]$ and $\tilde{u}_l = \tilde{u}|_{I_l}$.
By construction, $|\langle z(R_l), e\rangle_l| \to 0$ and $\|Q_l z(R_l)\|_l \to 0$. On the other hand, we can extract a subsequence so that $u(s_l, t)$ converges to a closed Reeb orbit γ'. Therefore, $\|Q_l z(s_l)\|_l \to 0$. Using lemmas 3.2 and 3.3, we then have

$$\sup_{s\in[R_l, s_l]} \|z(s)\|_l \to 0.$$

From this we can deduce the pointwise estimate

$$\sup_{(s,t)\in[R_l,s_l]\times S^1} |z_{in}(s,t)| \to 0.$$

Indeed, arguing by contradiction, if (s'_l, t'_l) is such that $|z_{in}(s'_l, t'_l)| \geq \delta$ with $s'_l \in [R_l, s_l]$, then $|z_{in}(s'_l, t)| \geq \frac{\delta}{2}$ for all $t \in S^1$, since $\sup_{(s,t)\in I_l\times S^1} |\partial z(s,t)| \to 0$. But then $\|z_{in}(s'_l)\| \geq C\frac{\delta}{2}$, contradicting the uniform convergence obtained above.

On the other hand, let us prove that we have

$$\sup_{s\in[R_l,s_l]} |\beta_l(s)| \to 0.$$

We already know that this is true for z_{in}, so we just have to prove that

$$\sup_{(s,t)\in[R_l,s_l]\times S^1} |\vartheta(s,t)-\vartheta(R_l,t)| \to 0.$$

In order to do this, consider the Cauchy-Riemann equations for a and ϑ :

$$\begin{cases} a_s-(\vartheta_t+xy_t)f &= 0, \\ a_t+(\vartheta_s+xy_s)f &= 0, \end{cases}$$

where $f=T+bz_{out}$. This can be rewritten as

$$\begin{cases} a_s-T\vartheta_t &= Txy_t+bz_{out}(\vartheta_t+xy_t), \\ a_t+T\vartheta_s &= -Txy_s-bz_{out}(\vartheta_s+xy_s). \end{cases}$$

The right hand side is bounded in norm by

$$C\max(\|Q_lz(R_l)\|,\|Q_lz(s_l)\|)\frac{\cosh(c_1(s-\frac{R_l+s_l}{2}))}{\cosh(c_1\frac{R_l-s_l}{2})}.$$

Indeed, $\|z_{out}\|_l\le C'\|Q_lz\|_l$, $\|z_t\|_l\le C''\|Q_lz_t\|_l$ and z_s can be expressed in terms of z_t and z_{out} using the Cauchy-Riemann equation. Therefore, integrating the second equation over t, we obtain

$$\int_0^1 \vartheta_s dt \le C\max(\|Q_lz(R_l)\|,\|Q_lz(s_l)\|)\frac{\cosh(c_1(s-\frac{R_l+s_l}{2}))}{\cosh(c_1\frac{R_l-s_l}{2})}.$$

Integrating over s, we get

$$\int_0^1 |\vartheta(s,\tau)-\vartheta(R_l,\tau)|d\tau \le C'\max(\|Q_lz(R_l)\|,\|Q_lz(s_l)\|).$$

We already know that $\vartheta_t-1\to 0$ uniformly, so the above estimate implies uniform convergence and not just L^1 convergence.

Therefore

$$\sup_{(s,t)\in\theta_l\times S^1}|z(s,t)|\to 0$$

because for l sufficiently large, $s_l = R_l + \delta_l$ since β_l will always be smaller than $\bar{\beta}$.

Therefore, $\delta_l = +\infty$ for l sufficiently large, and u converges uniformly to γ on the corresponding half-cylinder. □

3.3 Exponential decay

Knowing that the holomorphic cylinder converges to a given Reeb orbit, we can revisit our estimates from propositions 3.2 and 3.3, in order to get finer information about the decaying rate of z.

Proposition 3.5. *There exists $r > 0$ such that, for every multi-index I there is a constant c_I so that*

$$|\partial^I z(s,t)| \le c_I e^{-rs}$$

for all $s \ge s_0$.

Proof. By proposition 3.4, we can assume that $u(s,t)\in V$ for s sufficiently large. Let $M_\infty(t) = \lim_{s\to\infty} M(s,t)$ and $S_\infty(t) = \lim_{s\to\infty} S(s,t)$. Then

$$\begin{aligned} z_s &= A(s)z \\ &= A_\infty z + \bar{\Delta} z_t + \Delta z_{out} \end{aligned}$$

where $\Delta = S_\infty - S$ and $\bar{\Delta} = M_\infty - M$. Applying the projection Q corresponding to the limiting Reeb orbit, we obtain

$$w_s = A_\infty w + Q\bar{\Delta} w_t + Q\Delta w_{out}$$

where $w = Qz$. Let W be the vector obtained by catenating $(\frac{\partial}{\partial s})^a (A_\infty)^b w$ for $0 \leq a, b \leq k$. Then W satisfy an equation of the same type :

$$W_s = \mathcal{A}_\infty W + \mathcal{Q}\tilde{\Delta} W_t + \mathcal{Q}\hat{\Delta} W_{out}$$

where $\mathcal{A}_\infty = \mathrm{diag}(A_\infty, \ldots, A_\infty)$ and $\mathcal{Q} = \mathrm{diag}(Q, \ldots, Q)$. Therefore, using the same estimates as in proposition 3.2, we obtain

$$\|W(s)\| \leq e^{-r(s-s_0)} \|W(s_0)\|.$$

Next we estimate Pz and its derivatives. Applying P to the Cauchy-Riemann equation, we get

$$(Pz)_s = P\bar{\Delta}(Qz)_t + P\Delta(Qz)_{out}.$$

We can apply $(\frac{\partial}{\partial s})^a$ to this equation, and express the derivatives of Pz in terms of quantities converging exponentially to zero. Moreover, by integrating, Pz itself converges exponentially to zero (we already know its limit is zero, so there is no constant term). □

Similarly, we obtain decaying rates for a and ϑ.

Proposition 3.6. *For the same r as in proposition 3.5, there is a constant c'_I for every multi-index I so that*

$$\begin{aligned} |\partial^I(\vartheta(s,t) - t - \vartheta_0)| &\leq c'_I e^{-rs}, \\ |\partial^I(a(s,t) - Ts - a_0)| &\leq c'_I e^{-rs}. \end{aligned}$$

Proof. The proof is identical to the original argument of Hofer, Wysocki and Zehnder [16], since this is an estimate in 2 dimensions, independently of $\dim M$. Alternatively, we can use lemma 4.2 in [15]. □

3.4 Energy and area

The results of the last sections show the importance of the Hofer energy. However, it is sometimes more natural to work with the area of a J-holomorphic map.

Definition 3.7. *The area of a holomorphic map $\tilde{u} : (\Sigma, j) \to (\mathbb{R} \times M, J)$ is defined by $A(f) = \int_\Sigma f^* d\alpha$.*

Note that the area is a nonnegative quantity, since $d\alpha$ is positive on complex lines in the contact distribution.

The next lemma describes the relationship between finiteness of area and finiteness of energy.

Lemma 3.8. *Let $\tilde{u} : (\Sigma, j) \to (\mathbb{R} \times M, J)$ be a holomorphic map. Then the following are equivalent :*

(i) $A(\tilde{u}) < \infty$ and $\tilde{u}$ is a proper map,

(ii) $E(\tilde{u}) < \infty$ and Σ has no punctures with a bounded as in lemma 2.2.

Proof. (i) $\Rightarrow$ (ii) By properness of $\tilde{u}$, each puncture of Σ can be called positive or negative, according to the end of $\mathbb{R} \times M$ that it approaches. In a neighborhood U of a puncture $x_i^\pm$, let z be a complex coordinate vanishing at $x_i^\pm$. Let $D_r(x_i^\pm) = \{q \in U \,|\, |z(q)| \leq r\}$ and $C_r(x_i^\pm) = \mp\partial D_r(x_i^\pm)$. Consider $\int_{C_r(x_i^\pm)} u^*\alpha$ as a function of r. It is an increasing function of $r^{\mp 1}$, since $d\alpha \geq 0$ on complex lines.
On the other hand, it is a nonnegative function for a negative puncture x_i^-, because $\int_{C_r(x_i^-)} u^*\alpha = \frac{d}{dr}\int_{C_r(x_i^-)} a$ by holomorphicity, and the latter is nonnegative by properness of $\tilde{u}$. Hence, this is a decreasing function of $\frac{1}{r}$ bounded below and it has a nonnegative limit for $r \to 0$, for all negative punctures.
Consider now a positive puncture x_i^+; by Stokes theorem, $\sum_{i=1}^{s^+} \int_{C_r(x_i^+)} u^*\alpha \leq A(\tilde{u}) + \sum_{i=1}^{s^-} \int_{C_r(x_i^-)} u^*\alpha < C < \infty$. Hence, we obtain an increasing function of $\frac{1}{r}$ bounded above and it has a finite limit for $r \to 0$, for all positive punctures. Now, let $\phi \in \mathcal{C}$ and let $\phi_n \in \mathcal{C}$ such that $\phi_n \circ \tilde{u}$ is constant in $D_{\frac{1}{n}}(x_i^\pm)$ for all punctures $x_i^\pm$. Such functions exist, by properness of $\tilde{u}$. Moreover, we can choose ϕ_n so that

$\|\phi - \phi_n\|_{C^1} < \epsilon_n$, with $\epsilon_n \to 0$ for $n \to \infty$. By Stokes theorem, $\int_\Sigma \tilde{u}^* d(\phi_n \alpha) = \lim_{r\to 0} \sum_{i=1}^{s^+} \int_{C_r(x_i^+)} u^*\alpha < C < \infty$. Moreover, this integral is uniformly convergent in n. Hence, $\int_\Sigma \tilde{u}^* d(\phi\alpha) = \lim_{n\to\infty} \int_\Sigma \tilde{u}^* d(\phi_n\alpha) < C$ and $E(\tilde{u}) < C$.

(ii) $\Rightarrow$ (i) Take $\phi = 1$ in the expression for $E(\tilde{u})$ to obtain $A(\tilde{u}) \leq E(\tilde{u}) < \infty$. Moreover, $\tilde{u}$ has only positive and negative punctures by assumption. Therefore, for every compact set K in $\mathbb{R} \times M$, there exists, by proposition 3.4, a neighborhood of each puncture such that its image under $\tilde{u}$ is disjoint from K. Hence, $\tilde{u}^{-1}(K)$ is closed in Σ and away from the punctures, so it is compact. $\square$

In that case, the area and energy of a holomorphic map are easily computable using Stokes theorem.

Lemma 3.9. *Under the conditions of lemma 3.8, denote by $\gamma_1^+, \ldots, \gamma_{s^+}^+, \gamma_1^-, \ldots, \gamma_{s^-}^-$ the periodic Reeb orbits of M asymptotic to the positive and negative punctures of Σ. Then*

$$E(\tilde{u}) = \sum_{j=1}^{s^+} \mathcal{A}(\gamma_j^+)$$

and

$$A(\tilde{u}) = \sum_{j=1}^{s^+} \mathcal{A}(\gamma_j^+) - \sum_{j=1}^{s^-} \mathcal{A}(\gamma_j^+)$$

where $\mathcal{A}(\gamma) = \int_\gamma \alpha$ is the action functional.

Chapter 4

Compactness

4.1 Fixed asymptotics

4.1.1 Convergence of holomorphic maps

We first introduce in a more systematic way the types of holomorphic maps we will consider for the compactness theorem.

Definition 4.1. *A level k holomorphic map $(\Sigma, j, \tilde{u})$ to $(\mathbb{R} \times M, J)$ consists of the following data :*

(i) A labeling of the connected components of $\Sigma^* = \Sigma \setminus \{nodes\}$ by integers in $\{1, \dots, k\}$, called levels, such that two components sharing a node have levels differing by at most 1. We denote by Σ_i the union of connected components of Σ^* with level i.

(ii) Holomorphic maps $\tilde{u}_i : (\Sigma_i, j) \to (\mathbb{R} \times M, J)$ with $E(\tilde{u}_i) < \infty$, $i = 1, \dots, k$, such that each node shared by Σ_i and Σ_{i+1}, is a positive puncture for $\tilde{u}_i$, asymptotic to some periodic Reeb orbit γ and is a negative puncture for $\tilde{u}_{i+1}$, asymptotic to the same periodic Reeb orbit γ, and such that $\tilde{u}_i$ extends continuously across each node within Σ_i.

The area of a level k holomorphic map $(\Sigma, j, \tilde{u})$ is naturally defined by $A(\tilde{u}) = \sum_{i=1}^{k} A(\tilde{u}_i)$.

Similarly as in the setting of Gromov-Witten theory, there is a notion of stability for holomorphic maps.

Definition 4.2. *A level k holomorphic map $(\Sigma, j, \tilde{u})$ to $(\mathbb{R} \times M, J)$ is stable if, for every $i = 1, \dots, k$, either $A(\tilde{u}_i) > 0$ or Σ_i has a negative Euler characteristic (after removing marked points).*

We now define a notion of convergence for a sequence of stable level k holomorphic maps.

Definition 4.3. *A sequence of stable level k holomorphic maps $(\Sigma_n, j_n, \tilde{u}_n)$ converges to a stable level k' $(k' \geq k)$ holomorphic map $(\Sigma, j, \tilde{u})$ if there exist a sequence of maps $\phi_n : \Sigma_n \to \Sigma$ and sequences $t_n^{(i)} \in \mathbb{R}$ $(i = 1, \dots, k')$, such that*

(i) the maps ϕ_n are diffeomorphisms, except that they may collapse a circle in Σ_n to a node in Σ, and $\phi_{n*} j_n \to j$ away from the nodes of Σ.

(ii) the sequences of maps $(t_n^{(i)} + a_n \circ \phi_n^{-1}, u_n \circ \phi_n^{-1}) : \Sigma_i \to \mathbb{R} \times M$ converge in the C^∞ topology to $\tilde{u}_i : \Sigma_i \to \mathbb{R} \times M$ on every compact subset of Σ_i, for $i = 1, \dots, k'$.

(iii) for each node p of Σ between adjacent levels, consider a sequence of curves $\gamma_n : (-\epsilon, +\epsilon) \to \Sigma_n$ intersecting $\phi_n^{-1}(p)$ transversally at $t = 0$ and satisfying $\phi_n \circ \gamma_n = \gamma$ for all n. Then $\lim_{t \to 0^+} u(\gamma(t)) = \lim_{t \to 0^-} u(\gamma(t))$.

These conditions automatically imply that $\lim_{n \to \infty} A(\tilde{u}_n) = A(\tilde{u})$.

4.1.2 Compactness theorem

We will denote the minimal distance between two distinct values of the action spectrum by $\hbar > 0$.

Theorem 4.4. *Let $(\Sigma_n, j_n, \tilde{u}_n)$ be a sequence of stable level k holomorphic maps to $(\mathbb{R} \times M, J)$ of same genus such that $E(\tilde{u}_n) < C$. Then there exists a subsequence that converges to a stable level k' $(k' \geq k)$ holomorphic map $(\Sigma, j, \tilde{u})$ to $(\mathbb{R} \times M, J)$.*

In order to prove this theorem, we need the following lemmas.

Lemma 4.5. (Gromov-Schwarz) *Let $f : D^2(1) \to W$ be a pseudoholomorphic disk in an almost complex manifold, such that J is tamed by an exact symplectic form. If the image of f is contained in a compact set $K \subset W$, then*

$$\|\nabla^k f(x)\| \leq C(K,k) \qquad \text{for all } x \in D^2(\frac{1}{2}).$$

Lemma 4.6. (Monotonicity) *There are constants ϵ_0 and c_0 such that for every $\epsilon \leq \epsilon_0$ and for every holomorphic curve S, if $x \in S$ is such that $S \cap B(x,\epsilon)$ is compact with its boundary contained in $\partial B(x,\epsilon)$, then*

$$\int_{S \cap B(x,\epsilon)} \omega \geq \frac{\pi}{1 + c_0 \epsilon} \epsilon^2.$$

Lemma 4.7. (Hofer) *Let (X,d) be a complete metric space, $f : X \to \mathbb{R}$ be a non-negative continuous function, $x \in X$, and $\delta > 0$. Then there exist $y \in X$ and a positive number $\epsilon \leq \delta$ such that*

$$d(x,y) < 2\delta, \qquad \sup_{B_\epsilon(y)} f \leq 2f(y), \qquad \epsilon f(y) \geq \delta f(x).$$

Proofs of Gromov-Schwarz lemma and the monotonicity lemma can be found in [28]. A proof of Hofer's lemma is contained in [14].

Proof of theorem 4.4. First note that it is enough to consider $k = 1$, because we can handle each level separately. Next, after extracting a subsequence, we can assume that the maps $\tilde{u}_n$ are asymptotic to the same Reeb orbits. Indeed, the energy bound and the discreteness of the action spectrum guarantee that there are finitely many possibilities for the asymptotics of $\tilde{u}_n$. In order to prove the theorem for this reduced case, we proceed in 5 steps.

Step 1. Riemann surfaces.

Consider the sequence of Riemann surfaces (Σ_n, j_n). If Σ_n is a plane or a cylinder, choose additional marked points, so that the Euler characteristic is negative. Then, we have a unique hyperbolic metric of curvature -1 on each Σ_n, and we can extract a subsequence, such that (Σ_n, j_n) converges to a nodal curve (C, j). More precisely, there exist maps $\phi_n : \Sigma_n \to C$ that collapse a circle above some of the nodes of C, but otherwise are diffeomorphisms, such that $\phi_{n*} j_n \to j$ away from the nodes of C.

Step 2. Convergence away from the nodes.

Denote by C_i $(i = 1, \dots, m)$ the connected components of $C^* = C \backslash \{\text{nodes}\}$. Let $\epsilon > 0$ and denote the $\epsilon-$thick part of C_i by C_i^ϵ. On C_i^ϵ, the maps ϕ_n are diffeomorphisms, and we can define $\tilde{u}_{i,n}^\epsilon = (a_{i,n}^\epsilon, u_{i,n}^\epsilon) : C_i^\epsilon \to \mathbb{R} \times M$ by $\tilde{u}_{i,n}^\epsilon = \tilde{u}_n \circ \phi_n^{-1}$.

Lemma 4.8. *There exist sequences $y_n^{(1)}, \dots, y_n^{(2l)}$ of points in C_i^ϵ, where l is bounded by energy, so that $\|\nabla \tilde{u}_{i,n}^\epsilon\|$ is uniformly bounded on C_i^ϵ for the Poincaré metric on $C_i \setminus \{y_n^{(1)}, \dots, y_n^{(2l)}\}$.*

Proof. Let $x \in C_i^\epsilon$, and assume that for every open neighborhood U of x, the diameter of $a_{i,n}^\epsilon(U)$ is unbounded when $n \to \infty$. Therefore, there exists a sequence $x_n \to x$ such that $\|\nabla \tilde{u}_{i,n}^\epsilon(x_n)\| \to \infty$. Now, let $\delta_n > 0$ such that $\delta_n \to 0$ and $\delta_n \|\nabla \tilde{u}_{i,n}^\epsilon(x_n)\| \to \infty$. Applying Hofer's lemma, we obtain new sequences $y_n \in C_i^\epsilon$ and $0 < \epsilon_n \leq \delta_n$ such that $y_n \to x$ and

$$\sup_{B_{\epsilon_n}(y_n)} \|\nabla \tilde{u}_{i,n}^\epsilon\| \leq 2 \|\nabla \tilde{u}_{i,n}^\epsilon(y_n)\|, \qquad \epsilon_n \|\nabla \tilde{u}_{i,n}^\epsilon(y_n)\| \to \infty.$$

Denote $c_n = \|\nabla \tilde{u}_{i,n}^\epsilon(y_n)\|$ and $R_n = \epsilon_n c_n$. Consider the rescaled maps $\tilde{v}_n^x(z) = \tilde{u}_{i,n}^\epsilon(y_n + c_n^{-1} z)$. In this definition, we used a fixed complex coordinate on C_i^ϵ near x. This sequence satisfies the following properties :

$$\sup_{B_{R_n}} \|\nabla \tilde{v}_n^x\| \leq 2, \qquad R_n \to \infty$$

and $\int_{B_{R_n}} (v_n^x)^* d\alpha$ is uniformly bounded by the energy of $\tilde{u}_n$. Now, by Ascoli-Arzela, we can extract a converging subsequence and we obtain a finite energy plane $\tilde{v}^x$. By proposition 3.4, we deduce that $\tilde{v}^x$ is converging to a Reeb orbit γ for large radius. Hence, the energy of $\tilde{v}^x$ is equal to the action of γ, so it is bounded below by $\hbar$. Since the energy of $\tilde{u}_n$ is uniformly bounded, and such a point x requires a quantum $\hbar$ of

energy, there are only finitely many such points : $x_1, \dots, x_l$.
Add two marked points on Σ_n for each of these points x_i : $y_n^{(2i-1)} = \phi_n^{-1}(y_n)$ and $y_n^{(2i)} = \phi_n^{-1}(y_n + c_n^{-1})$. Since these 2 sequences converge to the same point, the new limiting curve C' is going to have an extra spherical component, with two marked points on it. So, considering the ϵ-thick parts of the components of C', we will have an extra component corresponding to the rescaled map $\tilde{v}^x$, and the component C'^{ϵ}_i that contained x will have a deleted disk around x.
Now consider $x \neq x_i$ ($i = 1, \dots, l$). There exists a neighborhood U_x of x such that the diameter of $a^{\epsilon}_{i,n}(U_x)$ is bounded when $n \to \infty$. In this case, we can directly apply Gromov-Schwarz lemma. Indeed, the assumption on $a^{\epsilon}_{i,n}$ implies that, after translating the map $\tilde{u}^{\epsilon}_{i,n}$, the image of U_x will be contained in a set of the form $I \times M$ where $I \subset \mathbb{R}$ is a fixed, bounded interval. As the symplectic form is exact on $\mathbb{R} \times M$, the lemma applies after replacing U_x by a smaller disk centered at x. Therefore, we obtain uniform gradient bounds in a neighborhood U_x of each point x of C^{ϵ}_i. Extracting a finite cover out of $\{U_x\}$, we get a finite uniform bound for $\|\nabla \tilde{u}^{\epsilon}_{i,n}\|$ on C^{ϵ}_i. □

By a repeated use of the Ascoli-Arzela theorem, we can extract a subsequence converging uniformly on C^{ϵ}_i in the C^r norm, for r as large as we want. Taking the diagonal sequence, we obtain smooth convergence to a map $\tilde{u}^{\epsilon}_i$. As $\epsilon > 0$ is arbitrary, we actually obtain smooth, uniform convergence to $\tilde{u}_i$ on each compact subset of C_i.

Step 3. Convergence in the thin part.

We have to understand the asymptotic behavior of the map $\tilde{u}$ on C_i near a node. First, if a is bounded near the node, then, by lemma 2.2, the map $\tilde{u}$ extends continuously on C_i across the node. On the other hand, if a is unbounded near the node, the behavior of $\tilde{u}$ is described by proposition 3.4 : there exists a closed Reeb orbit γ such that the map $\tilde{u}$ is asymptotic to γ near the node.
Then, given a node of C adjacent to the components C_i and C_j, the asymptotic behavior of $\tilde{u}$ on the 2 components might be different. For example, $\tilde{u}$ could be asymptotic to different Reeb orbits, or be asymptotic to a Reeb orbit on C_i and be mapped at finite distance on C_j, or be mapped at finite distance but to different points.

To each node of C, we can associate two objects γ^+ and γ^-, one for each component of C adjacent to the node. If the node is adjacent to only one component C_i, then we define the second object to be the asymptotic Reeb orbit of the maps $\tilde{u}_n$ at that node. When $\gamma^\pm$ is a point, we define its action $A(\gamma^\pm)$ to be zero.

Lemma 4.9. *There exist sequences $p_n^{(1)}, \dots, p_n^{(r)}$ of marked points on Σ_n, where r is bounded by energy, so that for every node in the limiting stable map $(C, j, \tilde{u})$, $T^+ = \mathcal{A}(\gamma^+)$ coincides with $T^- = \mathcal{A}(\gamma^-)$.*

Proof. We will work in cylindrical coordinates in the thin part of Σ_n. If the injectivity radius is sufficiently small, we know that the curve contains a cylindrical model with coordinates (s, t), where $t \in \mathbb{R}/\mathbb{Z}$, $s \in I_n \subset \mathbb{R}$, $z = s + it$ is a complex coordinate, and $\{s = 0\}$ is the shortest closed geodesic in the thin part.
We can translate the s coordinate in such a way that $\int_{\{s=0\}} \alpha = T^+ - \hbar$. Indeed, this integral is an increasing function of s, is very close to T^+ for s large and very close to T^- for s small. Now, for $n \to \infty$, we have $I_n \to \mathbb{R}$ since the injectivity radius converges to zero. Hence, we obtain a sequence of holomorphic curves on compact subsets of the cylinder. Add a marked point $p_n^{(i)}$ on S_n at the coordinates $s = 0, t = 0$. This will create an additional component for C^*, and we can obtain gradient bounds on it using lemma 4.8. It is clear that the Reeb orbit (or point) associated to the node for $s \to +\infty$ has action T^+, by definition of $\hbar$, and that the Reeb orbit (or point) associated to the node for $s \to -\infty$ has action $T \in (T^+ - \hbar, T^-]$. Hence, the number of new components we can generate in this way is finite, so that we end up in a situation where $T^+ = T^-$ for every node of C. □

Corollary 4.10. *If $T^+ = T^- = 0$, then $\tilde{u}$ extends continuously to a neighborhood of the node in C.*

Proof. The image of a node belonging to 2 components C_i and C_j of C consists a priori of 2 points p^+ and p^-. Assume that $p^+ \neq p^-$. Extract from Σ_n a cylinder $I_n \times S^1$ so that the images of the boundary circles lie in small neighborhoods of p^+ and p^- respectively. Let p_n be a point in the middle of the cylinder. Apply the monotonicity lemma with a ball centered at p_n. On one hand, the energy will then be bounded away from zero, but on the other hand, the area $\int_C d(e^t\alpha)$ converges to zero because $\int_{\{s=s_0\}} \alpha \to 0$ for all s_0. Hence, we obtain a contradiction, and $p^+ = p^-$. □

Step 4. Asymptotic convergence.

When $T^+ = T^- > 0$, we have to check that the Reeb orbits γ^+ and γ^- discussed in the previous step are geometrically the same.
The convergence of (Σ_n, j_n) to (C, j) specifies, for each node in C, a limiting angle allowing us to identify the tangent spaces on each component of C (modulo scaling). Therefore, given coordinates (s, t) on the 2 cylindral ends near a node, if makes sense to identify the t coordinates.

Lemma 4.11. *If $T^+ = T^- > 0$, then*

$$\lim_{s\to-\infty} u^+(s,t) = \lim_{s\to+\infty} u^-(s,t).$$

Proof. Let us first prove that the limiting Reeb orbits belong to the same path component of orbit space. By continuity of u_n we can find a point p_n in the thin part of Σ_n that is mapped to a point away from a fixed neighborhood of the spaces of orbits with period $T^+ = T^-$. Translate the coordinates (s, t) so that p_n corresponds to $s = 0, t = 0$. The area of the corresponding sequence of curves converges to zero, since $\int_{\{s=s_0\}} \alpha$ becomes independent of s_0. Then the sequence converges to a cylinder (without bubble, since no area is available) with zero area, so it is a vertical cylinder over a Reeb orbit of period $T^+ = T^-$. But this is a contradiction with the choice of the points p_n. Hence, the orbits γ^+ and γ^- belong to the same path component of the orbit space.
Next, let us check that the orbits γ^+ and γ^- agree. By contradiction, assume that $\gamma^+ \neq \gamma^-$. Use a local chart around γ^+ as in lemma 3.1, with $z_{in} = 0$ along γ^+. Then, for n sufficiently large, we can extract from (Σ_n, j_n) a finite cylinder $[a_n, b_n] \times S^1$ so that $z_{in}(a_n, 0) \to 0$ and b_n is the largest number satisfying $u(s,t) \in S^1 \times [-\sigma, +\sigma]^{2n-2}$ for all $s \in [a_n, b_n]$, $t \in S^1$. Reduce $\sigma > 0$ if necessary to make sure that γ^- is not contained in $S^1 \times [-\sigma, +\sigma]^{2n-2}$. Since $T^+ = T^-$, the energy of these cylinders converges to zero, and for every $c_n \in [a_n, b_n]$, the maps $\tilde{u}_n(c_n + s, t)$ converge uniformly

with their derivatives to a vertical cylinder over a closed Reeb orbit. Therefore, we can apply to this situation the estimates of lemmas 3.2 and 3.3. Then, proceeding as in proposition 3.4, it follows that

$$\sup_{s\in[a_n,s_n]} |\beta_n(s)| \to 0.$$

But this would imply that, for n large, $s_n = b_n = +\infty$, which is absurd.
Finally, note that the parametrizations agree as well. Indeed, suppose that

$$\lim_{s\to-\infty} u^+(s,t) = \lim_{s\to+\infty} u^-(s,t+\delta)$$

where $\delta \neq 0$. Then, we can repeat the above argument with $\bar{\beta}$ much smaller than $|\delta|$ and obtain a contradiction as well. $\square$

Step 5. Level structure.

Let us introduce an ordering on the set of components of C^*. For two components C_i and C_j, pick two points $x_i \in C_i$ and $x_j \in C_j$. Then, we will say that $C_i \leq C_j$ if $\lim_{n\to\infty} a_n(x_i) - a_n(x_j) < \infty$. If $C_i \leq C_j$ and $C_j \leq C_i$, then we will say that $C_i \sim C_j$. Clearly, this ordering is independent of the choice of points x_i and x_j. Now, we can label the components C_i with their level number as follows : the set of minimal components for the above ordering will be of level 1. Then, after removing these components, the set of minimal components will be of level 2, etc, ... Clearly, this labelling is constant across nodes that are mapped at finite distance. However, it may happen that the level number jumps by an integer $N > 1$ across a node at infinity. In that case, we have to insert $N-1$ additional components between these two components, each of them a vertical cylinder over the Reeb orbit corresponding to the above node. Finally, remove the marked points that we added in step 1 of the proof. If level i becomes unstable because of this, we remove it and decrease by 1 the labeling of higher levels. Hence, we obtain a level structure that satisfies all the necessary conditions for a stable level k' holomorphic map in a symplectization. $\square$

4.2 Degenerating the asymptotics

The above compactness results are valid for a sequence J_n of almost complex structures on the symplectic cobordism (W, ω) such that $J_n \to J$ and J_n is independent of n near the ends of the cobordism.

However, it would be very useful to extend our compactness results to the case in which the complex structure on the contact distribution ξ near the boundary is fixed, but the Reeb dynamics (and hence J_n) vary.

In particular, we wish to consider the case of $\alpha_n = (1 + \lambda_n \bar{f}_T)\alpha$, where α is a contact form of Morse-Bott type, α_n has nondegenerate periodic Reeb orbits, $\bar{f}_T$ descends to a Morse function f_T on each orbit space S_T for α, and $\lambda_n \to 0$. Moreover, we assume that J is invariant under the Reeb flow along the submanifolds N_T.

4.2.1 Convergence of generalized holomorphic maps

For degenerating asymptotics, it turns out that we need to generalize the concept of level k holomorphic curves in order to obtain a limit. First, we need the following definition when $k = 1$.

Definition 4.12. *A generalized level* 1 *holomorphic map* $\tilde{u}$ *from* (Σ, j) *to* $(\mathbb{R} \times M, J)$ *with Morse functions* f_T *consists of the following data :*

(i) A labeling of the connected components of $\Sigma^* = \Sigma \setminus \{nodes\}$ *by integers in* $\{1, \dots, l\}$*, called sublevels, such that two components sharing a node have sublevels differing by at most 1. We denote by* Σ_i *the union of connected components of sublevel* i*.*

(ii) Positive numbers t_i*,* $i = 1, \dots, l-1$*.*

(iii) Holomorphic maps $\tilde{u}_i : (\Sigma_i, j) \to (\mathbb{R} \times M, J)$ *with* $E(\tilde{u}_i) < \infty$*,* $i = 1, \dots, l$*, such that*

- *each node shared by* Σ_i *and* Σ_{i+1}*, is a positive puncture for* $\tilde{u}_i$*, asymptotic to some periodic Reeb orbit* $\gamma \in S_T$ *and is a negative puncture for* $\tilde{u}_{i+1}$*,* *asymptotic to a periodic Reeb orbit $\delta \in S_T$, such that $\varphi_{t_i}^{f_T}(\gamma) = \delta$, where $\varphi_t^{f_T}$ is the gradient flow of f_T.*

- *$\tilde{u}_i$ extends continuously across each node within Σ_i.*

We say that the positive asymptotics of $\tilde{u}$ are the critical points of f_T obtained by following ∇f_T from the periodic Reeb orbits corresponding to the positive punctures of $\tilde{u}_l$. Similarly, the negative asymptotics of $\tilde{u}$ are the critical points of f_T obtained by following $-\nabla f_T$ from the periodic Reeb orbits corresponding to the negative punctures of $\tilde{u}_1$. Next, we extend the definition to level k as in section 4.1.1.

Definition 4.13. *A generalized level k holomorphic map $\tilde{u}$ from (Σ, j) to $(\mathbb{R} \times M, J)$ with Morse functions f_T consists of k generalized level 1 holomorphic maps $\tilde{u}_i$, $i = 1, \ldots, k$, such that the positive asymptotics of $\tilde{u}_i$ coincide with the negative asymptotics of $\tilde{u}_{i+1}$.*

We now extend the definition of stability to generalized holomorphic maps.

Definition 4.14. *A generalized level k holomorphic map $(\Sigma, j, \tilde{u})$ to $(\mathbb{R} \times M, J)$ is stable if, for every $i = 1, \ldots, k$, either $A(\tilde{u}_i) > 0$, or $\tilde{u}_i$ contains at least one non-constant gradient trajectory of f_T, or Σ_i has a negative Euler characteristic (after removing marked points).*

We now define a notion of convergence for a sequence of stable generalized level k holomorphic maps.

Definition 4.15. *A sequence of stable level k J_n-holomorphic maps $(\Sigma_n, j_n, \tilde{u}_n)$ converges to a stable generalized level k' $(k' \geq k)$ J-holomorphic map $(\Sigma, j, \tilde{u})$ if there exist a sequence of maps $\phi_n : \Sigma_n \to \Sigma$ and sequences $t_n^{(i,j)} \in \mathbb{R}$ $(i = 1, \ldots, k', j = 1, \ldots, l_i)$, where l_i is the number of sublevels in level i, such that*

(i) the maps ϕ_n are diffeomorphisms, except that they may collapse a circle in Σ_n to a node in Σ, and $\phi_{n*} j_n \to j$ away from the nodes of Σ.

(ii) the sequences of maps $(t_n^{(i,j)} + a_n \circ \phi_n^{-1}, u_n \circ \phi_n^{-1}) : \Sigma_{i,j} \to \mathbb{R} \times M$ converge in the C^∞ topology to $\tilde{u}_{i,j} : \Sigma_i \to \mathbb{R} \times M$ on every compact subset of $\Sigma_{i,j}$, for $i = 1, \ldots, k'$, $j = 1, \ldots, l_i$.

(iii) for each node p of Σ between adjacent (sub)levels, consider a sequence of curves $\gamma_n : (-\epsilon, +\epsilon) \to \Sigma_n$ intersecting $\phi_n^{-1}(p)$ transversally at $t = 0$ and satisfying $\phi_n \circ \gamma_n = \gamma$ for all n. Then $\lim_{t\to 0^+} u(\gamma(t))$ and $\lim_{t\to 0^-} u(\gamma(t))$ lie on the same gradient trajectory of $\bar{f}_T$ in N_T.

4.2.2 Compactness theorem

The goal of this section is to prove the following compactness theorem.

Proposition 4.16. *Let $\tilde{u}_n : (\Sigma_n, j_n) \to (\mathbb{R} \times M, J_{\lambda_n})$ be a sequence of holomorphic curves of fixed genus and asymptotics, such that $\lim_{n\to\infty} \lambda_n = 0$ and $E(\tilde{u}_n) < C$. Then there exists a subsequence that converges to a generalized holomorphic map $\tilde{u}$ with Morse functions f_T.*

Clearly, the arguments of section 4.1 are enough to obtain convergence of each sublevel. Moreover, the study of the asymptotics in section 4.1 show that, in each sublevel, the holomorphic maps converge to closed Reeb orbits for contact form α. We just need to show that these orbits are related by the gradient flow of $\bar{f}_T$ for any pair of adjacent levels.

We therefore need to modify our arguments in the study of the asymptotics. Let us write down the Cauchy-Riemann equations in the local coordinates provided by lemma 3.1. The difference with the equations of the last chapter lies in the fact that $J_l \frac{\partial}{\partial t} = R_{\alpha_l} = R_\alpha + X_l$. We obtain

$$\begin{aligned} z_s + M z_t + \frac{1}{(1+\lambda_l \bar{f})} S z_{out} + \frac{\lambda_l}{f(1+\lambda_l \bar{f})^2}(a_t I - a_s M) v &= 0, \\ a_s - (\vartheta_t + x \cdot y_t) f &= 0, \\ a_t + (\vartheta_s + x \cdot y_s) f &= 0, \end{aligned}$$

where

$$v = \begin{bmatrix} \bar{f}_y - x \bar{f}_\vartheta \\ -\bar{f}_x \end{bmatrix}.$$

Note that, on N_T, $\bar{f}_\vartheta = 0$ so that $Mv = \nabla \bar{f}$ with respect to the metric $\omega(\cdot, J\cdot)$.

Moreover, since this vector field is tangent to N_T and invariant under the Reeb flow, it lies in $\ker A_x$ for all $x \in N_T$. In particular, $Q_x M v = 0$.

Let $w = \begin{bmatrix} a \\ \vartheta \end{bmatrix}$; we will also write $\tilde{w} = \begin{bmatrix} \tilde{a} \\ \tilde{\vartheta} \end{bmatrix}$ where $\tilde{a} = a - (1+\lambda_l \bar{f})Ts$ and $\tilde{\vartheta} = \vartheta - t$. On the other hand let

$$\tilde{A}_l = \begin{pmatrix} 0 & T(1+\lambda_l \bar{f}) \\ -\frac{1}{T(1+\lambda_l \bar{f})} & 0 \end{pmatrix} \frac{d}{dt}.$$

Clearly, $\ker \tilde{A}_l$ consists of the functions $\tilde{w}(t)$ that are independent of t. Let $\tilde{P}$ be the orthonogal projection from $L^2(S^1)$ to $\ker \tilde{A}_l$ and $\tilde{Q} = I - \tilde{P}$.

We can rewrite the last two Cauchy-Riemann equations as :

$$w_s = \tilde{A}_l w + B z_{out} + B z_t + B(Q_l z) + C\langle P_l z, z_s \rangle_l.$$

Here, B stands for a matrix that is bounded with all its derivatives, and C is a constant. Similarly, for $\tilde{w}$, we have an equation of the form

$$\tilde{w}_s = \tilde{A}_l \tilde{w} + B z_{out} + B z_t + B(Q_l z) + C\langle P_l z, z_s \rangle_l + C' \lambda_l s \langle Mv, z_s \rangle_l$$

where the last term comes from $\lambda_l s \frac{\partial \bar{f}}{\partial s}$ in w_s.

We first need to generalize the estimates of lemma 3.2.

Lemma 4.17. *There exists $l_0 > 0$, $\bar{\beta} > 0$ and $\delta > 0$ such that for $l \geq l_0$ and $s \in I_l$ satisfying $|\beta_l(s)| \leq \bar{\beta}$, the function $H_l(s) = \|Q_l z(s)\|_l^2 + \delta^2 \|\tilde{w}\|^2$ satisfies*

$$H_l''(s) \geq K^2 H_l(s)$$

where $K > 0$ is a constant independent of l.

Proof. Applying Q_l to the first Cauchy-Riemann equation, we obtain

$$Q_l z_s = A_l Q_l z + B(Q_l z)_{out} + B(Q_l z)_t + \lambda_l B \tilde{Q} \tilde{w}$$

where we took into account the fact that $Q_l v = 0$. The last term involving $\tilde{w}$ comes from the last term of the Cauchy-Riemann equation, after substracting from a_s the part that is independent of t and that is killed by Q_l.
On the other hand, applying $\tilde{Q}$ to the equation for $\tilde{w}_s$, we obtain

$$\tilde{Q}\tilde{w}_s = \tilde{A}_l\tilde{Q}\tilde{w} + B(Q_l z) + B(Q_l z)_t + C\tilde{Q}\langle P_l z, z_s\rangle_l + C'\lambda_l s\tilde{Q}\langle Mv, z_s\rangle_l.$$

Replacing z_s by its expression in the first Cauchy-Riemann equation, we obtain an equation of the form

$$\tilde{Q}\tilde{w}_s = \tilde{A}_l\tilde{Q}\tilde{w} + B(Q_l z) + B(Q_l z)_t + \lambda_l B\tilde{Q}\tilde{w}$$

where we used the fact that $\tilde{Q}\langle P_l z, Mv\rangle_l = \tilde{Q}\langle Mv, Mv\rangle_l = 0$.
From these expression for $Q_l z_s$ and $\tilde{Q}\tilde{w}_s$ we deduce the following inequalities :

$$\begin{aligned} \|Q_l z_s\|_l &\leq \|A_l Q_l z\|_l + \tilde{\epsilon}(l)\|Q_l z_t\|_l + \tilde{\epsilon}(l)\|Q_l z\|_l + \epsilon(l)C\|\tilde{Q}\tilde{w}\|, \\ \|\tilde{Q}\tilde{w}_s\| &\leq (1-\epsilon(l))\|\tilde{A}_l\tilde{Q}\tilde{w}\| + C\|Q_l z\| + C\|Q_l z_t\|, \end{aligned}$$

where $\tilde{\epsilon}(l)$ represents at term of the form $\epsilon(l) + C|\beta_l|$ following the notations of lemma 3.2.
Taking the derivative of the expression for $Q_l z_s$ with respect to s, we obtain

$$Q_l z_{ss} = A_l(Q_l z) + \tilde{\epsilon}(l)Q_l z + \tilde{\epsilon}(l)(Q_l z)_s + \tilde{\epsilon}(l)(Q_l z)_t + \tilde{\epsilon}(l)(Q_l z)_{st} + \epsilon(l)\tilde{Q}\tilde{w} + \epsilon(l)\tilde{Q}\tilde{w}_s.$$

Now consider the scalar product of this expression with $Q_l z$:

$$\begin{aligned} \frac{d^2}{ds^2}\|Q_l z\|_l &\geq \langle Q_l z, Q_l z_{ss}\rangle_l \\ &\geq \|A_l Q_l z\|_l^2 - \tilde{\epsilon}(l)\|Q_l z\|_l^2 - \tilde{\epsilon}(l)\|Q_l z_s\|_l\|Q_l z_t\|_l - \epsilon(l)\|Q_l z\|_l\|\tilde{Q}\tilde{w}\| \\ &\quad -\epsilon(l)\|Q_l z\|_l\|\tilde{Q}\tilde{w}_s\|. \end{aligned}$$

Using the inequality for $\|\tilde{Q}\tilde{w}_s\|$ on the last term, we obtain

$$\frac{d^2}{ds^2}\|Q_l z\|_l \geq c_1^2(1-\tilde{\epsilon}(l))\|Q_l z\|_l^2 - \epsilon(l)\|Q_l z\|_l\|\tilde{Q}\tilde{w}\|.$$

Next, taking the derivative of the expression for $\tilde{Q}\tilde{w}_s$ with respect to s, we obtain

$$\tilde{Q}\tilde{w}_{ss} = \tilde{A}_l\tilde{Q}\tilde{w}_s + B(Q_l z) + B(Q_l z_s) + B(Q_l z)_t + B(Q_l z)_{st} + \lambda_l B\tilde{Q}\tilde{w} + \lambda_l B\tilde{Q}\tilde{w}_s.$$

Now consider the scalar product of this expression with $\tilde{Q}\tilde{w}$:

$$\begin{aligned}
\frac{d^2}{ds^2}\|\tilde{Q}\tilde{w}\|^2 &\geq \langle\tilde{Q}\tilde{w}, \tilde{Q}\tilde{w}_{ss}\rangle \\
&\geq \|\tilde{A}_l\tilde{Q}\tilde{w}\|^2 - \epsilon(l)\|\tilde{Q}\tilde{w}\|^2 - B\|\tilde{Q}\tilde{w}\|\|Q_l z\| - B\|\tilde{Q}\tilde{w}\|\|Q_l z_s\| \\
&\quad -B\|\tilde{Q}\tilde{w}\|\|Q_l z_t\| - B\|\tilde{Q}\tilde{w}_t\|\|Q_l z_s\| - \epsilon(l)\|\tilde{Q}\tilde{w}_s\|\|\tilde{Q}\tilde{w}\|.
\end{aligned}$$

Using the inequalities for $\|Q_l z_s\|$ and $\|\tilde{Q}\tilde{w}\|$, we obtain

$$\frac{d^2}{ds^2}\|\tilde{Q}\tilde{w}\|^2 \geq c_2^2(1-\epsilon(l))\|\tilde{Q}\tilde{w}\|^2 - B\|\tilde{Q}\tilde{w}\|\|Q_l z\|.$$

Let $F_l = \|Q_l z\|$ and $G_l = \|\tilde{Q}\tilde{w}\|$. We just obtained the 2 inequalities

$$(F_l^2)'' \geq \frac{c_1^2}{2}F_l^2 - \epsilon(l)F_l G_l,$$
$$(G_l^2)'' \geq \frac{c_2^2}{2}G_l^2 - B\,F_l G_l.$$

Let $H_l = F_l^2 + \delta^2 G_l^2$ for some $\delta > 0$. Combining the above inequalities, we obtain

$$\begin{aligned}
H_l'' &\geq cH_l - (\epsilon(l) + \delta^2 B)F_l G_l \\
&\geq cH_l - \frac{\epsilon(l)+\delta^2 B}{2\delta}H_l
\end{aligned}$$

where $c = \frac{\min(c_1^2, c_2^2)}{2}$. In order to obtain the desired inequality for H_l'', we need to choose $\delta > 0$ so that $\frac{\epsilon(l)+\delta^2 B}{2\delta} < c$. Clearly, for l_0 sufficiently large, $\epsilon(l_0)$ will be small enough so that we can choose a small $\delta > 0$ satisfying this condition. Then, for

$\epsilon(l) < \epsilon(l_0)$, then condition will still be satisfied with the same $\delta > 0$. $\square$

Therefore, under the assumptions of lemma 4.17, the function H_l satisfies an estimate

$$H_l(s) \leq \max(H_l(a_l), H_l(b_l)) \frac{\cosh(c(s - \frac{a_l+b_l}{2})}{\cosh(c\frac{a_l-b_l}{2})}$$

where $\beta_l(s) \leq \bar{\beta}$ for all $s \in [a_l, b_l]$. After extracting a subsequence, we can assume that $\max(H_l(a_l), H_l(b_l)) \to 0$ for $l \to \infty$.

Next, let us generalize the estimates of lemma 3.3 in order to understand the behavior of z_{in}.

Lemma 4.18. *For $l \geq l_0$ and $s \in I_l$ satisfying $|\beta_l(s)| \leq \bar{\beta}$, we have*

$$|\langle z_{in}(s) - \varphi_{\lambda_l(s-a_l)}^{f_T/(1+\lambda_l f_T)} z_{in}(a_l), e\rangle_l| \leq \frac{4d'}{c}\sqrt{\max(H_l(a_l), H_l(b_l))}.$$

Proof. Let e be a constant unit vector in some of the z_{in} directions. Taking the scalar product of the first Cauchy-Riemann equation with e, we obtain

$$\frac{d}{ds}\langle z, e\rangle + \langle Mz_t, e\rangle + \langle S' z_{out}, e\rangle + \langle \lambda_l B\tilde{Q}\tilde{w}, e\rangle - \langle \frac{\lambda_l}{(1+\lambda_l f_T)^2} Mv, e\rangle = 0.$$

The second and third terms are estimated as in lemma 3.3. The fourth term is estimated using the upper bound of H_l. We then obtain

$$|\langle \frac{d}{ds} z_{in} - \frac{\lambda_l}{(1+\lambda_l f_T)^2}\nabla f_T, e\rangle| \leq d'\sqrt{\max(H_l(a_l), H_l(b_l))}\sqrt{\frac{\cosh(c(s - \frac{a_l+b_l}{2}))}{\cosh(c\frac{a_l-b_l}{2})}}.$$

Integrating with respect to s, we obtain, as in lemma 3.3,

$$\begin{aligned} |\langle z_{in}(s) - \varphi_{\lambda_l(s-a_l)}^{f_T/(1+\lambda_l f_T)} z_{in}(a_l), e\rangle| &\leq \frac{4d'}{c}\sqrt{\max(H_l(a_l), H_l(b_l))}\frac{\sinh(c\frac{a_l-b_l}{4})}{\sqrt{\cosh(c\frac{a_l-b_l}{2})}} \\ &\leq \frac{4d'}{c}\sqrt{\max(H_l(a_l), H_l(b_l))}. \end{aligned}$$

$\square$

With these results, we can now prove the generalized compactness theorem.

Proof of proposition 4.16. Let p_n be a point in the thin part of Σ_n. Choose cylindrical coordinates (s, t) near p_n so that $s = 0$ and $t = 0$ at p_n. We can extract a subsequence so that $\tilde{u}_n$ converges to a vertical cylinder over a closed Reeb orbit $\gamma \in S_T$. Fix a small neighborhood of γ where we have the coordinates of lemma 3.1. Let I_n be the largest interval such that the assumptions of lemma 4.17 and 4.18 are satisfied for $s \in I_n$. As a consequence of these lemmas, we obtain that $\tilde{u}_n$ converges to a cylinder in N_T over a fragment of gradient trajectory of f_T in S_T. Repeating this argument with the closed orbits at the endpoints of the obtained trajectory, we enlarge the fragment of gradient trajectory. Indeed, the size of the tubular neighborhood of a Reeb orbit such that we can apply lemmas 4.17 and 4.18 is bounded away from zero on a given orbit space.

Proceeding this way, we obtain a maximal gradient flow trajectory, i.e. such that for every choice of p_n as above, $\tilde{u}_n$ converges to a vertical cylinder over a closed Reeb orbit on the obtained trajectory. It follows that the endpoints of the maximal trajectory coincide with the closed Reeb orbits to which the upper and lower levels of the holomorphic curve converge.

Finally, let us show that the gradient trajectories between 2 given sublevels have the same length. By contradiction, assume that we have lengths l_1 and l_2 with $l_1 < l_2$. Then, it is always possible to find t_n such that one portion of cylinder of $\tilde{u}_n$ near $a_n = t_n$ converges to a portion of gradient trajectory (of total length l_2), but another portion does not. Hence, by lemma 4.18, the second portion of cylinder cannot be contained in a tubular neighborhood of a closed Reeb orbit. Therefore, this portion of cylinder must converge to a J-holomorphic cylinder with positive area. But this contradicts the fact that all positive area fragments of the limit were already taken care of. Therefore, we must have $l_1 = l_2$. □

Chapter 5

Fredholm theory

5.1 Fredholm property

5.1.1 Banach manifold with exponential weights

Let us reformulate the results of chapter 3. First, we know that a finite energy holomorphic map on a genus g Riemann surface Σ with $s^+ + s^-$ punctures to the symplectization of (M, α) is converging to closed Reeb orbits near a puncture. Let $x_1^+, \dots, x_{s^+}^+$ be the punctures so that the map converges to Reeb orbits $\gamma_1^+ \in S_1^+, \dots, \gamma_{s^+}^+ \in S_{s^+}^+$ for $t \to +\infty$ and $x_1^-, \dots, x_{s^-}^-$ be the punctures so that the map converges to Reeb orbits $\gamma_1^- \in S_1^-, \dots, \gamma_{s^-}^- \in S_{s^-}^-$ for $t \to -\infty$. Let $\mathcal{B}_k^{p,d}(g; S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$ be the Banach manifold of maps $\tilde{u} : \Sigma_{\tilde{u}} \to \mathbb{R} \times M$ with the prescribed asymptotics at the punctures, which are locally in L_k^p and so that, near each puncture, $a(s,t) - Ts - a_0, \vartheta(s,t) - t - \vartheta_0, z(s,t) \in L_k^{p,d} = \{f(s,t) | f(s,t)e^{d|s|/p} \in L_k^p\}$.

Corollary 5.1. *If $\tilde{u} : (\Sigma_{\tilde{u}}, j) \to (\mathbb{R} \times M, J)$ is holomorphic and has finite energy, then $\tilde{u} \in \mathcal{B}_k^{p,d}(g; S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$ for some choice of orbit spaces $S_i^{\pm}$, for all $k \geq 0$, all $p > 2$ and $0 < d < r$.*

Let us now define a Banach bundle $\mathcal{E}$ over $\mathcal{B}_k^{p,d}(g; S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$ so that the Cauchy-Riemann operator can be considered as a section $\overline{\partial}_J$ of $\mathcal{E}$. The fiber $\mathcal{E}_{\tilde{u}}$ over $\tilde{u}$ will be the Banach space $L_{k-1}^{p,d}(\Lambda^{0,1}(\tilde{u}))$ of $L_{k-1}^{p,d}$ $(0,1)$-forms over Σ with values in $E = \tilde{u}^* T(\mathbb{R} \times M)$. In order to keep notation simple, we will also use $L_k^{p,d}(\tilde{u})$ for the $L_k^{p,d}$ sections of E.

We want to allow the conformal structure of the Riemann surface to vary, therefore we have to consider the enlarged Banach manifold $\tilde{\mathcal{B}} = \mathcal{T}_{g,s^+ + s^-} \times \mathcal{B}$ where $\mathcal{T}_{g,s^+ + s^-}$ is the Teichmüller space for Riemann surfaces of genus g and $s^+ + s^-$ punctures (or marked points). Similarly, we have a Banach bundle $\tilde{\mathcal{E}}$ over $\tilde{\mathcal{B}}$ and the Cauchy-Riemann operator induces a section $\overline{\partial}_J : \tilde{\mathcal{B}} \to \tilde{\mathcal{E}}$.

The zero set $\overline{\partial}_J^{-1}(0)$ of the Cauchy-Riemann section $\overline{\partial}_J : \tilde{\mathcal{B}} \to \tilde{\mathcal{E}}$ is the set of holomorphic maps. We will consider holomorphic maps modulo the following equivalence relation : $\tilde{u} : (\Sigma_{\tilde{u}}, j) \to (\mathbb{R} \times M, J)$ is equivalent to $\tilde{u}' : (\Sigma_{\tilde{u}'}, j') \to (\mathbb{R} \times M, J)$ if there exists a biholomorphism $h : (\Sigma_{\tilde{u}}, j) \to (\Sigma_{\tilde{u}'}, j')$ such that $\tilde{u} = \tilde{u}' \circ h$ and $h(x_i^\pm) = x_i^{\pm\prime}$ $(i = 1, \ldots, s^\pm)$.

The moduli space of holomorphic maps consists of the equivalence classes in $\overline{\partial}_J^{-1}(0) \subset \tilde{\mathcal{B}}$; we will denote it by

$$\mathcal{M}^A_{g,s^+,s^-}(S_1^+, \ldots, S_{s^+}^+; S_1^-, \ldots, S_{s^-}^-)$$

where $A \in H_2(M)$ is the homology class of the holomorphic curves, defined as in section 2.1. It follows by elliptic regularity that holomorphic maps are smooth, therefore the moduli space is independent of the values of p, k and d as long as they satisfy the assumptions of corollary 5.1.

5.1.2 Linearized operator

Consider the linearization of the section $\overline{\partial}_J : \mathcal{B} \to \mathcal{E}$ near some $\tilde{u} \in \mathcal{B}$. We obtain a linear operator $\overline{\partial}_{\tilde{u}}$ between Banach spaces of sections of E :

$$\overline{\partial}_{\tilde{u}} : T_{\tilde{u}}\mathcal{B} \to \mathcal{E}_{\tilde{u}} = L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u})).$$

Near a puncture $x_i^{\pm}$, this operator is given by

$$\frac{\partial}{\partial s} + J_0 \frac{\partial}{\partial t} + S_i^{\pm}(s,t)$$

where (s,t) are cylindrical coordinates, J_0 is the standard complex structure and $S_i^{\pm}(s,t)$ are symmetric matrices. Let us define $S_i^{\pm}(t) = \lim_{s\to\pm\infty} S_i^{\pm}(s,t)$.

We identify $T_{\tilde{u}}\mathcal{B}$ with $\mathbb{R}^N \oplus L_k^{p,d}(\tilde{u})$, where $N = \sum_{i=1}^{s^+}(\dim S_i^+ + 2) + \sum_{j=1}^{s^-}(\dim S_j^- + 2)$. The terms 2 account for $\dim \operatorname{span}(\frac{\partial}{\partial t}, R_\alpha)$. Let $v_j^{(\pm,i)}$ $(j = 1,\dots,\dim S_i^{\pm} + 2)$ be a basis of solutions for the equation $J_0\partial_t v + S_i^{\pm}(t)v = 0$ on the circle. Let $\rho_i^{\pm}$ be a function with support in a small neighborhood of $x_i^{\pm}$, depending only on s and equal to 1 for $\pm s$ large. Then the summand $\mathbb{R}^N$ is spanned by the functions $\rho_i(s)v_j^{(\pm,i)}(t)$.

Note that, because of the exponential behavior of $\tilde{u}$, the linear operator is exponentially converging to its asymptotic value at each puncture. Hence, the image of the $\mathbb{R}^N$ summand in the domain is contained in $L_{k-1}^{p,d}$.

Proposition 5.2. *The linear operator*

$$\overline{\partial}_{\tilde{u}} : \mathbb{R}^N \oplus L_k^{p,d}(\tilde{u}) \to L_{k-1}^{p,d}(\Lambda^{0,1}(\tilde{u}))$$

is Fredholm.

Proof. Clearly, using the elementary properties of Fredholm operators, it is equivalent to prove the Fredholm property for $\overline{\partial}_1 = \overline{\partial}_{\tilde{u}|L_k^{p,d}(\tilde{u})}$. Moreover, let

$$\varphi : L_k^{p,d}(\tilde{u}) \to L_k^p(\tilde{u}) \text{ and } \varphi' : L_k^{p,d}(\Lambda^{0,1}(\tilde{u})) \to L_k^p(\Lambda^{0,1}(\tilde{u}))$$

be the multiplication by $e^{d|s|/p}$; clearly, this map is an isomorphism for every k. Let now $\overline{\partial}' = \varphi' \circ \overline{\partial}_1 \circ \varphi^{-1}$. Given the explicit expression $\overline{\partial}_1 = \partial_s + J_0\partial_t + S_i^{\pm}(s,t)$ near the puncture $x_i^{\pm}$, we obtain the expression $\overline{\partial}' = \partial_s + J_0\partial_t + S_i^{\pm}(s,t) \mp d/p$. Note that with these perturbed matrices near the ends, the operator $\overline{\partial}' : L_k^p(\tilde{u}) \to L_{k-1}^p(\Lambda^{0,1}(\tilde{u}))$ has nondegenerate asymptotics. Hence, the usual Fredholm theory applies to $\overline{\partial}'$. Finally, $\overline{\partial}_1$ is Fredholm as well since it is conjugate to a Fredholm operator. $\square$

5.2 Fredholm index

5.2.1 Reduction to Riemann-Roch

The Fredholm index of holomorphic cylinders in Floer homology is classically computed using the analysis of the spectral flow [30]. Later, a gluing method was used to extend these results to holomorphic curves with different topologies [32]. The strategy was to glue punctured holomorphic curves in order to obtain a compact Riemann surface, and then apply the Riemann-Roch theorem. However, those results were still dependent of the original computation using the spectral flow.
We know explain a refinement of the gluing construction that completely reduces the index computation to the Riemann-Roch theorem, without using the spectral flow techniques.

Let $\overline{\partial} : L^p_k(E) \to L^p_{k-1}(\Lambda^{0,1}(E))$ be a linearized Cauchy-Riemann operator for sections of a hermitian vector bundle E over a punctured Riemann surface Σ, with non-degenerate asymptotics. We already know by classical elliptic results that this operator is Fredholm.
Near a puncture $x_i^\pm$, the operator $\overline{\partial}$ has the form $\partial_s + J_0\partial_t + S_i^\pm(s,t)$ where J_0 is the standard complex structure, in an appropriate trivialization. We can assume without loss of generality that, for $\pm s$ large enough, $S_i^\pm(s,t) = S_i^\pm(t)$ is a loop of symmetric matrices. Let $\Psi_i^\pm(t)$ be the path of symplectic matrices defined by $\dot{\Psi}_i^\pm(t) = J_0 S_i^\pm(t)\Psi_i^\pm(t)$ and $\Psi_i^\pm(0) = I$.
Let us now construct another linearized Cauchy-Riemann operator $\overline{\partial}$ for sections of E, almost identical to $\overline{\partial}$, but with slightly different asymptotics.
Note that $\Psi_i^\pm(1)$ and its inverse can be joined by a path of symplectic matrices such that 1 is never an eigenvalue. Indeed, $Sp^*(2n) = \{\Psi \in Sp(2n) \,|\, \det(\Psi - I) \neq 0\}$ has exactly 2 connected components, distinguished by the sign of $\det(\Psi - I)$, and $\det(\Psi - I) = \det(\Psi)\det(I - \Psi^{-1}) = \det(\Psi^{-1} - I)$. Therefore, we can homotope $\Psi_i^\pm(t)$ to a path $\tilde{\Psi}_i^\pm(t)$ with the same Maslov index and satisfying $\tilde{\Psi}_i^\pm(0) = \Psi_i^\pm(0) = I$ and $\tilde{\Psi}_i^\pm(1) = \Psi_i^\pm(1)^{-1}$. Define $\tilde{S}_i^\pm(t)$ by $\dot{\tilde{\Psi}}_i^\pm(t) = J_0\tilde{S}_i^\pm(t)\tilde{\Psi}_i^\pm(t)$.
Near the puncture $x_i^\pm$, the operator $\overline{\partial}'$ will have the form $\partial_s + J_0\partial_t + \tilde{S}_i^\pm(s,t)$ with

$\tilde{S}(s,t) = \tilde{S}_\infty(t)$ for $\pm s$ large enough. It is clear that $\overline{\partial}'$ and $\overline{\partial}$ are in the same path component of the set of Fredholm operators, and therefore have the same index.

We now want to glue two copies of the Riemann surface Σ. Near each puncture $x_i^\pm$, take a complex coordinate $z_i^\pm$ vanishing at the puncture. Cut out a small disk of radius $\epsilon > 0$. Then glue the two boundary circles $|z_i^\pm| = \epsilon$ on each copy of Σ by identifying $z_i^\pm$ on one copy of Σ with $\frac{\epsilon^2}{z_i^\pm}$ on the other copy of Σ. If Σ has genus g and $s^+ + s^-$ punctures, then the glued Riemann surface $\Sigma\sharp\Sigma$ has genus $2g + s^+ + s^- - 1$.

Next, we want to glue the vector bundles E over each copy on Σ in order to obtain a vector bundle $E\sharp E$ over $\Sigma\sharp\Sigma$. To that end, define a clutching function $\Phi_i^\pm$ in a neighborhood of $|z_i^\pm| = \epsilon$ by $\Phi_i^\pm(t) = \Psi_i^\pm(t)\tilde{\Psi}_i^\pm(1-t)^{-1}$, where $t = \mp\frac{\arg z}{2\pi}$. This is well defined, since $\Phi_i^\pm(0) = \Phi_i^\pm(1)$ by construction. An elementary computation shows that the operator $\overline{\partial}$ and $\overline{\partial}'$ then match over the glued boundary and give a glued operator $\overline{\partial}\sharp\overline{\partial}'$.

The first Chern class of the glued vector bundle $E\sharp E$ is given by $c_1(E\sharp E) = 2c_1(E) + \sum_{i=1}^{s^+} \mu(\Phi_i^+) - \sum_{j=1}^{s^-} \mu(\Phi_j^-)$, where $\mu(\Phi_i^\pm)$ is the Maslov index of the loop of symplectic matrices $\Phi_i^\pm(t)$. Let us compute this in terms of the Conley-Zehnder index of $\Psi_i^\pm$. The loop $\Psi_i^\pm(t)\tilde{\Psi}_i^\pm(1-t)^{-1}$ is homotopic to the catenation of the paths $\Psi_i^\pm(t)$ and $\tilde{\Psi}_i^\pm(1-t)^{-1}$. By the catenation property of the Conley-Zehnder index, we then have

$$\begin{aligned} 2\mu(\Phi_i^\pm) &= \mu_C Z(\Psi_i^\pm(t)) - \mu_{CZ}(\tilde{\Psi}_i^\pm(t)^{-1}) \\ &= \mu_C Z(\Psi_i^\pm(t)) + \mu_{CZ}(\tilde{\Psi}_i^\pm(t)) \\ &= 2\mu_C Z(\Psi_i^\pm). \end{aligned}$$

Hence, by the Riemann-Roch theorem, the Fredholm index of the glued operator $\overline{\partial}\sharp\overline{\partial}'$ is given by

$$n(2 - 4g - 2s^+ - 2s^- + 2) + 4c_1(E) + \sum_{i=1}^{s^+} 2\mu_{CZ}(\Psi_i^+) - \sum_{j=1}^{s^-} 2\mu_{CZ}(\Psi_j^-).$$

Now we use theorem 3.2.12 of [32], stating that the Fredholm index is additive under the gluing operation we used above. This theorem was proved using standard analytic estimates, and is completely independent of the spectral flow.

Since, $\overline{\partial}$ and $\overline{\partial}'$ have the same Fredholm index, we just need to divide the above expression by 2 in order to obtain

$$\text{index}(\overline{\partial}) = n(2 - 2g - s^+ - s^-) + 2c_1(E) + \sum_{i=1}^{s^+} \mu_{CZ}(\Psi_i^+) - \sum_{j=1}^{s^-} \mu_{CZ}(\Psi_j^-).$$

This is the classical formula that was derived already in [32].

5.2.2 The generalized Maslov index

The Fredholm index of the $\overline{\partial}$ operator is usually computed in terms of the Conley-Zehnder index corresponding to the asymptotic conditions. Here however, those asymptotics are degenerate, so the Conley-Zehnder index is not defined. Robbin and Salamon [29] introduced a Maslov index for general paths of symplectic matrices. Let $\Psi(t)$ be a path of symplectic matrices such that $\Psi(0) = I$; assume that there are a finite number of values of t $(0 < t < 1)$, $t_1, \ldots, t_l$, called crossings, such that $V_t = \ker(\Psi(t) - I) \neq 0$, and that $J_0 \frac{d}{dt}\Psi(t)$, the crossing form, is nondegenerate on V_t. Denote the signature of that symmetric form by $\sigma(t)$. Then, the Maslov index $\mu(\Psi)$ can be defined by :

$$\mu(\Psi) = \frac{1}{2}\sigma(0) + \sum_{i=1}^{l} \sigma(t_i) + \frac{1}{2}\sigma(1)$$

where $\sigma(1)$ is defined to be zero if $\Psi(1) - I$ is invertible. Then, the Maslov index is half-integer valued, invariant under homotopy with fixed ends, additive under catenation of paths, and $\mu(\Psi) + \frac{1}{2}\text{dim}V_1 \in \mathbb{Z}$.

5.2.3 Computation of the index

The index of $\overline{\partial}$ is given by $\text{index}(\overline{\partial}_1) + N$, by elementary properties of the Fredholm index. Now, $\overline{\partial}_1$ and $\overline{\partial}'$ have the same Fredholm index, since they are conjugate to each other. Using the index formula from section 5.2.1, we have :

$$\mathrm{index}(\overline{\partial}') = n(2-2g-s^+-s^-) + 2c_1(E) + \sum_{i=1}^{s^+} \mu_{CZ}({\Psi^+}'_i) - \sum_{j=1}^{s^-} \mu_{CZ}({\Psi^-}'_j).$$

In this equation, the paths of matrices ${\Psi^{\pm}}'_i$ are the solutions of

$$\left\{ \begin{array}{rcl} J_0 \partial_t {\Psi^{\pm}}'_i(t) + (S_i^{\pm}(s,t) \mp d/p){\Psi^{\pm}}'_i(t) & = & 0, \\ {\Psi^{\pm}}'_i(0) & = & I. \end{array} \right.$$

The paths of symplectic matrices ${\Psi^{\pm}}'_i$ for $\overline{\partial}'$ are related to the paths of symplectic matrices $\Psi_i^{\pm}$ for $\overline{\partial}$ in the following way :

$${\Psi^{\pm}}'(t) = \left\{ \begin{array}{ll} \Psi^{\pm}(2t) & \text{for } t \leq 1/2, \\ \Psi^{\pm}(1) e^{\mp d(2t-1)/p} & \text{for } t > 1/2. \end{array} \right.$$

For $d > 0$ sufficiently small, there will be no crossing for $t > 1/2$. At $t = 1/2$, however, there is a crossing with crossing form $\mp d/pI$. Using the definition of the Maslov index and its catenation property, we deduce that $\mu_{CZ}({\Psi^{\pm}}') = \mu(\Psi^{\pm}) \mp \frac{1}{2}\dim\ker(\Psi^{\pm} - I)$. Hence, substituting into the index formula for $\overline{\partial}'$, we obtain :

$$\mathrm{index}(\overline{\partial}_1) = n(2-2g-s^+-s^-) + 2c_1(E) + \sum_{i=1}^{s^+} \mu(\Psi_i^+) - \sum_{j=1}^{s^-} \mu(\Psi_j^-) - \frac{1}{2}N.$$

Therefore we have proved

Proposition 5.3. *The Fredholm index of the linear operator*

$$\overline{\partial} : \mathbb{R}^N \oplus L_k^{p,d}(E) \to L_{k-1}^{p,d}(\Lambda^{0,1}(E))$$

is given by the formula

$$n(2-2g-s^+-s^-) + 2c_1(E) + \sum_{i=1}^{s^+} \mu(\Psi_i^+) - \sum_{j=1}^{s^-} \mu(\Psi_j^-) + \frac{1}{2}N.$$

If we apply this formula to the linearization of the Cauchy-Riemann operator at some $\tilde{u} \in \mathcal{B}_k^{p,d}(g; S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$ then we have $\mu(\Psi_i^+) = \mu(S_i^+)$, $\mu(\Psi_j^-) = \mu(S_j^-)$ and since $N = \sum_{i=1}^{s^+}(\dim S_i^+ + 2) + \sum_{j=1}^{s^-}(\dim S_j^- + 2)$, we obtain :

$$n(2-2g) - (n-1)(s^+ + s^-) + 2c_1(E) + \sum_{i=1}^{s^+} \mu(S_i^+) + \tfrac{1}{2}\sum_{i=1}^{s^+} \dim S_i^+ \\ - \sum_{j=1}^{s^-} \mu(S_j^-) + \tfrac{1}{2}\sum_{j=1}^{s^-} \dim S_t^-.$$

But for the moduli space we have to consider instead the Banach manifold

$$\mathcal{T}_{g,s^+ + s^-} \times \mathcal{B}_k^{p,d}(g; S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-).$$

Adding $\dim \mathcal{T}_{g,s^+ + s^-} = 6g - 6 + 2(s^+ + s^-)$ to the above index formula, we obtain

Corollary 5.4. *The predicted dimension of the moduli space of holomorphic maps*

$$\mathcal{M}_{g,s^+,s^-}^A(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$$

is given by

$$(n-3)(2-2g-s^+-s^-)+2c_1(A)+\sum_{i=1}^{s^+} \mu(S_i^+)+\frac{1}{2}\sum_{i=1}^{s^+} \dim S_i^+ -\sum_{j=1}^{s^-} \mu(S_j^-)+\frac{1}{2}\sum_{j=1}^{s^-} \dim S_t^-.$$

Note that, if $6g - 6 + 2(s^+ + s^-) \leq 0$, the Teichmüller space is trivial but the automorphism group of (Σ, j) fixing the punctures has dimension $6 - 6g - 2(s^+ + s^-)$. Otherwise, this group is discrete.

5.3 Gluing estimates

In this section, we derive the estimates that are necessary to study the structure of the moduli space near a split (generalized) holomorphic map.

5.3.1 Holomorphic maps

We first work out the estimates for holomorphic maps. Let $\tilde{u} \in \mathcal{B}_k^{p,d}(g; S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-)$ and $\tilde{v} \in \mathcal{B}_k^{p,d}(g'; S_1^{'+}, \dots, S_{s'^+}^{'+}; S_1^{'-}, \dots, S_{s'^-}^{'-})$ such that $\tilde{u}$ and $\tilde{v}$ are asymptotic to the same Reeb orbit $\gamma_i \in S_{s^-+1-i}^- = S_i^{'+}$ for $i = 1, \dots, t \leq \min(s^-, s'^+)$. We want to construct a glued map

$$\tilde{u}\sharp_R\tilde{v} \in \mathcal{B}_k^{p,d}(g + g' + t - 1; S_1^+, \dots, S_{s^+}^+, S_{t+1}^{'+}, \dots, S_{s'^+}^{'+}; S_1^-, \dots, S_{s^--t}^-, S_1^{'-}, \dots, S_{s'^-}^{'-}).$$

First, we construct a glued Riemann surface $\Sigma_R = \Sigma_{\tilde{u}}\sharp_R\Sigma_{\tilde{v}}$, for $R \in \mathbb{R}^+$ sufficiently large. Take cylindrical coordinates (s_i, t_i) near $x_{s^-+1-i}^-$ and (s_i', t_i') near $y_i^{'+}$, for $i = 1, \dots, t$. Pick a Riemannian metric g on M; we could choose the metric induced by ω and J. We have

$$\begin{aligned} a_{\tilde{u}}(s_i, t_i) &= Ts_i + a_{\tilde{u}}^0 + \eta(s_i, t_i), \\ a_{\tilde{v}}(s_i', t_i') &= Ts_i' + a_{\tilde{v}}^0 + \eta'(s_i', t_i'), \\ u(s_i, t_i) &= \exp_{\gamma_i(t_i)}(U_i(s_i, t_i)), \\ v(s_i', t_i') &= \exp_{\gamma_i(t_i')}(V_i(s_i', t_i')), \end{aligned}$$

where U_i and η are decaying exponentially for $s_i \to -\infty$ and V_i and η' are decaying exponentially for $s_i' \to +\infty$.

For $R > R_0$, cut out the punctured disks $s_i < -\frac{R+a_{\tilde{u}}^0}{T}$ near the negative punctures and the punctured disks $s_i' > \frac{R-a_{\tilde{v}}^0}{T}$ near the positive punctures, and identify the boundaries of the remaining surfaces via $t_i = t_i'$. In each neck, we obtain cylindrical coordinates (s_i'', t_i''), where $t_i'' = t_i = t_i'$ and $s_i'' = s_i + \frac{R+a_{\tilde{u}}^0}{T} = s_i' - \frac{R-a_{\tilde{v}}^0}{T}$.

Let us define the preglued map $\tilde{u}\sharp_R\tilde{v}$ on Σ_R. Away from the necks, this map coincides with $\tilde{u}$ on $\Sigma_{\tilde{u}}$ and with $\tilde{v}$ on $\Sigma_{\tilde{v}}$. In the neck i, the map $\tilde{u}\sharp_R\tilde{v}(s_i'', t_i'')$ is given by

$$\begin{cases} (Ts_i'' + \beta(s_i'' - 1)\eta(s_i'', t_i''), \exp_{\gamma_i(t_i)}(\beta(s_i'' - 1)U_i(s_i'', t_i''))) & \text{if } s_i'' > +1, \\ (Ts'', \gamma_i(t_i)) & \text{if } -1 \leq s_i'' \leq +1, \\ (Ts_i'' + \beta(-s_i'' - 1)\eta'(s_i'', t_i''), \exp_{\gamma_i(t_i)}(\beta(-s_i'' - 1)V_i(s_i'', t_i''))) & \text{if } s_i'' < -1, \end{cases}$$

where $\beta : \mathbb{R} \to [0,1]$ be a smooth function such that $\beta(s) = 0$ if $s < 0$, $\beta(s) = 1$ if $s > 1$ and $0 \leq \beta'(s) \leq 2$.
Note that, if we vary $\tilde{u}$ and $\tilde{v}$, this pregluing map varies smoothly with the matching Reeb orbits $\gamma_i \in S^-_{s^-+1-i} = S'^+_i$.

We want to show that the glued map $\tilde{u}\sharp_R\tilde{v}$ is approximately J-holomorphic, or more precisely that $\overline{\partial}_J(\tilde{u}\sharp_R\tilde{v})$ is small in $L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}\sharp_R\tilde{v}))$. Note that on $L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}\sharp_R\tilde{v}))$, we will not use the Banach norm introduced in section 5.1, but rather the Banach norm $\|\cdot\|_R$, with the usual exponential weights near the punctures and the additional weights $e^{d(\frac{R-R_0}{T}-|s''_i|)}$ in the necks.

Lemma 5.5. *With the above Banach structure on $L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}\sharp_R\tilde{v}))$, if $\tilde{u}$ and $\tilde{v}$ are J-holomorphic, then*

$$\lim_{R\to\infty} \|\overline{\partial}_J(\tilde{u}\sharp_R\tilde{v})\| = 0.$$

Proof. Since $\overline{\partial}_J\tilde{u} = 0$ and $\overline{\partial}_J\tilde{v} = 0$, it follows from the definition of the glued map that $\overline{\partial}_J(\tilde{u}\sharp_R\tilde{v}) = 0$ except in the necks when $s''_i \in [-2,+2]$. Therefore, we have the estimate

$$\|\overline{\partial}_J(\tilde{u}\sharp_R\tilde{v})\| \leq C\sum_i (\|(\nabla U_i, \nabla\eta)\|_{[-\frac{R+a^0_{\tilde{u}}}{T}, -\frac{R+a^0_{\tilde{u}}}{T}+2]} + \|(\nabla V_i, \nabla\eta')\|_{[\frac{R-a^0_{\tilde{v}}}{T}-2, \frac{R-a^0_{\tilde{v}}}{T}]}).$$

But since $\tilde{u}$ and $\tilde{v}$ are maps in $\mathcal{B}^{p,d}_k$ with $k \geq 1$, the right hand side converges to zero when $R \to \infty$. □

Let $V_{\tilde{u}}$ be the vector space generated by the sections $\rho_i(s)v^{(-,i)}_j(t) \in C^\infty(\tilde{u})$ for $j = 1, \ldots, \dim S^-_{s^-+1-i} + 2$ and $i = 1, \ldots, t$, and $V_{\tilde{v}}$ be the vector space generated by the sections $\rho_i(s)v^{(+,i)}_j(t) \in C^\infty(\tilde{u})$ for $j = 1, \ldots, \dim S'^+_i + 2$ and $i = 1, \ldots, t$. The vector space $V_{\tilde{u}}$ is naturally a summand of $T_{\tilde{u}}\tilde{\mathcal{B}}$. Let Δ be the diagonal in $V_{\tilde{u}} \oplus V_{\tilde{v}}$.

Consider the linearized Cauchy-Riemann operators $\overline{\partial}_{\tilde{u}} : V_{\tilde{u}} \oplus L^{p,d}_k(\tilde{u}) \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}))$ and $\overline{\partial}_{\tilde{v}} : V_{\tilde{v}} \oplus L^{p,d}_k(\tilde{v}) \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{v}))$. Pick a finite dimensional subspace W of $C^\infty_0(\Lambda^{0,1}(\tilde{u})) \oplus C^\infty_0(\Lambda^{0,1}(\tilde{v})) \subset L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u})) \oplus L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{v}))$, such that

$$W + \overline{\partial}_{\tilde{u}}(L^{p,d}_k(\tilde{u})) + \overline{\partial}_{\tilde{v}}(L^{p,d}_k(\tilde{v})) + (\overline{\partial}_{\tilde{u}} \oplus \overline{\partial}_{\tilde{v}})(\Delta) = L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u})) + L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{v})).$$

We define the stabilization of $\overline{\partial}_{\tilde{u}} \oplus \overline{\partial}_{\tilde{v}}$ by

$$\begin{array}{rcl} \overline{\partial}^W_{\tilde{u},\tilde{v}} : W \oplus \Delta \oplus L^{p,d}_k(\tilde{u}) \oplus L^{p,d}_k(\tilde{v}) & \to & L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u})) \oplus L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{v})) \\ (w, (v, v), \xi_{\tilde{u}}, \xi_{\tilde{v}}) & \to & w + \overline{\partial}_{\tilde{u}}(v + \xi_{\tilde{u}}) + \overline{\partial}_{\tilde{v}}(v + \xi_{\tilde{v}}). \end{array}$$

With such a choice of W, the operator $\overline{\partial}^W_{\tilde{u},\tilde{v}}$ is surjective and has a bounded right inverse Q_∞.

Note that, on $L^{p,d}_k(\tilde{u}\sharp_R\tilde{v})$, we will not use the Banach norm introduced in section 5.1, but a modified norm. For $\xi \in L^{p,d}_k(\tilde{u}\sharp_R\tilde{v})$, we define $\bar{\xi}_i = \int_{s''_i=0} \pi_i \xi \, dt''_i$, where π_i is the orthonormal projection to $\mathbb{R} \times N_i$. We then multiply the vector $\bar{\xi}_i$ by the function $\rho_i^- \sharp_R \rho_i^+$ having support near the neck i; we denote by $\bar{\xi}$ the sum of these sections. We define the norm of ξ to be $\|\xi - \bar{\xi}\|_R + \sum_{i=1}^t |\bar{\xi}_i|$, where the Banach norm $\|\cdot\|_R$ has the usual exponential weights near the punctures and additional weights $e^{d(\frac{R-R_0}{T} - |s''_i|)}$ in the necks.

Proposition 5.6. *Assume that W is chosen so that the operator $\overline{\partial}^W_{\tilde{u},\tilde{v}}$ is surjective. Then the operator*

$$\overline{\partial}_R = \overline{\partial}^W_{\tilde{u}\sharp_R\tilde{v}} : W \oplus L^{p,d}_k(\tilde{u}\sharp_R\tilde{v}) \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}\sharp_R\tilde{v}))$$

has a uniformly bounded right inverse Q_R, if R is sufficiently large.

In order to prove this proposition, we adapt the gluing construction of McDuff and Salamon [25].

Let $\gamma_R : \mathbb{R} \to [0, 1]$ be a smooth function such that $\gamma_R(s) = 1$ for $s \geq \frac{R/2-R_0}{T}$, $\gamma_R(s) = 0$ for $s \leq 1$ and $0 \leq \frac{d}{ds}\gamma_R(s) \leq \frac{2T}{R/2-R_0-T}$ for $1 \leq s \leq \frac{R/2-R_0}{T}$.

Let us define the gluing map

$$\begin{array}{rcl} g_R : \Delta \oplus L^{p,d}_k(\tilde{u}) \oplus L^{p,d}_k(\tilde{v}) & \to & L^{p,d}_k(\tilde{u}\sharp_R\tilde{v}) \\ ((\sum_i v_i(t)\rho_i^-(s), \sum_i v_i(t)\rho_i^+(s)), \xi_{\tilde{u}}, \xi_{\tilde{v}}) & \to & \xi = \xi^0 + \sum_i v_i(t)\rho_i^- \sharp_R \rho_i^+(s) \end{array}$$

where, in the neck with coordinates (s_i'', t_i''),

$$\xi^0(s_i'', t_i'') = \begin{cases} \xi_{\tilde{u}}(s_i'', t_i'') + \gamma_R(s_i'')\xi_{\tilde{v}}(s_i'', t_i'') & \text{if } s_i'' > +1, \\ \xi_{\tilde{u}}(s_i'', t_i'') + \xi_{\tilde{v}}(s_i'', t_i'') & \text{if } -1 \leq s_i'' \leq +1, \\ \gamma_R(-s_i'')\xi_{\tilde{u}}(s, t) + \xi_{\tilde{v}}(s_i'', t_i'') & \text{if } s < -1, \end{cases}$$

and ξ^0 coincides with $\xi_{\tilde{u}}$ (resp. $\xi_{\tilde{v}}$) on the rest of $\Sigma_{\tilde{u}}$ (resp. $\Sigma_{\tilde{v}}$).

Let us define the splitting map

$$\begin{array}{rcl} s_R : L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}\sharp_R\tilde{v})) & \to & L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u})) \oplus L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{v})) \\ \eta & \to & (\eta_{\tilde{u}}, \eta_{\tilde{v}}) \end{array}$$

where, near the puncture $x^-_{s^-+1-i}$ (resp. $y_i^{'+}$),

$$\left\{ \begin{array}{rcl} \eta_{\tilde{u}}(s_i, t_i) & = & \beta(s_i'')\eta(s_i'', t_i''), \\ \eta_{\tilde{v}}(s_i', t_i') & = & (1 - \beta(s_i''))\eta(s_i'', t_i''), \end{array} \right.$$

and $\eta_{\tilde{u}}, \eta_{\tilde{v}}$ coincide with η away from these punctures.
Note that the operators g_R and s_R are uniformly bounded in R.

Let us define an approximate right inverse $\tilde{Q}_R$ for $\overline{\partial}_R$ using the following commutative diagram :

$$\begin{array}{ccc} L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}\sharp_R\tilde{v})) & \xrightarrow{\tilde{Q}_R} & W \oplus W' \oplus L^{p,d}_k(\tilde{u}\sharp_R\tilde{v}) \\ {\scriptstyle s_R}\big\downarrow & & \big\uparrow{\scriptstyle g_R} \\ L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u})) \oplus L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{v})) & \xrightarrow{Q_\infty} & W \oplus W' \oplus \Delta \oplus L^{p,d}_k(\tilde{u}) \oplus L^{p,d}_k(\tilde{v}). \end{array}$$

Note that $\tilde{Q}_R$ is uniformly bounded in R, since g_R and s_R are.

Proof of proposition 5.6. By construction, $\overline{\partial}_R \tilde{Q}_R \eta = \eta$ away from the necks. On the other hand, in the neck i, we have

$$\begin{aligned}
\overline{\partial}_R \tilde{Q}_R \eta &= \overline{\partial}_R \xi \\
&= \overline{\partial}_R (v_i(t) + \gamma_R(s_i'')\xi_{\tilde{u}} + \gamma_R(-s_i'')\xi_{\tilde{v}}) \\
&= \gamma_R(s_i'')\overline{\partial}_{\tilde{u}}\xi_{\tilde{u}} + \gamma_R(-s_i'')\overline{\partial}_{\tilde{v}}\xi_{\tilde{v}} + \frac{d}{ds}\gamma_R(s_i'')\xi_{\tilde{u}} - \frac{d}{ds}\gamma_R(-s_i'')\xi_{\tilde{v}} \\
&\quad + \gamma_R(s_i'')(\overline{\partial}_R - \overline{\partial}_{\tilde{u}})\xi_{\tilde{u}} + \gamma_R(-s_i'')(\overline{\partial}_R - \overline{\partial}_{\tilde{v}})\xi_{\tilde{v}}.
\end{aligned}$$

But

$$\begin{aligned}
\gamma_R(s_i'')\overline{\partial}_{\tilde{u}}\xi_{\tilde{u}} + \gamma_R(-s_i'')\overline{\partial}_{\tilde{v}}\xi_{\tilde{v}} &= \gamma_R(s_i'')\eta_{\tilde{u}} + \gamma_R(-s_i'')\eta_{\tilde{v}} \\
&= \eta_{\tilde{u}} + \eta_{\tilde{v}} \\
&= \eta
\end{aligned}$$

because $\gamma_R(s) = 1$ (resp. $\gamma_R(-s) = 1$) when $\eta_{\tilde{u}}$ (resp. $\eta_{\tilde{v}}$) is not zero.
Therefore,

$$|\overline{\partial}_R \tilde{Q}_R \eta - \eta| \leq \frac{2T}{R/2 - R_0 - T}(|\xi_{\tilde{u}}| + |\xi_{\tilde{v}}|) + \|A_{\tilde{u}}\||\xi_{\tilde{u}}| + \|A_{\tilde{v}}\||\xi_{\tilde{v}}|$$

where $A_{\tilde{u}} = \overline{\partial}_R - \overline{\partial}_{\tilde{u}}$ and $A_{\tilde{v}} = \overline{\partial}_R - \overline{\partial}_{\tilde{v}}$ are matrices. Note that, because of the (exponential) convergence of $\tilde{u}$ and $\tilde{v}$ to closed Reeb orbits, the norms of these matrices uniformly converge to zero when $R \to \infty$.
Hence, we can rewrite the above pointwise estimate as

$$|\overline{\partial}_R \tilde{Q}_R \eta - \eta| \leq C(R)(\gamma_R(s_i'')|\xi_{\tilde{u}}| + \gamma_R(-s_i'')|\xi_{\tilde{v}}|)$$

where $C(R)$ is a constant depending only on R such that $\lim_{R\to\infty} C(R) = 0$.
We now have to integrate this estimate on the neck i, using the appropriate weight $e^{d(\frac{R-R_0}{T} - |s_i''|)}$ for norm $\|\cdot\|_R$. We obtain

$$\|\overline{\partial}_R \tilde{Q}_R \eta - \eta\| \leq C(R)(\|\xi_{\tilde{u}}\| + \|\xi_{\tilde{v}}\|).$$

Therefore, if R is sufficiently large, then

$$\|\overline{\partial}_R \tilde{Q}_R - I\| \leq \frac{1}{2}.$$

Hence, the operator $\overline{\partial}_R \tilde{Q}_R$ is invertible. Let $Q_R = \tilde{Q}_R(\overline{\partial}_R \tilde{Q}_R)^{-1}$. By construction, Q_R is a right inverse for $\overline{\partial}_R$, and it is uniformly bounded in R. □

We now turn to the Fredholm section defined by the linearized Cauchy-Riemann operators over the extended spaces $\tilde{\mathcal{B}}$, including changes in the conformal structure. Let us denote by S the subspace of $V_{\tilde{u}} \simeq V_{\tilde{v}}$ spanned by the sections $\rho_i(s)v_j^{(\pm,i)}(t)$ that are tangent to the orbit spaces. Then we have natural maps $\pi_{\tilde{u}} : T_{\tilde{u}}\mathcal{B} \to S$ and $\pi_{\tilde{v}} : T_{\tilde{v}}\mathcal{B} \to S$.

Corollary 5.7. *Suppose that the operators* $\overline{\partial}_{\tilde{u}}^{W} : W \oplus T_{\tilde{u}}\tilde{\mathcal{B}} \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}))$ *and* $\overline{\partial}_{\tilde{v}}^{W'} : W' \oplus T_{\tilde{v}}\tilde{\mathcal{B}} \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{v}))$ *are surjective, and that* $\pi_{\tilde{u}}(\ker \overline{\partial}_{\tilde{u}}^{W}) + \pi_{\tilde{v}}(\ker \overline{\partial}_{\tilde{v}}^{W'}) = S$. *Then, the operator*

$$\overline{\partial}_{\tilde{u}\sharp_R\tilde{v}}^{W\oplus W'} : W \oplus W' \oplus T_{\tilde{u}\sharp_R\tilde{v}}\tilde{\mathcal{B}} \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}\sharp_R\tilde{v}))$$

has a uniformly bounded right inverse Q_R, if R is sufficiently large.

Proof. In view of proposition 5.6, we just have to show that the restriction of the operator $\overline{\partial}_{\tilde{u}}^{W} \oplus \overline{\partial}_{\tilde{v}}^{W'}$ to $W \oplus W' \oplus N \oplus (T_{\tilde{u}}\tilde{\mathcal{B}} \oplus_V T_{\tilde{v}}\tilde{\mathcal{B}})$ is surjective, where N denotes the orthogonal complement to $T_{\tilde{u}}\mathcal{T} \oplus T_{\tilde{v}}\mathcal{T}$ in $T_{\tilde{u}\sharp_R\tilde{v}}\mathcal{T}$.
The summand N has 2 real dimensions for each glued pair of punctures. Those degrees of freedom correspond to varying the radius ϵ and the angle of identification when gluing $\Sigma_{\tilde{u}}$ and $\Sigma_{\tilde{v}}$.
The angular degree of freedom amounts to replace the complex structure $j_{\tilde{v}}$ on $\Sigma_{\tilde{v}}$ with $\phi_{\mu*}j_{\tilde{v}}$, where $\phi_\mu(s,t) = (s, t + \mu\rho(s))$ and ρ is a decreasing function that vanishes

for s large and is equal to 1 for s small. We then compute

$$\begin{aligned}\frac{d}{d\mu}|_{\mu=0}(J\circ d\tilde{v}\circ\phi_{\mu*}\circ j\partial_s+d\tilde{v}\partial_s) &= J\circ d\tilde{v}\circ(\mathcal{L}_{\rho\partial_t}j)\partial_s\\ &= -J\circ d\tilde{v}\circ j[\rho\partial_t,\partial_s]\\ &= \rho' J\circ d\tilde{v}\circ j\partial_t\\ &= -\rho' R_\alpha.\end{aligned}$$

The radial degree of freedom amounts to replace the complex structure $j_{\tilde{v}}$ on $\Sigma_{\tilde{v}}$ with $\psi_{\mu*}j_{\tilde{v}}$, where $\psi_{\mu*}(s,t)=(s+\mu\rho(s),t)$. We compute

$$\begin{aligned}\frac{d}{d\mu}|_{\mu=0}(J\circ d\tilde{v}\circ\psi_{\mu*}\circ j\partial_s+d\tilde{v}\partial_s) &= J\circ d\tilde{v}\circ(\mathcal{L}_{\rho\partial_s}j)\partial_s\\ &= -J\circ d\tilde{v}\circ j[\rho\partial_s,\partial_s]\\ &= \rho' J\circ d\tilde{v}\circ j\partial_s\\ &= -\rho'\partial_t.\end{aligned}$$

These computations show that the cokernel of $\overline{\partial}_{\tilde{u}}^{W}\oplus\overline{\partial}_{\tilde{v}}^{W'}$ does not increase when we replace the span of $(0,\rho_i R_\alpha)$ and $(0,\rho_i\partial_t)$ with N in the domain.
On the other hand, the restriction of the domain relative to S decreases, by assumption, the dimension of the kernel by $\dim S$. Therefore, it does not increase the dimension of the cokernel, so the above operator is surjective. □

5.3.2 Generalized holomorphic maps

We now turn to generalized holomorphic maps, including fragments of gradient flow trajectories on the orbit spaces when $\lambda\to 0$. We first need to understand the asymptotic behavior of these maps, in order to introduce the appropriate Banach structures.

Proposition 5.8. *Let $\tilde{u}$ be a J_λ-holomorphic map with finite energy. Suppose that $\tilde{u}$ converges, near a puncture, to a closed Reeb orbit γ corresponding to a critical point* *of f_T in S_T. Then there exists $r > 0$ sufficiently small and independent of λ so that*

$$\begin{aligned} |\partial^I(a(s,t) - Ts - a_0)| &\leq C_I e^{-rs}, \\ |\partial^I(\vartheta(s,t) - t - \vartheta_0)| &\leq C_I e^{-rs}, \\ |\partial^I z_{out}(s,t)| &\leq C_I e^{-rs}, \\ |\partial^I(z_{in}(s,t) - \varphi_{\lambda s}^{f_T/(1+\lambda f_T)}(z_{in}^0))| &\leq C_I e^{-\lambda rs}, \end{aligned}$$

for every multi-index I and for some constants $C_I > 0$.

Proof. The first three inequalities follow from lemma 4.17 applied to a half-cylinder mapping to a small tubular neighborhood of the Reeb orbit γ.
The last inequality follows from the proof of lemma 4.18 after replacing the hyperbolic cosine with a negative exponential. □

Let $\mathcal{B}_k^{p,d,\lambda}(g;\gamma_1^+,\dots,\gamma_{s^+}^+;\gamma_1^-,\dots,\gamma_{s^-}^-)$ be the Banach manifold of maps $\tilde{u} : \Sigma_{\tilde{u}} \to \mathbb{R}\times M$ with the prescribed asymptotics at the punctures, which are locally in L_k^p and so that, near each puncture, $a(s,t) - T(1+\lambda f_T)s - a_0$, $\vartheta(s,t) - t - \vartheta_0$, $z_{out}(s,t)$, $z_{in}(s,t) - \varphi_{\lambda s}^{f_T/(1+\lambda f_T)}(z_{in}^0) \in L_k^{p,d}$.

Let $\gamma : \mathbb{R} \to S_T$ be a gradient trajectory for $\frac{f_T}{1+\lambda f_T}$. Note that this is just a reparametrized gradient trajectory of f_T. Let $\beta_{a,b} : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that $\beta_{a,b}(s) = s$ for $s \in [a+1, b-1]$, $\beta_{a,b}(s) = a$ for $s \leq a$ and $\beta_{a,b}(s) = b$ for $s \geq b$. We choose a family of such functions depending smoothly on $a < b \in \overline{\mathbb{R}}$.
Let $\tilde{u}_{\lambda,\gamma,a,b} : \mathbb{R}\times S^1 \to \mathbb{R}\times M$ be the map characterized by

$$\begin{aligned} a_{\lambda,\gamma,a,b}(s,t) &= T(1+\lambda f_T)s, \\ u_{\lambda,\gamma,a,b}(s,t) &\in N_T, \\ \pi_T \circ u_{\lambda,\gamma,a,b}(\mathbb{R}\times S^1) &= \gamma([a,b]), \\ \frac{\partial}{\partial s} u_{\lambda,\gamma,a,b}(s,t) &= \lambda \beta'_{\frac{a}{\lambda},\frac{b}{\lambda}}(s) \frac{\nabla f_T}{(1+\lambda f_T)^2}, \\ \frac{\partial}{\partial t} u_{\lambda,\gamma,a,b}(s,t) &= T R_\alpha. \end{aligned}$$

First consider the case $a = -\infty$ and $b = +\infty$; we abbreviate the corresponding map by $\tilde{u}_{\lambda,\gamma}$. This is the case of a holomorphic cylinder degenerating into a gradient trajectory. We denote by D_γ the linearized operator in Morse theory corresponding to this trajectory. We want to show that this map $\tilde{u}_{\lambda,\gamma}$ is approximately J_λ-holomorphic for small λ.

Lemma 5.9. *The map* $\tilde{u}_{\lambda,\gamma} \in \mathcal{B}_k^{p,d,\lambda}(\gamma(+\infty);\gamma(-\infty))$ *and*

$$\lim_{\lambda\to 0} \|\overline{\partial}_{J_\lambda} \tilde{u}_{\lambda,\gamma}\| = 0.$$

Proof. Note that

$$-J_\lambda(TR_\alpha) = T(1+\lambda f_T)\frac{\partial}{\partial t} + \frac{\lambda}{1+\lambda f_T}\nabla f_T.$$

Therefore,

$$\overline{\partial}_{J_\lambda} \tilde{u}_{\lambda,\gamma} = \frac{\lambda}{(1+\lambda f_T)^2}\nabla f_T.$$

But since $\|\nabla f_T\| = \int_{-\infty}^{+\infty} |\nabla f_T(\gamma(s))|^p e^{d|s|} ds < \infty$ for $d > 0$ sufficiently small, we have

$$\|\overline{\partial}_{J_\lambda} \tilde{u}_{\lambda,\gamma}\| \le C\lambda \|\nabla f_T\|$$

and the latter clearly converges to zero as $\lambda \to 0$. □

Consider now the linearized Cauchy-Riemann operator $\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$ at the maps $\tilde{u}_{\lambda,\gamma,a,b}$. We will use the following Banach structures :

$$\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}} : L_k^{p,d}(\tilde{u}) \oplus V_- \oplus V_+ \to L_{k-1}^{p,d}(\Lambda^{0,1}(\tilde{u})).$$

The finite dimensional vector spaces V_- and V_+ are defined as follows : if $a = -\infty$, V_- is spanned by $\rho_-(s)\frac{\partial}{\partial t}, \rho_-(s)R_\alpha$ and $\rho_-(s)v_-^i(s)$, $i = 1, \dots, \dim W^u(\gamma(-\infty))$, where the functions $v_-^i(s)$ span the tangent space of the stable manifold of $\gamma(-\infty)$ along γ and satisfy $D_\gamma v_-^i(s) = 0$.
If a is finite, V_- is spanned by $\rho_-(s)\frac{\partial}{\partial t}, \rho_-(s)R_\alpha$ and $\rho_-(s)w_i$, $i = 1, \dots, \dim S_T$, where the vectors w_i span TS_T. The space V_+ is defined similarly, distinguishing between

$b = +\infty$ and b finite, and using the unstable manifold of $\gamma(+\infty)$.

Proposition 5.10. *The Fredholm index of* $\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$ *is given by*

$$\begin{array}{ll} \mathrm{index}_{f_T}(\gamma(+\infty)) - \mathrm{index}_{f_T}(\gamma(-\infty)) + 2 & \text{if } a = -\infty, b = +\infty, \\ \mathrm{index}_{f_T}(\gamma(+\infty)) + 2 & \text{if } a > -\infty, b = +\infty, \\ \dim S_T - \mathrm{index}_{f_T}(\gamma(-\infty)) + 2 & \text{if } a = -\infty, b < +\infty, \\ \dim S_T + 2 & \text{if } a > -\infty, b < +\infty. \end{array}$$

Proof. The index of the operator $\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$ restricted to $L^{p,d}_k(\tilde{u})$ is always equal to $-\dim S + 2$. This is proved as before by conjugating with multiplication by $e^{\pm|d|s}$. The perturbation terms due to the Morse functions f_T do not contribute since $\lambda > 0$ is very small.
Then, the above index formulas follow from adding to $-\dim S + 2$ the dimension of $V_- \oplus V_+$ in the various cases. □

We now want to glue holomorphic maps and gradient trajectories. Consider maps $\tilde{u} \in \mathcal{B}^{p,d}_k(g; S^+_1, \dots, S^+_{s^+}; S^-_1, \dots, S^-_{s^-})$ and $\tilde{v} \in \mathcal{B}^{p,d}_k(g'; S^{'+}_1, \dots, S^{'+}_{s^{'+}};\ S^{'-}_1, \dots, S^{'-}_{s^{'-}})$ such that $\tilde{u}$ is asymptotic to the Reeb orbits $\gamma^+_i \in S^+_i$ for $i = 1, \dots, s^+$ and $\gamma^-_i \in S^-_i$ for $i = 1, \dots, s^-$; similarly, $\tilde{v}$ is asymptotic to the Reeb orbits $\gamma^{'+}_i \in S^{'+}_i$ for $i = 1, \dots, s^{'+}$ and $\gamma^{'-}_i \in S^{'-}_i$ for $i = 1, \dots, s^{'-}$. Assume that there exists $t' > 0$ for $i = 1, \dots, t \leq \min(s^-, s^{'+})$, such that $\varphi^{f_T}_{t'}(\gamma^-_{s^- - i + 1}) = \gamma^{'+}_i$.

We denote $\lim_{t \to \pm\infty} \varphi^{f_T}_t(\gamma^\pm_i)$ by $\tilde{\gamma}^\pm_i$ and similarly $\lim_{t \to \pm\infty} \varphi^{f_T}_t(\gamma^{'\pm}_i)$ by $\tilde{\gamma}^{'\pm}_i$.
We then construct a map

$$\tilde{u} \sharp_\lambda \tilde{v} \in \mathcal{B}^{p,d,\lambda}_k(g + g' + t - 1; \tilde{\gamma}^+_1, \dots, \tilde{\gamma}^+_{s^+}, \tilde{\gamma}^{'+}_{t+1}, \dots, \tilde{\gamma}^{'+}_{s^{'+}}; \tilde{\gamma}^-_1, \dots, \tilde{\gamma}^-_{s^- - t}, \tilde{\gamma}^{'-}_1, \dots, \tilde{\gamma}^{'-}_{s^{'-}})$$

by gluing $\tilde{u}$ and $\tilde{v}$ with fragment of gradient trajectories.
We construct the map $\tilde{u} \sharp_\lambda \tilde{v}$ by gluing $\tilde{u}$ and $\tilde{v}$ with the trajectories $\tilde{u}_{\lambda,\gamma,a,b}$ and $\tilde{v}_{\lambda,\gamma,a,b}$. Therefore, we just need to know how to glue a holomorphic map $\tilde{u}$ to a gradient trajectory $\tilde{u}_{\lambda,\gamma,a,b}$ with b finite.
Assume that $R = \frac{1}{\sqrt{\lambda}}$ is sufficiently large, so that $\tilde{u}$ is very close to vertical cylinders for $t \leq -R$, and we can cut the map at $t = -R$. On the other hand, the map

$\tilde{u}_{\lambda,\gamma,a,b}$ will be cut at $s = \frac{b}{\lambda}$. We glue the corresponding Riemann surfaces along the boundary circles, and we define as in the previous section the glued map $\tilde{u}\sharp_\lambda \tilde{u}_{\lambda,\gamma,a,b}$ using a cutoff function.

Lemma 5.11. *We have*

$$\lim_{\lambda\to 0} \|\overline{\partial}_{J_\lambda}(\tilde{u}\sharp_\lambda \tilde{v})\| = 0.$$

Proof. Since $\overline{\partial}_J \tilde{u} = 0$, we have

$$\|\overline{\partial}_{J_\lambda}\tilde{u}\| \leq |J - J_\lambda| \|\nabla \tilde{u}\|$$

where the norms are computed on $\Sigma_{\tilde{u}}$ with disks removed aroud the punctures. Since $R = \frac{1}{\sqrt{\lambda}}$ and the factor $\|\nabla\tilde{u}\|$ grows proportionally to R near the punctures, the right hand size is estimated by $C\sqrt{\lambda}$ for some constant $C > 0$.
On the other hand, the map $\tilde{u}_{\lambda,\gamma,a,b}$ is approximately J_λ-holomorphic between $s = \frac{a}{\lambda}+1$ and $s = \frac{b}{\lambda} - 1$, by lemma 5.9.
Finally, in the finite portion of cylinder where the maps are glued, the glued map converges to a vertical cylinder as $\lambda \to 0$ and $R \to \infty$. □

Note that the vector spaces $V_{\tilde{u}}$ for $\tilde{u}$ and V_+ for $\tilde{u}_{\lambda,\gamma,a,b}$ are isomorphic. Let Δ be the diagonal in $V_{\tilde{u}} \oplus V_+$.
Consider the linearized Cauchy-Riemann operators $\overline{\partial}_{\tilde{u}} : V_{\tilde{u}} \oplus L^{p,d}_k(\tilde{u}) \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}))$ and $\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}} : V_+ \oplus L^{p,d}_k(\tilde{u}_{\lambda,\gamma,a,b}) \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}_{\lambda,\gamma,a,b}))$. Pick a finite dimensional subspace W of $C^\infty_0(\Lambda^{0,1}(\tilde{u})) \oplus C^\infty_0(\Lambda^{0,1}(\tilde{u}_{\lambda,\gamma,a,b})) \subset L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u})) \oplus L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}_{\lambda,\gamma,a,b}))$, such that

$$\begin{aligned} W + \ & \overline{\partial}_{\tilde{u}}(L^{p,d}_k(\tilde{u})) + \overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}(L^{p,d}_k(\tilde{u}_{\lambda,\gamma,a,b})) + (\overline{\partial}_{\tilde{u}} \oplus \overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}})(\Delta) \\ = \ & L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u})) + L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}_{\lambda,\gamma,a,b})). \end{aligned}$$

We define the stabilization $\overline{\partial}^W_{\tilde{u},\tilde{u}_{\lambda,\gamma,a,b}}$ of $\overline{\partial}_{\tilde{u}} \oplus \overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$ by

$$\begin{aligned} W \oplus \Delta \oplus L^{p,d}_k(\tilde{u}) \oplus L^{p,d}_k(\tilde{u}_{\lambda,\gamma,a,b}) &\to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u})) \oplus L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}_{\lambda,\gamma,a,b})) \\ (w, (v,v), \xi_{\tilde{u}}, \xi_{\tilde{u}_{\lambda,\gamma,a,b}}) &\to w + \overline{\partial}_{\tilde{u}}(v + \xi_{\tilde{u}}) + \overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}(v + \xi_{\tilde{u}_{\lambda,\gamma,a,b}}). \end{aligned}$$

With such a choice of W, the operator $\overline{\partial}^W_{\tilde{u},\tilde{u}_{\lambda,\gamma,a,b}}$ is surjective and has a bounded right inverse Q_∞.

On $L^{p,d}_k(\tilde{u}\sharp_\lambda\tilde{u}_{\lambda,\gamma,a,b})$, we will use a modified Banach norm : the norm of an element ξ is given by $\|\xi-\bar{\xi}\|_\lambda+\sum_{i=1}^t|\bar{\xi}_i|$, where $\bar{\xi}_i$ is defined as in the last section, and the Banach norm $\|\cdot\|_\lambda$ has exponential weights $e^{d(\frac{1}{\sqrt{\lambda}}-|s''_i|)}$ in the necks.

Proposition 5.12. *Assume that W is chosen so that the operator $\overline{\partial}^W_{\tilde{u},\tilde{u}_{\lambda,\gamma,a,b}}$ is surjective. Then the operator*

$$\overline{\partial}_\lambda=\overline{\partial}^W_{\tilde{u}\sharp_\lambda\tilde{u}_{\lambda,\gamma,a,b}}:W\oplus L^{p,d}_k(\tilde{u}\sharp_\lambda\tilde{u}_{\lambda,\gamma,a,b})\to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}\sharp_\lambda\tilde{u}_{\lambda,\gamma,a,b}))$$

has a uniformly bounded right inverse Q_λ, if λ is sufficiently small.

Proof. This is just proposition 5.6 applied to $R=\frac{1}{\sqrt{\lambda}}$. It is indeed clear that both operators $\overline{\partial}_{\tilde{u}}$ and $\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$ converge, on the cylinders $[-R,+R]\times S^1$ in the center of the necks, to the linearized operator of a vertical cylinder over a closed Reeb orbit. □

Next, we want to show that realizing transversality for these Cauchy-Riemann operators, and obtaining a uniformly bounded right inverse, is not harder than realizing transversality for the corresponding gradient flow.

Lemma 5.13. *If the pair (f_T,g_T) is Morse-Smale and $\lambda>0$ is sufficiently small, then the linearized Cauchy-Riemann operator*

$$\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}:L^{p,d}_k(\tilde{u})\oplus V_-\oplus V_+\to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}))$$

is surjective.

Proof. First, the linearized Cauchy-Riemann operator will be of the form

$$\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}\xi=\frac{\partial}{\partial s}\xi+J_0\frac{\partial}{\partial t}\xi+S(\lambda s)\xi_N+\lambda A(\lambda s)\xi_T$$

where ξ_T (resp. ξ_N) is the tangent (resp. normal) component of ξ with respect to the submanifold $\mathbb{R}\times N_T$. Moreover, the operator $D_{\gamma\circ\beta_{a,b}}=\frac{\partial}{\partial s}+A(s)$ is the linearized gradient flow equation for $\frac{f_T}{1+\lambda f_T}$ on N_T.

Claim. If $\xi \in \ker \overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$ then $\xi_N = 0$.

Let $F(s) = J_0 \frac{\partial}{\partial t} + S(\lambda s) + \lambda A(\lambda s)$. If $\xi \in \ker \overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$, we have

$$\frac{\partial^2}{\partial s^2}\xi - F^2(s)\xi + [\frac{\partial}{\partial s}, F(s)]\xi = 0.$$

Taking the scalar product with ξ_N, we obtain

$$\langle \xi_N, \frac{\partial^2}{\partial s^2}\xi_N\rangle - \|F(s)\xi_N\|^2 + \langle \xi_N, \frac{\partial}{\partial s}S(\lambda s)\xi_N\rangle = 0$$

since $\langle \xi_N, A(s)\xi_T\rangle = 0$.

By the Morse-Bott assumption, we have $\|F(s)\xi_N\| \geq k\|\xi_N\|$. Therefore, after rearranging, we obtain

$$\frac{\partial^2}{\partial s^2}\|\xi_N\|^2 \geq \frac{1}{2}k^2\|\xi_N\|^2$$

if λ is sufficiently small. Suppose that ξ_N does not vanish. Then, since ξ_N must vanish for $s \to \pm\infty$, it must have a maximum for some s. But this contradicts the above inequality, therefore $\xi_N = 0$.

Claim. If $\xi = \xi_T \in \ker \overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$ then $\frac{\partial}{\partial t}\xi_T = 0$.

This is a variant of a proposition due to Salamon and Zehnder [31].

Note that, since $\xi_N = 0$, we can replace $S(\lambda s)$ with 0. Let us denote the resulting operator by $\overline{\partial}_T$, so that we have $\overline{\partial}_T\xi = 0$.

Next, if $\xi \in \ker \overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$, then $\xi_0(s) = \int_0^1 \xi(s,t)\,dt \in \ker \overline{\partial}_{\tilde{u}_{\lambda,\gamma}}$, and $\xi - \xi_0 \in \ker \overline{\partial}_{\tilde{u}_{\lambda,\gamma}}$. Therefore, we can assume without loss of generality that $\xi_0 = 0$. We have

$$\xi(s,t') - \xi(s,t) = \int_t^{t'} \frac{\partial}{\partial t}\xi(s,\theta)\,d\theta.$$

After integrating with respect to t, we obtain

$$\begin{aligned}
|\xi(s,t')| &= \Big|\int_0^1 \int_t^{t'} \frac{\partial}{\partial t}\xi(s,\theta)\, d\theta dt\Big| \\
&\leq \int_0^1 |\frac{\partial}{\partial t}\xi(s,\theta)|\, d\theta \\
&\leq \Big(\int_0^1 |\frac{\partial}{\partial t}\xi(s,\theta)|^2\, d\theta\Big)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, after integrating with respect to s and t', with appropriate exponential weights,

$$\|\xi\|_{L^{2,d}} \leq \|\frac{\partial}{\partial t}\xi\|_{L^{2,d}}.$$

Hence,

$$\begin{aligned}
\|\frac{\partial}{\partial t}\xi\| &\leq \|\nabla\xi\| \\
&= \|\frac{\partial}{\partial s}\xi + J_0\frac{\partial}{\partial t}\xi\| \\
&\leq \|\overline{\partial}_T\xi\| + \|\lambda A(\lambda s)\xi_T\| \\
&\leq C\lambda\|\xi\| \\
&\leq C\lambda\|\frac{\partial}{\partial t}\xi\|.
\end{aligned}$$

If λ is sufficiently small, this inequality forces $\frac{\partial}{\partial t}\xi = 0$ and therefore $\xi = 0$. Hence, in general $\xi(s,t) = \xi_0(s)$.

When $a = -\infty$ and $b = +\infty$, we conclude that $\ker\overline{\partial}_{\tilde{u}_{\lambda,\gamma}} = \ker D_\gamma \oplus \mathrm{span}(R_\alpha, \partial_t)$. The second summand corresponds to reparametrization of the cylinder and is not counted in the index formula. Hence, the Fredholm operators $\overline{\partial}_{\tilde{u}_{\lambda,\gamma}}$ and D_γ have the same index, therefore $\overline{\partial}_{\tilde{u}_{\lambda,\gamma}}$ is surjective if (f_T, g_T) is Morse-Smale.

When a and b are finite, the Cauchy-Riemann operator is approximated, for λ small, by

$$\frac{\partial}{\partial s}\xi + J_0\frac{\partial}{\partial t}\xi + S(\lambda s)\xi_N.$$

The kernel of this operator is isomorphic to $\mathbb{R} \times TN_T$. Hence, by the index formula, this operator is surjective. When λ is sufficiently small, the operator $\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$ is surjective as well.

When exactly one of a and b is finite, we can approximate the Cauchy-Riemann operator by cutting off the term $\lambda A(\lambda s)$ near the corresponding end. The kernel of the approximate operator is then spanned by the stable or unstable manifold of $\gamma(\pm\infty)$, $\frac{\partial}{\partial t}$ and R_α. Again, it follows that $\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$ is surjective. $\square$

Proposition 5.14. *Under the assumptions of lemma 5.13, the linearized Cauchy-Riemann operator*

$$\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}} : L^{p,d}_k(\tilde{u}) \oplus V_- \oplus V_+ \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}))$$

has a right inverse that is bounded uniformly in λ.

Proof. When S is independent of s and t, lemma 5.13 shows that the corresponding linear Cauchy-Riemann operator has a right inverse Q_S. This is true for all operators $\overline{\partial}_S$ obtained as linearized Cauchy-Riemann operator at a vertical cylinder over a closed orbit in the gradient trajectory γ. Since the image of that trajectory is compact, there is a uniform bound for the norm of all right inverses Q_S.

Consider now a linear Cauchy-Riemann operator $\overline{\partial}$ with matrix $S(s)$ such that $\|S(s)-S\| + \sum_{i=1}^k \|\frac{d^i}{ds^i}S(s)\| \leq \frac{1}{2\|Q_S\|}$, then $\overline{\partial}$ has a right inverse Q satisfying $\|Q\| \leq 2\|Q_S\|$. Indeed, the assumption implies that $\|\overline{\partial} - \overline{\partial}_S\| \leq \frac{1}{2\|Q_S\|}$. Then, we have $\|\overline{\partial}Q_S - I\| = \|\overline{\partial}Q_S - \overline{\partial}_S Q_S\| \leq \frac{1}{2}$, so that $\overline{\partial}Q_S$ is invertible and $\|(\overline{\partial}Q_S)^{-1}\| \leq 2$. We can define the right inverse as $Q = Q_S(\overline{\partial}Q_S)^{-1}$.

This assumption is certainly satisfied by $S(\lambda s)$ when $\lambda > 0$ is sufficiently small and if we restrict ourselves to a sufficiently small portion of the gradient trajectory γ. We therefore subdivide γ into a finite number of small enough portions, obtain a uniform bound in λ for the right inverses of the corresponding operators, and glue these Cauchy-Riemann operators. In order to glue trajectories $\tilde{u}_{\lambda,\gamma,a,b}$ and $\tilde{u}_{\lambda,\gamma',a',b'}$ with a and b' finite, we cut the first map at $s = \frac{a}{\lambda}$ and the second map at $s = \frac{b'}{\lambda}$. Then, we can apply the estimates of proposition 5.12 to see that the glued operator has a right inverse that is uniformly bounded in λ as well. Since only trajectories are involved,

the Banach structures have no exponential weights in the necks. Moreover, by lemma 5.13, the kernels of the operators to be glued are transversal in TS. Therefore, we can choose $W = 0$ in proposition 5.12.
Finally, the glued operator can be made arbitrarily close to the actual operator $\overline{\partial}_{\tilde{u}_{\lambda,\gamma,a,b}}$, by choosing λ sufficiently small. Therefore, we also obtain a uniformly bounded right inverse for that operator. □

We now return to the holomorphic maps $\tilde{u}$ and $\tilde{v}$ and the corresponding glued map $\tilde{u}\sharp_\lambda\tilde{v}$.
Let $S = \bigoplus_{i=1}^{s^+} T_{\gamma_i^+}S + \bigoplus_{i=t+1}^{s'^+} T_{\gamma_i'^+}S + \bigoplus_{i=1}^{s^-} T_{\gamma_i^-}S + \bigoplus_{i=1}^{s'^-} T_{\gamma_i'^-}S$. We define the projection $\pi_{\tilde{u}} : T_{\tilde{u}}\mathcal{B} \to S$ by restriction to the punctures of $\Sigma_{\tilde{u}}$. We also define the projection $\pi'_{\tilde{v}} : T_{\tilde{v}}\mathcal{B} \to S$ by restriction to the punctures of $\Sigma_{\tilde{v}}$, followed by the linearized gradient flow $\varphi^{f_T}_{t'*} : T_{\gamma_i'^+}S \to T_{\gamma^-_{s^-+1-i}}S$, when $i = 1, \dots, t$.
Combining propositions 5.12 and 5.14, we obtain the following result.

Corollary 5.15. *Let $\tilde{u}$ and $\tilde{v}$ be as above. Suppose that the operators $\overline{\partial}^W_{\tilde{u}} : W \oplus T_{\tilde{u}}\tilde{\mathcal{B}} \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}))$ and $\overline{\partial}^{W'}_{\tilde{v}} : W' \oplus T_{\tilde{v}}\tilde{\mathcal{B}} \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{v}))$ are surjective, and that*

$$\begin{aligned}&\pi_{\tilde{u}}(\ker \overline{\partial}^W_{\tilde{u}}) + \pi'_{\tilde{v}}(\ker \overline{\partial}^{W'}_{\tilde{v}}) + \textstyle\sum_{i=1}^{s^+} T_{\gamma_i^+}W^u(\tilde{\gamma}_i^+) + \sum_{i=t+1}^{s'^+} T_{\gamma_i^+}W^u(\tilde{\gamma}_i'^+)\\ &+ \textstyle\sum_{i=1}^{s^- - t} T_{\gamma_i^-}W^s(\tilde{\gamma}_i^-) + \sum_{i=1}^{s'^-} T_{\gamma_i'^-}W^s(\tilde{\gamma}_i'^-) + \sum_{i=1}^{t} \nabla f_T(\gamma^-_{s^-+1-i}) = S.\end{aligned}$$

Then, for λ sufficiently small, the operator

$$\overline{\partial}^{W\oplus W'}_{\tilde{u}\sharp_\lambda\tilde{v}} : W \oplus W' \oplus T_{\tilde{u}\sharp_\lambda\tilde{v}}\tilde{\mathcal{B}} \to L^{p,d}_{k-1}(\Lambda^{0,1}(\tilde{u}\sharp_\lambda\tilde{v}))$$

has a uniformly bounded right inverse Q_λ.

Proof. By proposition 5.14, it is not necessary to stabilize the linearized operators corresponding to gradient trajectories. Therefore, only W and W' are needed.
Moreover, lemma 5.13 and the above transversality assumption on the evaluation maps show that the cokernel does not grow when we restrict ourselves to the diagonal Δ of the space of sections not vanishing at a puncture where we glue.
Therefore, we can apply proposition 5.12 with stabilization by $W \oplus W'$. □

5.3.3 Implicit function theorem

In this section, we show how to use the gluing estimates from the previous sections in order to glue holomorphic curves. We need the following result, used in [21], that is a consequence of the Banach fixed point theorem.

Proposition 5.16. *Assume that a smooth map $f : E \to F$ of Banach spaces has a Taylor expansion*

$$f(\xi) = f(0) + Df(0)\xi + N(\xi)$$

such that $Df(0)$ has a finite dimensional kernel and a right inverse Q satisfying

$$||QN(\xi) - QN(\eta)|| \leq C(||\xi|| + ||\eta||)||\xi - \eta||$$

for some constant C. Let $\delta = \frac{1}{8C}$. If $||Qf(0)|| \leq \frac{\delta}{2}$, then $f^{-1}(0) \cap B_\delta(\xi)$ is a smooth manifold of dimension equal to $\dim\ker Df(0)$.
More precisely, there exists a smooth function

$$\phi : \ker Df(0) \cap B_\delta(\xi) \to Q(F)$$

such that $f(\xi + \phi(\xi)) = 0$ and all zeroes of f in $B_\delta(\xi)$ are of the form $\xi + \phi(\xi)$.

We want to apply this proposition to the section $\overline{\partial}_J : \tilde{\mathcal{B}} \to \tilde{\mathcal{E}}$. By proposition 5.2, the differential $\overline{\partial}_{\tilde{u}}$ of $\overline{\partial}_J$ is Fredholm. By propositions 5.6 and 5.12, this linear operator has a uniformly bounded right inverse Q.
It follows from standard results [6] that the remainder N of the Taylor expansion satisfies an estimate

$$||Q(\xi) - Q(\eta)|| \leq C'(||\xi|| + ||\eta||)||\xi - \eta||.$$

Indeed, the operators $\overline{\partial}_J$ and $\overline{\partial}_{\tilde{u}}$ have the same form as in Floer homology, and this result is independent of the nature of the asymptotics.
The constant C is then given by $||Q||C' < \infty$. By lemmata 5.5, 5.9 and 5.11, we can find glued maps with $f(0)$ as small as we want. Therefore, we can apply proposition 5.16 in order to find nearby holomorphic maps.

Chapter 6

Transversality

6.1 Transversality conditions

In order to realize our moduli spaces as nice geometric objects with the virtual dimension predicted by the Fredholm index, we have to make sure that the Fredholm operators obtained by linearizing the Cauchy-Riemann equation are surjective.

Using the results of the previous chapter, we can write sufficient conditions to ensure that $\overline{\partial}_J : \mathcal{B} \to \mathcal{E}$ is everywhere transverse to the zero section. Let $\mathcal{M} = \overline{\partial}_J^{-1}(0)$ be the set of J-holomorphic curves in $\mathcal{B}$.
The following properties must hold for every moduli space of level k (generalized) holomorphic maps.

(i) For every level 1 map $\tilde{u} \in \mathcal{M}$, the linearized Cauchy-Riemann operator $\overline{\partial}_{\tilde{u}}$ is surjective.

(ii) The pair (f_T, g) is Morse-Smale.

(iii) For every $\mathcal{M}$ and $\mathcal{M}'$, for every collection $S_1, \ldots, S_k$ of orbit spaces in the negative asymptotics of $\mathcal{M}$ and in the positive asymptotics of $\mathcal{M}'$, the evaluation maps $ev^- : \mathcal{M} \to S_1 \times \ldots \times S_k$ and $ev^+ : \mathcal{M}' \to S_1 \times \ldots \times S_k$ are transverse.

(iv) For every $\mathcal{M}$, its positive (resp. negative) evaluation map to the product of its positive (resp. negative) asymptotic orbit spaces is transverse to any product of unstable (resp. stable) manifolds for f_T.

(v) For every $\mathcal{M}$ and $\mathcal{M}'$, for every collection $S_1, \dots, S_k$ of orbit spaces in the negative asymptotics of $\mathcal{M}$ and in the positive asymptotics of $\mathcal{M}'$, the evaluation map $ev^- : \mathcal{M} \to S_1 \times \ldots \times S_k$ and the generalized evaluation map $\varphi^{f_T} \circ ev^+ : \mathbb{R}^+ \times \mathcal{M}' \to S_1 \times \ldots \times S_k : (t, \tilde{u}) \to \varphi_t^{f_T} \circ ev^+(\tilde{u})$ are transverse.

When these conditions are satisfied, we can construct the moduli spaces $\mathcal{M}^{f_T}$ of generalized holomorphic maps, using the moduli spaces $\mathcal{M}$ with degenerate asymptotics and the gradient flow of f_T, in the following way :

$$
\begin{aligned}
&\mathcal{M}^{f_T}(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \\
&= \quad \mathcal{M}(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \\
&\cup \quad \mathcal{M}(S_1^+, \dots, S_{s^+}^+; S_1', \dots, S_{s'}') \times_{S_1' \times \ldots \times S_{s'}'} \mathbb{R}^+ \times \mathcal{M}(S_1', \dots, S_{s'}'; S_1^-, \dots, S_{s^-}^-) \\
&\cup \quad \ldots
\end{aligned}
$$

where we use ev^- and $\varphi^{f_T} \circ ev^+$ in the fibered products, and we take the union over any number of successive fibered products. Note that this union is finite, since the action spectrum is discrete.
Then, we make the transition to non-degenerate asymptotics using the stable and unstable manifolds of f_T :

$$
\begin{aligned}
&\mathcal{M}^{f_T}(\gamma_1^+, \dots, \gamma_{s^+}^+; \gamma_1^-, \dots, \gamma_{s^-}^-) \\
&= \quad (W^u(\gamma_1^+) \times \ldots \times W^u(\gamma_{s^+}^+)) \times_{S_1^+ \times \ldots \times S_{s^+}^+} \\
&\quad\quad \mathcal{M}^{f_T}(S_1^+, \dots, S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \times_{S_1^- \times \ldots \times S_{s^-}^-} (W^s(\gamma_1^-) \times \ldots \times W^s(\gamma_{s^-}^-)).
\end{aligned}
$$

If the above transversality conditions are not automatically satisfied, we have to modify the section $\overline{\partial}_J : \mathcal{B} \to \mathcal{E}$ so that the perturbed section satisfies these properties.

In order to construct the moduli spaces with degenerate asymptotics, we choose to keep the almost complex structure J fixed and perturb the right hand side of the Cauchy-Riemann equation. Indeed, the greatest benefit of this Morse-Bott setup is to work with symmetric Reeb dynamics and symmetric almost complex structures.

Therefore, we prefer to keep this symmetry during all steps of the construction of the moduli spaces : it is probably much easier to solve the Cauchy-Riemann equations for a natural J, and then understand the obstruction bundle in this natural setup, than to solve those equations for generic J.
Moreover, a generic J is generally not enough in contact homology to guarantee transversality, because of multiply covered cylinders, for example. The almost complex structure would have to depend on the points of the Riemann surface as well, which would make things even harder for computations.

Therefore, we will use the virtual cycle techniques developed in [20] and in [21], for example, in order to obtain a branched, labeled pseudo-manifold with corners, as explained in [24].

6.2 Virtual neighborhood

6.2.1 Holomorphic maps

We now describe the space $\overline{\mathcal{B}}$ in which we will construct the virtual cycle.

$$\begin{aligned}\overline{\mathcal{B}}_k^{p,d}\,(g,m;S_1^+,\dots,S_{s^+}^+;S_1^-,\dots,S_{s^-}^-) = \{&\text{stable maps of genus } g\\ &\text{with } m \text{ marked points and the given asymptotics, each level in } \mathcal{B}_k^{p,d}\}/\sim\end{aligned}$$

where $(\Sigma, j, \tilde{u}) \sim (\Sigma', j', \tilde{u}')$ if and only if they have the same number k of levels, and for each level i, $i = 1, \dots, k$, there is $\delta_i \in \mathbb{R}$ and a biholomorphism $\phi_i : (\Sigma_i, j) \to (\Sigma_i', j')$ preserving the marked points such that $u_i = u_i' \circ \phi_i$ and $a_i + \delta_i = a_i' \circ \phi_i$.

The space $\overline{\mathcal{B}}$ can be described as the union of finitely many strata : $\overline{\mathcal{B}} = \cup_D \overline{\mathcal{B}}^D$, where D are the patterns corresponding to holomorphic maps in $\mathcal{B}$.
The pattern of a stable holomorphic map $(\Sigma, j, \tilde{u})$ consists of the following data : the intersection pattern of the domain Σ, with its labeling by levels, the number of marked points on each component of Σ, and the orbit space corresponding to each node or puncture of Σ.

Lemma 6.1. *There exists a neighborhood $\mathcal{W}$ of the set of holomorphic maps in $\overline{\mathcal{B}}$ admitting the structure of a stratified Banach orbifold.*

Proof. Fix a pattern D; for each holomorphic map $(\Sigma, j, \tilde{u}) \in \overline{\mathcal{B}}^D$, let us construct a uniformization chart $\pi_U^D : \tilde{U}^D \to U^D$ containing it.
First consider the domain Σ. We can add finitely many points $x_1, \dots, x_l$ to make it a stable curve. Let H_j $(j = 1, \dots, l)$ be a small piece of real codimension 2 hypersurface in $\mathbb{R} \times M$ that is transversal to $\tilde{u}(\Sigma)$ at $\tilde{u}(x_l)$. Such hypersurfaces H_j, $j = 1, \dots, l$, can always be found, provided the corresponding marked points are chosen generically.
We add in each level Σ_i one more marked point $\tilde{x}_i$; let $\tilde{H}_i$ be a small piece of real codimension 3 hypersurface in $\mathbb{R} \times M$ that is transversal to $\mathbb{R} \times \tilde{u}(\Sigma)$ at $\tilde{u}(\tilde{x}_i)$. For this, we need to choose $\tilde{x}_i$ so that $\frac{\partial}{\partial t}$ is not tangent to $\tilde{u}(\Sigma)$ at $\tilde{u}(\tilde{x}_i)$. If this cannot be achieved in level i, then this level consists of one or more vertical cylinders over a closed Reeb orbit. By the stability condition, there is at least one marked point on each cylinder. We then choose one of these points, call it $\tilde{x}_i$, and let $\tilde{H}_i$ be a small piece of real codimension 1 hypersurface in $\mathbb{R} \times M$ that is transversal to $\mathbb{R}$ at $\tilde{u}(\tilde{x}_i)$.
Each component of Σ with its marked points is represented by a point in $\mathcal{M}_{g,n}$ for appropriate g and n. Pick a uniformization chart of $\mathcal{M}_{g,n}$ centered at this point, and let $\pi_V^D : \tilde{V}^D \to V^D$ be their product.
We define

$$\begin{aligned}\tilde{U}^D &= \{(\Sigma', j', \tilde{u}') \in \overline{\mathcal{B}}^D \mid (\Sigma', j') \in \tilde{V}^D, \|\tilde{u} - \tilde{u}'\| \leq \epsilon, \\ &\qquad \tilde{u}'(x_j) \in H_j, \tilde{u}'(\tilde{x}_i) \in \tilde{H}_i\}.\end{aligned}$$

We work with $k \geq 2$ so that that the $L_k^{p,d}$ norm is stronger than the C^1 norm. In particular, if $\epsilon > 0$ is sufficiently small, all maps in $\tilde{U}^D$ keep the transversality properties as above with respect to the hypersurfaces H_j, $j = 1, \dots, l$, and $\tilde{H}_i$, $i = 1, \dots, k$.
Let Γ be the automorphism group of $(\Sigma, j, \tilde{u})$. We define an action of $\sigma \in \Gamma$ on $\tilde{U}^D$ by $\sigma(\Sigma', j', \tilde{u}') = (\Sigma'', j'', \tilde{u}'')$ where $\Sigma'' = \sigma(\Sigma')$, $j'' = \sigma_* j'$, $\tilde{u}'' = \tilde{u}' \circ \sigma^{-1}$. If $\epsilon > 0$ is chosen sufficiently small, then for $j = 1, \dots, l$, there is a unique point p_j in a small neighborhood of $\sigma(x_j)$ such that $\tilde{u}''(p_j) \in H_j$. We define $x'_j = p_j$ and we proceed similarly to define $\tilde{x}'_i$, $i = 1, \dots, k$.

Let $U^D = \tilde{U}^D/\Gamma$ and let $\pi_U^D : \tilde{U}^D \to U^D$ be the corresponding projection map.

Next, let us construct a uniformization chart $\pi_U : \tilde{U} \to U$ across the strata.
First, we need to take care of the nodes inside each stratum. For each node q_i, $i = 1, \ldots, v$, we introduce a complex coordinate c_i that is used to glue (Σ, j) near q_i : on the 2 components of Σ adjacent to p_i, we cut out small disks centered on q_i of radius $|c_i|$ and identify the boundaries with relative angle $\arg c_i$. We denote the obtained Riemann surface by $(\Sigma_{\bar{c}}, j)$. Given $(\Sigma, j, \tilde{u}) \in \overline{\mathcal{B}}$, we define the glued map $\tilde{u}_{\bar{c}}$ on $(\Sigma_{\bar{c}}, j)$ in the standard way using the exponential map at p_i and a cutoff function.
Now, suppose that $(\Sigma, j, \tilde{u}) \in \overline{\mathcal{B}}$ has two levels $(\Sigma_1, j, \tilde{u}_1)$ and $(\Sigma_2, j, \tilde{u}_2)$, and t ends need to be glued together. We now define the twisted map $\tau_{\bar{\theta},\bar{l}}\tilde{u}_1$, where $\bar{\theta} = (\theta_1, \ldots, \theta_t)$ and $\bar{l} = (l_1, \ldots, l_t) \in \mathbb{R}^t$. Near puncture x_i^-, the map $\tau_{\bar{\theta},\bar{l}}\tilde{u}_1$ is given by

$$
\begin{aligned}
\tau_{\bar{\theta},\bar{l}}a(s,t) &= a(s,t) + l_i\rho_i(s), \\
\tau_{\bar{\theta},\bar{l}}\vartheta(s,t) &= \vartheta(s,t) + \theta_i\rho_i(s), \\
\tau_{\bar{\theta},\bar{l}}z(s,t) &= z(s,t).
\end{aligned}
$$

We define the map $\tilde{u}_{R,\bar{\theta},\bar{l}} = \left(\tau_{\bar{\theta},\bar{l}}\tilde{u}_1\right)\sharp_R\tilde{u}_2$. We denote the domain of this map by $(\Sigma_{R,\bar{\theta},\bar{l}}, j)$.
We define

$$
\begin{aligned}
\tilde{U} = \{&(\Sigma'_{R,\bar{\theta},\bar{l},\bar{c}}, j', \tilde{u}') \in \overline{\mathcal{B}} \,|\, (\Sigma', j') \in \tilde{V}, R > R_0, \bar{\theta}, \bar{l} \in B_\epsilon^t, \bar{c} \in B_\epsilon^{2t}, \\
&\|\tilde{u}' - \tilde{u}_{R,\bar{\theta},\bar{l},\bar{c}}\| \leq \epsilon, \tilde{u}'(x_j) \in H_j, \tilde{u}'(\tilde{x}_i) \in \tilde{H}_i\}
\end{aligned}
$$

where B_ϵ^k is the ball of radius ϵ centered on the origin of $\mathbb{R}^k$.
We then extend the action of the automorphism group Γ to $\tilde{U}$ in the following way. For every element $(\Sigma'_{R,\bar{\theta},\bar{l},\bar{c}}, j', \tilde{u}')$ of $\tilde{U}$, the Riemann surface $(\Sigma'_{R,\bar{\theta},\bar{l},\bar{c}}, j')$ is in a small uniformization chart $\tilde{V}$ around (Σ, j) in $\overline{\mathcal{M}}_{g,n}$. Since Γ consists of automorphisms of (Σ, j) and the latter act on $\tilde{V}$, we can define the action of $\sigma \in \Gamma$ as : $\sigma(\Sigma', j', \tilde{u}') = (\Sigma'', j'', \tilde{u}'')$ where $\Sigma'' = \sigma(\Sigma')$, $j'' = \sigma_* j'$, $\tilde{u}'' = \tilde{u}' \circ \sigma^{-1}$. The images of the additional marked points, associated to pieces of hyperplanes, are defined as before.

Finally, we let $U = \tilde{U}/\Gamma$ and let $\pi_U : \tilde{U} \to U$ be the corresponding projection map. $\square$

For holomorphic curves in a symplectic cobordism, the same constructions can be used. The only difference is that no marked point $\tilde{x}$ is introduced in the level corresponding to the cobordism itself.

There is a bundle $\overline{\mathcal{E}} \to \overline{\mathcal{B}}$ that is obtained from the vector bundles $\tilde{\mathcal{E}} \to \tilde{\mathcal{B}}$ introduced in the previous chapter, over each stratum $\overline{\mathcal{B}}^D$. There is no topology on $\overline{\mathcal{E}}$ across the strata. All we need for the virtual cycle construction is the natural way to glue sections of $\tilde{\mathcal{E}}$ vanishing in a neighborhood of the nodes.

6.2.2 Generalized holomorphic maps

We now describe the space $\overline{\mathcal{B}}$ that is appropriate for generalized holomorphic maps.

$$\begin{aligned}\overline{\mathcal{B}}_k^{f_T,p,d}\,(g, m; \gamma_1^+, \dots, \gamma_{s^+}^+; \gamma_1^-, \dots, \gamma_{s^-}^-) = \{&\text{generalized stable maps of genus } g \\ &\text{with } m \text{ marked points and the given asymptotics, each sublevel in } \mathcal{B}_k^{p,d}\}/\sim\end{aligned}$$

where the equivalence relation $\sim$ is defined as in the last section.

As before, we have a stratification of this space according to the pattern D of a generalized stable map. In addition to the information of the last section, the pattern D also contains the labeling of the sublevels.

Lemma 6.2. *There exists a neighborhood $\mathcal{W}^{f_T}$ of the set of generalized holomorphic maps in $\overline{\mathcal{B}}^{f_T}$ admitting the structure of a stratified Banach orbifold.*

Proof. This is just a variant of lemma 6.1, and we just have to take care of the extra variables $t_{i,j}$, for $j = 1, \dots, l_i - 1$ and $i = 1, \dots, k$, where l_i is the number of sublevels in level i and k is the number of levels.
We stabilize each sublevel of a stable map as before, and we let

$$\begin{aligned}\tilde{U}^{f_T,D} \;\; = \;\; &\{(\Sigma', j', t'_{i,j}, \tilde{u}') \in \overline{\mathcal{B}}^{f_T,D} \,|\, (\Sigma', j') \in \tilde{V}^D, t'_{i,j} \in \mathbb{R}^+, |t'_{i,j} - t_{i,j}| \le \epsilon, \\ &\|\tilde{u} - \tilde{u}'\| \le \epsilon, \tilde{u}'(x_j) \in H_j, \tilde{u}'(\tilde{x}_i) \in \tilde{H}_i\}\end{aligned}$$

so that $\tilde{U}^{f_T,D}$ has codimension $k-1$ in $\overline{\mathcal{B}}^{f_T,D}$, independently of the number of sublevels. The quotient by the automorphism group Γ is defined as before.

For the uniformization charts $\pi_U^{f_T} : \tilde{U}^{f_T} \to U^{f_T}$ across the strata, we need to glue gradient trajectories of f_T at critical points, between adjacent levels. We use a large gluing parameter R_i between level i and level $i+1$, so that all glued trajectories have length R_i. The result is a generalized level 1 stable curve with $\sum_{i=1}^k l_i$ sublevels. Therefore, when all $t_{i,j} > 0$, we define

$$\begin{aligned}\tilde{U}^{f_T} &= \{(\Sigma'_{\bar{c}}, j', t'_{i,j}, R_i, \tilde{u}') \in \overline{\mathcal{B}}^{f_T} \mid (\Sigma', j') \in \tilde{V}, \bar{c} \in B_\epsilon^{2t}, \\ &\qquad t'_{i,j} \in \mathbb{R}^+, |t'_{i,j} - t_{i,j}| \leq \epsilon, R_i > R_{0,i}, \|\tilde{u}' - \tilde{u}_{\bar{c}}\| \leq \epsilon, \\ &\qquad \tilde{u}'(x_j) \in H_j, \tilde{u}'(\tilde{x}_i) \in \tilde{H}_i\}.\end{aligned}$$

On the other hand, when some variable $t_{i,j} = 0$, we must allow for 2 types of deformation : $t_{i,j}$ can become positive, and the sublevels j and $j+1$ in level i can be glued into a single sublevel, with a large gluing parameter $R_{i,j}$. This degree of freedom can be parametrized by a single variable $\tau_{i,j} \in \mathbb{R}$, given by $\tau_{i,j} = t_{i,j}$ when $\tau_{i,j} \geq 0$ and $\tau_{i,j} = -\frac{1}{R_{i,j}}$ when $\tau_{i,j} < 0$.
We will construct the open set $\tilde{U}^{f_T}$ in the case of one variable $t_{i,j} = 0$, the general case being analogous. Then, we have $\tilde{U}^{f_T} = \tilde{U}^{f_T}_{t_{i,j}} \cup \tilde{U}^{f_T}_{R_{i,j}}$, where $\tilde{U}^{f_T}_{t_{i,j}}$ is defined as $\tilde{U}^{f_T}$ above, with the restriction $t_{i,j} \geq 0$, and $\tilde{U}^{f_T}_{R_{i,j}}$ is defined as $\tilde{U}$ in lemma 6.1, with the additional variables $t_{i',j'}$.

$$\begin{aligned}\tilde{U}^{f_T}_{R_{i,j}} &= \{(\Sigma'_{R_{i,j},\bar{\theta},\bar{l},\bar{c}}, j', t'_{i',j'}, \tilde{u}') \in \overline{\mathcal{B}}^{f_T} \mid (\Sigma', j') \in \tilde{V}, R_{i,j} > R_0, \bar{\theta}, \bar{l} \in B_\epsilon^t, \bar{c} \in B_\epsilon^{2t}, \\ &\qquad t'_{i',j'} \in \mathbb{R}^+, |t'_{i',j'} - t_{i',j'}| \leq \epsilon, \|\tilde{u}' - \tilde{u}_{R_{i,j},\bar{\theta},\bar{l},\bar{c}}\| \leq \epsilon, \\ &\qquad \tilde{u}'(x_j) \in H_j, \tilde{u}'(\tilde{x}_i) \in \tilde{H}_i\}\end{aligned}$$

where the vectors $\bar{\theta}$ and $\bar{l}$ correspond to the nodes between sublevels j and $j+1$ in level i only.
The quotient by the automorphism group is defined as before. $\square$

Next, we describe the space $\overline{\mathcal{B}}$ for J_λ-holomorphic maps, $0 < \lambda \leq \lambda_0$.

$$\begin{aligned}\overline{\mathcal{B}}_k^{(0,\lambda_0],p,d}\,(g, m; \gamma_1^+, \dots, \gamma_{s^+}^+; \gamma_1^-, \dots, \gamma_{s^-}^-) = \{(\lambda, \tilde{u}) \,|\, 0 < \lambda \leq \lambda_0, \tilde{u} \text{ stable maps}\\ \text{of genus } g \text{ with } m \text{ marked points and the given asymptotics,}\\ \text{each level in } \mathcal{B}_k^{p,d,\lambda}\}/\sim\end{aligned}$$

with the equivalence relation $\sim$ as before.

We want to construct the relative virtual cycle in the following compactification of that space :

$$\overline{\mathcal{B}}_k^{[0,\lambda_0],p,d} = \overline{\mathcal{B}}_k^{(0,\lambda_0],p,d} \cup \overline{\mathcal{B}}_k^{f_T,p,d}.$$

Lemma 6.3. *There exists a neighborhood $\mathcal{W}^{[0,\lambda_0]}$ of the set of J_λ-holomorphic maps in $\overline{\mathcal{B}}^{[0,\lambda_0]}$ admitting the structure of a stratified Banach orbifold.*

Proof. The definition of the uniformization charts $\tilde{U}^D$ when $\lambda = 0$ is identical to lemma 6.2. When $\lambda > 0$, we use the uniformization charts $\tilde{U}^D$ and $\tilde{U}$ from lemma 6.1, multiplied by a small interval in λ.

Therefore, we just have to construct uniformization charts $\pi_U^{[0,\lambda_0]} : \tilde{U}^{[0,\lambda_0]} \to U^{[0,\lambda_0]}$ near a stable map with $\lambda = 0$. Note that we must have $\tilde{U}^{[0,\lambda_0]} \cap \{\lambda = 0\} = \tilde{U}^{f_T}$. Then, for any $\tilde{u}' \in \tilde{U}^{f_T}$, we have to glue its adjacent sublevels using $\tilde{u}'_{i,j+1}\sharp_\lambda \tilde{u}'_{i,j}$. We denote the result of this operation by $\tilde{u}'_\lambda$, with domain Σ'_λ.
We then define

$$\begin{aligned}\tilde{U}^{[0,\lambda_0]} &= \{(\lambda, \Sigma'', j'', \tilde{u}'') \in \overline{\mathcal{B}}^{[0,\lambda_0]} \mid 0 \leq \lambda \leq \epsilon, \Sigma'' = \Sigma'_\lambda,\\ &\qquad \|\tilde{u}'' - \tilde{u}'_\lambda\| \leq \epsilon \text{ for some } \tilde{u}' \in \tilde{U}^{f_T}\}.\end{aligned}$$

Since Σ'_λ is obtained by gluing Σ' with long cylinders at some of the nodes, it is still true that Σ'_λ is in a small uniformization chart around (Σ, j) in $\overline{\mathcal{M}}_{g,n}$. Therefore, we can extend the action of the automorphism group Γ as before, and define $U^{[0,\lambda_0]} = \tilde{U}^{[0,\lambda_0]}/\Gamma$. $\square$

6.3 Relative virtual cycle

We now want to use virtual cycle techniques to obtain the following results.

Proposition 6.4. *There exists a relative cycle $\mathcal{M}^{vir}$ in $\overline{\mathcal{B}}_k^{p,d}$ of dimension equal to the Fredholm index of $\overline{\partial}_J$ and such that*

$$\partial \mathcal{M}^{vir} = \bigcup \mathcal{M}^{vir} \times_S \mathcal{M}^{vir}$$

where all fibered products are transverse. The isotopy class of these relative virtual cycles depends only on (M, α, J).

Proposition 6.5. *There exists a relative virtual cycle $\mathcal{M}^{vir}_{[0,\lambda_0]}$ in $\overline{\mathcal{B}}_k^{[0,\lambda_0],p,d}$ of dimension equal to the Fredholm index of $\overline{\partial}_J$ and such that*

$$\mathcal{M}^{vir}_{[0,\lambda_0]} \cap \{\lambda = 0\} = \mathcal{M}^{vir}$$

and

$$\mathcal{M}^{vir}_{[0,\lambda_0]} \cap \{\lambda = \lambda_0\} = \mathcal{M}^{vir}_{\lambda_0}.$$

The isotopy class of these relative virtual cycles depends only on $(M, \{\alpha_\lambda, J_\lambda\}_{\lambda \in [0,\lambda_0]})$.

These results will be proved using the same techniques.
First, for each holomorphic map $\tilde{u}$, we can choose a finite dimensional subspace $W_{\tilde{u}}$ of the target space of $\overline{\partial}_J$, consisting of smooth functions with support away from the nodes, in order to make the linear operator $\overline{\partial}_{\tilde{u}}$ surjective. There exists a small neighborhood $U^D_{\tilde{u}}$ of $\tilde{u}$ in $\mathcal{W}^D$ such that, for all $\tilde{v} \in U^D_{\tilde{u}}$, the stabilized operator $\overline{\partial}_{\tilde{v}}^{W_{\tilde{u}}}$ is surjective.
By propositions 5.6 and 5.12, there exists a small neighborhood $U_{\tilde{u}}$ of $\tilde{u}$ in $\mathcal{W}$ such that, for all $\tilde{v} \in U_{\tilde{u}}$, the stabilized operator $\overline{\partial}_{\tilde{v}}^{W_{\tilde{u}}}$ is surjective.

On the other hand, for each holomorphic map $\tilde{u}$, there exists a finite dimensional subspace $W'_{\tilde{u}}$ of $T_{\tilde{u}}\overline{\mathcal{B}}$, consisting of smooth functions with support away from the nodes, such that the differential at $\tilde{u}$ of the evaluation map $ev^+ \times ev^-$ maps $W'_{\tilde{u}}$ surjectively to the direct sum of the tangent spaces of the asymptotic orbit spaces of $\tilde{u}$.

If the neighborhood $U_{\tilde{u}}$ is sufficiently small, the same will hold for any $\tilde{v} \in U_{\tilde{u}}$. Let $\tilde{W}_{\tilde{u}} = \overline{\partial}_{\tilde{u}}(W'_{\tilde{u}})$.

We define $R_{\tilde{u}} = W_{\tilde{u}} \oplus \tilde{W}_{\tilde{u}}$. By the compactness results 4.4 and 4.16, we can extract from the collection $U_{\tilde{u}}$ a finite covering U_i of the set of holomorphic maps in $\mathcal{W}$. Let $R = \oplus_i R_i$ be the corresponding finite dimensional space. By proposition 5.16, the set $\mathcal{M}^{vir}_{\tilde{U},R} = (\overline{\partial}_J)^{-1}(R)$ of maps $\tilde{u}$ in a uniformization chart $\tilde{U}$ for $\mathcal{W}$ satisfying $\overline{\partial}_J \tilde{u} \in R$ is a smooth manifold with corners, of dimension index $\overline{\partial}_J + \dim R$. In order to obtain a local virtual cycle, we need to choose a generic element $\nu \in R$ and define $\mathcal{M}^{vir}_{\tilde{U}} = (\overline{\partial}_J + \nu)^{-1}(0) \cap \tilde{U}$.

We are now in position to use the construction of multi-sections from [21] in order to construct the relative virtual cycle. Indeed, this construction is purely topological and is based on the stratified Banach orbifold structures that we obtained in lemmata 6.1, 6.2 and 6.3. Each element of R corresponds to a multi-section on $\mathcal{W}$. We now explain how to choose a generic element of R in order to obtain relative virtual cycles satisfying the propositions 6.4 and 6.5.

Let E and A be the energy and the area of the stable curves, as in lemma 3.9. Each virtual neighborhood $\mathcal{W}$ with given asymptotics has a well-defined pair (E, A). We use the lexicographic ordering on these pairs : $(E_1, A_1) < (E_2, A_2)$ if $E_1 < E_2$ or if $E_1 = E_2$ and $A_1 < A_2$. Note that if $\tilde{u}$ is a stable map consisting of several adjacent levels in a stable map $\tilde{v}$, then $(E_{\tilde{u}}, A_{\tilde{u}}) < (E_{\tilde{v}}, A_{\tilde{v}})$.

We construct relative virtual cycles in the virtual neighborhoods $\mathcal{W}$ in the lexicographic order for the pairs (E, A). Therefore, when we construct $\mathcal{M}^{vir}$, the relative virtual cycles that should appear in its boundary $\partial\mathcal{M}^{vir}$ are already constructed. In other words, we already have a generic multi-section ν on the strata D corresponding to stable maps with several levels. Note that there exist generic multi-sections in $\mathcal{W}$ extending the given multi-section ν. Indeed, by propositions 5.6 and 5.12, we keep surjectivity of the linearized operator if we extend ν in the natural way on glued holomorphic curves.

Hence, by Sard's theorem it is possible to choose a generic element $\bar{\nu} \in R$ extending ν and such that, for each $\tilde{U}_i$, the preimage of $\bar{\nu}$ under the natural evaluation map

$\mathcal{M}^{vir}_{\tilde{U}_i,R} \to R$ is a branched manifold with corners, of dimension index $\overline{\partial}_J$.
On the other hand, we want to choose $\bar{\nu} \in R$ so that the evaluation maps to the orbit spaces satisfy the transversality conditions (iii), (iv) and (v) from section 6.1. When constructing the virtual cycle $\mathcal{M}^{vir}$, we need to enforce these conditions with respect to the finite number of virtual cycles we already constructed. Hence, each evaluation map $ev : \mathcal{M}^{vir} \to S$ must be transverse to finitely many given maps $f : X \to S$. By construction of R, it is clear that the map $ev_R : \mathcal{M}^{vir}_R \to S$ is transverse to any map $f : X \to S$. Therefore, we just need to find a regular value of the natural evaluation map $\mathcal{M}^{vir}_R \times_S X \to R$. By Sard's theorem, we can achieve this if we choose $\bar{\nu}$ outside a set of measure zero in R. Therefore, we can find $\bar{\nu} \in R$ satisfying all the above requirements simultaneously.

Finally, once the virtual cycles $\mathcal{M}^{vir}$ are constructed, we can also construct the virtual cycles $\mathcal{M}^{f_T,vir}$ as explained in section 6.1. Then, we extend the corresponding multi-sections from $\mathcal{W}^{[0,\lambda_0]} \cap \{\lambda = 0\}$ to $\mathcal{W}^{[0,\lambda_0]}$ as above. This way, we can construct all the virtual cycles $\mathcal{M}^{vir}_{[0,\lambda_0]}$.

In what follows, we will refer to the moduli spaces of holomorphic maps or the virtual cycles as $\mathcal{M}$, in order to keep the notation simple.

6.4 Free actions on virtual neighborhood

In this section, we assume that a compact Lie group G acts on the contact manifold M, preserving the contact form α and the almost complex structure J. We also assume that this action on M induces a free action on the virtual neighborhood $\mathcal{W}$ of (a connected component of) the set of J-holomorphic maps to $\mathbb{R} \times M$.

Proposition 6.6. *Under the above assumptions, if* $\dim G > index_{\mathcal{W}}(\overline{\partial}_J)$*, then the relative virtual cycle in* $\mathcal{W}$ *is empty.*

Proof. We adapt the construction of section 6.2 to construct a stratified Banach orbifold structure on $\mathcal{W}/G$. In addition to the marked points $\tilde{x}_i$, $i = 1, \ldots, k$, designed to eliminate the free action of $\mathbb{R}$ by translation, we introduce marked points to elimatinate the free action of G. Let $v_1, \ldots, v_h$ be a basis for the Lie algebra $\mathcal{G}$ of G.

Let $v_1^M, \dots, v_h^M$ be the corresponding vector fields on M. Choose marked points $\tilde{x}_j^G$, $j = 1, \dots, h$ on Σ, and small pieces of real codimension 3 hyperplanes H_j^G passing through $\tilde{u}(\tilde{x}_j^G)$ in such a way that $\tilde{u}_*(T_{\tilde{x}_j^G}\Sigma) \oplus \mathbb{R}v_j^M \oplus H_j^G = T_{\tilde{u}(\tilde{x}_j^G)}M$. Since the action of G on $\mathcal{W}$ is free, it is always possible to find these marked points and hyperplanes. We then define the uniformization charts $\tilde{U}^{D,G}$ and $\tilde{U}^G$ as before, with the additional requirement that $\tilde{u}'(x_j^G) \in H_j^G$ for $j = 1, \dots, h$. The action of the automorphism group Γ of $\tilde{u}$ is extended to $\tilde{U}^{D,G}$ and $\tilde{U}^G$ as before, and we define the quotients $U^{D,G}$ and U^G. Note that, if those local slices near $\tilde{u}$ are chosen sufficiently small, then 2 distincts points are not in the same G-orbit, because the action of G on $\mathcal{W}$ is free. Hence, $U^{D,G}$ and U^G are neighborhoods of $\tilde{u}$ in $\mathcal{W}/G$, and they induce as before a stratified Banach orbifold structure.

Since the almost complex structure J is preserved by the action of G, the Fredholm section $\overline{\partial}_J$ descends to $\mathcal{W}/G$. We are therefore in the same situation as in the previous section, and we can construct a relative virtual cycle for the Fredholm section in $\mathcal{W}/G$. Moreover, the relative virtual cycle in $\mathcal{W}$ is the preimage under the natural projection $\pi : \mathcal{W} \to \mathcal{W}/G$ of the relative virtual cycle in $\mathcal{W}/G$. But $\mathrm{index}_{\mathcal{W}/G}(\overline{\partial}_J) = \mathrm{index}_{\mathcal{W}}(\overline{\partial}_J) - \dim G < 0$, hence both relative virtual cycles are empty. $\square$

Chapter 7

Coherent orientations

7.1 Nondegenerate asymptotics

The construction of a set of coherent orientations for the moduli spaces in Symplectic Field theory has been carried out in a joint work with Klaus Mohnke [1].
This abstract construction is done at the level of Fredholm operators. Therefore, we will consider the space $\mathcal{O}(\gamma_1^+, \dots, \gamma_{s^+}^+; \gamma_1^-, \dots, \gamma_{s^-}^-)$ of Fredholm operators with analytical expression near the punctures corresponding to the given closed Reeb orbits. For fixed asymptotics, this space is contractible.
Moreover, $\mathcal{O}$ carries a natural real line bundle $\mathcal{L}$, the determinant line bundle of the Fredholm operators, defined as the top external power of the index bundle of these operators. Since $\mathcal{O}$ is contractible, $\mathcal{L}$ is trivial and has a global nonvanishing section. The choice of such a section (up to homotopy) is equivalent to the choice of an orientation on the corresponding moduli space $\mathcal{M}$, because $\pi^*\mathcal{L}$ is naturally isomorphic to $\Lambda^{\text{top}} T\mathcal{M}$, where $\pi : \mathcal{M} \to \mathcal{O}$ associates to every holomorphic map its linearized Cauchy-Riemann operator.
It is important to realize that the abstract construction in $\mathcal{O}$ is independent of the construction of the moduli spaces $\mathcal{M}$, and in particular of the specific way we achieve transversality.
Coherent orientations satisfy the following axioms :

(i) The coherent orientation of $\mathcal{O}(\gamma_1^+,\dots,\gamma_k^+,\gamma_{k+1}^+,\dots,\gamma_{s^+}^+;\gamma_1^-,\dots,\gamma_{s^-}^-)$ and the coherent orientation of $\mathcal{O}(\gamma_1^+,\dots,\gamma_{k+1}^+,\gamma_k^+,\dots,\gamma_{s^+}^+;\gamma_1^-,\dots,\gamma_{s^-}^-)$ coincide up to a factor $(-1)^{|\gamma_k^+|\cdot|\gamma_{k+1}^+|}$, where $|\gamma_i^\pm| = \mu_{CZ}(\gamma_i^\pm)+n-3$.
A similar statement holds for reordering of negative punctures.

(ii) The disjoint union map u

$$\begin{aligned}&\mathcal{O}(\gamma_1^+,\dots,\gamma_{s^+}^+;\gamma_1^-,\dots,\gamma_{s^-}^-)\times\mathcal{O}(\gamma'^+_1,\dots,\gamma'^+_{s'^+};\gamma'^-_1,\dots,\gamma'^-_{s'^-})\\ &\to\mathcal{O}(\gamma_1^+,\dots,\gamma_{s^+}^+,\gamma'^+_1,\dots,\gamma'^+_{s'^+};\gamma_1^-,\dots,\gamma_{s^-}^-,\gamma'^-_1,\dots,\gamma'^-_{s'^-})\end{aligned}$$

preserves coherent orientations up to a factor

$$(-1)^{(|\gamma_1^-|+\dots+|\gamma_{s^-}^-|)(|\gamma'^+_1|+\dots+|\gamma'^+_{s'^+}|)}.$$

(iii) The gluing map ϕ

$$\begin{aligned}&\mathcal{O}(\gamma_1^+,\dots,\gamma_{s^+}^+;\gamma_1^-,\dots,\gamma_{s^-}^-)\times\mathcal{O}(\gamma'^+_1,\dots,\gamma'^+_{s'^+};\gamma'^-_1,\dots,\gamma'^-_{s'^-})\\ &\to\mathcal{O}(\gamma_1^+,\dots,\gamma_{s^+}^+,\gamma'^+_{t+1},\dots,\gamma'^+_{s'^+};\gamma_1^-,\dots,\gamma_{s^--t}^-,\gamma'^-_1,\dots,\gamma'^-_{s'^-})\end{aligned}$$

that is defined when $\gamma^-_{s^-+1-i}=\gamma'^+_i$ for $i=1,\dots,t$, preserves coherent orientations up to a factor

$$(-1)^{(|\gamma'^+_{t+1}|+\dots+|\gamma'^+_{s'^+}|)(|\gamma_1^-|+\dots+|\gamma_{s^--t}^-|)}.$$

7.2 Degenerate asymptotics

We now explain how to generalize the construction of coherent orientations to the Morse-Bott case.

First, the definition of coherent orientations requires that the orbit spaces S_i are orientable. Indeed, in order to induce an orientation on $A\times_S B$ from orientations on A and B, we also need an orientation on S. Then, we define an orientation of $A\times_S B$

so that the isomorphism

$$T_{(a,b)}(A \times_S B) \oplus T_s S \simeq T_a A \oplus T_b B$$

changes the orientations by a sign $(-1)^{\dim B \dim S}$. This sign is necessary to make the fibered product associative.

Then, note that the moduli spaces are not always orientable. Indeed, when the asymptotic expression of the linearized Cauchy-Riemann operator is not fixed, theorem 2 of [7] shows that the determinant line bundle over the space $\mathcal{O}$ of Cauchy-Riemann operators is not trivial. Therefore, a non contractible loop in N_T may induce a "disorienting loop" of asymptotic linearized Cauchy-Riemann operators that makes the determinant line bundle non orientable.
If the projection of that disorienting loop to S_T is contractible, then the original loop in N_T is homotopic to a closed Reeb orbit with period dividing T. That Reeb orbit is then *bad* in the following sense :

Definition 7.1. *A Reeb orbit γ is said to be bad if it is the $2m$-cover of a simple orbit $\gamma' \in S_T$ and if $(\mu(S_{2T}) \pm \frac{1}{2} \dim S_{2T}) - (\mu(S_T) \pm \frac{1}{2} \dim S_T)$ is odd. If a Reeb orbit γ is not bad, then we say it is good.*

This definition extends the definition of bad orbits in the non-degenerate case that was formulated in [1].

Note that there are no bad orbits if and only if there are no orbits $\gamma \in S_T$ so that $(\mu(S_{2T}) - \frac{1}{2} \dim S_{2T}) - (\mu(S_T) - \frac{1}{2} \dim S_T)$ is odd and if $\dim S_{2T} - \dim S_T$ is even. If $\dim S_{2T} - \dim S_T$ is odd, then the Poincaré return map of a Reeb orbit contained in N_T is orientation reversing in N_{2T}. This implies that N_{2T} is not orientable.

Assume that there are no bad orbits. Then a disorienting loop in N_T for the determinant line bundle of the linearized Cauchy-Riemann operator projects to a non-contractible loop in S_T. Therefore, in order to guarantee that the moduli spaces are orientable, we also have to assume that $\pi_1(S_T)$ has no disorienting loops.

Summing up, we have

Lemma 7.2. *Assume that, for all $T \in \sigma(\alpha)$, N_T and S_T are orientable, $\pi_1(S_T)$ has no disorienting loop, and all elements of S_T are good. Then the moduli spaces $\mathcal{M}$ of holomorphic maps are orientable.*

We now assume that the moduli spaces are orientable. Let us define a gluing map for the kernel of linearized Cauchy-Riemann operators with degenerate asymptotics. Let $\mathcal{O}$ and $\mathcal{O}'$ be spaces of such operators such that some orbit spaces S in the negative asymptotics of $\mathcal{O}$ are also present in the positive asymptotics of $\mathcal{O}'$. Then we can define $\mathcal{O}\sharp_S\mathcal{O}'$ to be the space of operators $\overline{\partial}\sharp_S\overline{\partial}'$, where $\overline{\partial} \in \mathcal{O}$ and $\overline{\partial}' \in \mathcal{O}'$. Clearly, the choice of an orientation on $\mathcal{O}\sharp_S\mathcal{O}'$ is equivalent to the choice of an orientation on $\mathcal{M} \times_S \mathcal{M}'$, where $\mathcal{M}$ and $\mathcal{M}'$ are the corresponding moduli spaces. Indeed, if the operators $\overline{\partial}$ and $\overline{\partial}'$ are surjective (for example after stabilization), then the kernel of $\mathcal{O}\sharp_S\mathcal{O}'$ is isomorphic to $\ker \overline{\partial} \oplus_S \ker \overline{\partial}'$.
We want to define a gluing map ϕ inducing from this an orientation on the space $\mathcal{O}''$ of operators obtained by gluing operators in $\mathcal{O}$ and operators in $\mathcal{O}'$ at punctures asymptotic to orbit spaces S.

Proposition 7.3. *There is a natural isomorphism*

$$\phi : \ker \overline{\partial} \oplus_S \ker \overline{\partial}' \to \ker(\overline{\partial}\sharp\overline{\partial}')$$

that is defined up to homotopy.

Proof. Let Q_R be the uniformly bounded right inverse for the glued operator $\overline{\partial}_R \in \mathcal{O}''$, as in proposition 5.6. We define ϕ to be the composition of the map g_R, as in section 5.3, and the projection map $I - Q_R\overline{\partial}_R$ from the domain of $\overline{\partial}_R$ to its kernel, along the image of Q_R. In other words, $\phi = (I - Q_R\overline{\partial}_R) \circ g_R$.
We claim that the restriction of ϕ to $\ker \overline{\partial} \oplus_S \ker \overline{\partial}'$ is an isomorphism. By the index formula, the dimensions of both spaces agree, and it is enough to show that ϕ is injective.
By contradiction, assume that for any large R, we can find $\xi_R \in \ker \overline{\partial}$ and $\xi'_R \in \ker \overline{\partial}'$ such that $\|\xi_R\| + \|\xi'_R\| = 1$ and $g_R(\xi_R, \xi'_R) = Q_R\eta_R$, for some η_R. Applying $\overline{\partial}_R$ to the last equation, we see that $\lim_{R\to\infty} \overline{\partial}_R g_R(\xi_R, \xi'_R) = 0$. But since $\overline{\partial}_R Q_R = I$,

it follows that $\eta_R \to 0$ when $R \to \infty$. Using this in the original equation gives $\lim_{R\to\infty} g_R(\xi_R, \xi'_R) = 0$. But this contradicts $\|\xi_R\| + \|\xi'_R\| = 1$. $\square$

With this definition of the gluing map, we can construct, as in [1], a set of coherent orientations on the moduli spaces.
Coherent orientations satisfy the following axioms :

(i) The coherent orientation of $\mathcal{O}(S_1^+, \ldots, S_k^+, S_{k+1}^+, \ldots, S_{s^+}^+; S_1^-, \ldots, S_{s^-}^-)$ and the coherent orientation of $\mathcal{O}(S_1^+, \ldots, S_{k+1}^+, S_k^+, \ldots, S_{s^+}^+; S_1^-, \ldots, S_{s^-}^-)$ coincide up to a factor $(-1)^{|S_k^+|.|S_{k+1}^+|}$, where $|S_i^\pm| = \mu(S_i^\pm) \pm \frac{1}{2}\dim S_i^\pm + n - 3$.
A similar statement holds for reordering of negative punctures.

(ii) The disjoint union map u

$$\begin{aligned}&\mathcal{O}(S_1^+, \ldots, S_{s^+}^+; S_1^-, \ldots, S_{s^-}^-) \times \mathcal{O}(S'^+_1, \ldots, S'^+_{s'^+}; S'^-_1, \ldots, S'^-_{s'^-})\\ &\to \mathcal{O}(S_1^+, \ldots, S_{s^+}^+, S'^+_1, \ldots, S'^+_{s'^+}; S_1^-, \ldots, S_{s^-}^-, S'^-_1, \ldots, S'^-_{s'^-})\end{aligned}$$

preserves coherent orientations up to a factor

$$(-1)^{(|S_1^-|+\ldots+|S_{s^-}^-|)(|S'^+_1|+\ldots+|S'^+_{s'^+}|)}.$$

(iii) The gluing map ϕ

$$\begin{aligned}&\mathcal{O}(S_1^+, \ldots, S_{s^+}^+; S_1^-, \ldots, S_{s^-}^-)\sharp_{S'^+_1\times\ldots\times S'^+_t}\mathcal{O}(S'^+_1, \ldots, S'^+_{s'^+}; S'^-_1, \ldots, S'^-_{s'^-})\\ &\to \mathcal{O}(S_1^+, \ldots, S_{s^+}^+, S'^+_{t+1}, \ldots, S'^+_{s'^+}; S_1^-, \ldots, S_{s^--t}^-, S'^-_1, \ldots, S'^-_{s'^-})\end{aligned}$$

that is defined when $S^-_{s^-+1-i} = S'^+_i$ for $i = 1, \ldots, t$, preserves coherent orientations up to a factor

$$(-1)^{(|S'^+_{t+1}|+\ldots+|S'^+_{s'^+}|)(|S_1^-|+\ldots+|S^-_{s^--t}|)}\ (-1)^{\sum_{i=1}^t(\dim S^-_{s^--t+i}\sum_{j=i+1}^t |S^-_{s^--t+j}|)}.$$

Lemma 7.4. *Under the assumptions of lemma 7.2, the moduli spaces of holomorphic maps can be equipped with coherent orientations.*

We construct coherent orientations following the same steps as in [1].

Step 0. Operators on closed Riemann surfaces are oriented using their natural complex orientation.

Step 1. Operators on spheres with one positive puncture corresponding to asymptotics in the orbit space S are given an arbitrary orientation.

Step 2. Operators on spheres with one negative puncture corresponding to asymptotics in the orbit space S are given the orientation so that the induced orientation on $\mathcal{O}(\emptyset; S)\sharp_S\mathcal{O}(S;\emptyset)$ is the complex orientation of step 0.

Step 3. Operators on general Riemann surfaces with positive and negative punctures are given the orientation so that the induced orientation on

$$\begin{aligned}\mathcal{O}(\emptyset; S^+_{s^+})\sharp_{S^+_{s^+}} \dots \quad & \mathcal{O}(\emptyset; S^+_1)\sharp_{S^+_1} \\ & \mathcal{O}(S^+_1, \dots, S^+_{s^+}; S^-_1, \dots, S^-_{s^-})\sharp_{S^-_{s^-}}\mathcal{O}(S^-_{s^-};\emptyset)\dots\sharp_{S^-_1}\mathcal{O}(S^-_1;\emptyset)\end{aligned}$$

is the complex orientation of step 0.

From now on, we will use directly the moduli spaces in our notation, since that makes it easier to see the fibered products, keeping in mind that our construction is actually at the level of operators in $\mathcal{O}$.

Proof of lemma 7.4. We need to check that the orientations constructed above satisfy the 3 axioms for coherent orientations.
First, for reordering punctures, we use the identity

$$A \times_{S_1} B_1 \times_{S_2} B_2 = A \times_{S_2} B_2 \times_{S_1} B_1 (-1)^{(\dim S_1 + \dim B_1)(\dim S_2 + \dim B_2)}$$

and apply it to $A = \mathcal{M}$ and B_i moduli spaces of holomorphic spheres with one positive puncture. Since $\dim B_i = n - 3 + \frac{1}{2}\dim S_i + \mu(S_i)$, the parity of a negative puncture is given by $|S^-| = n - 3 - \frac{1}{2}\dim S^- + \mu(S^-)$. For a positive puncture, we use B_i moduli spaces of holomorphic spheres with one positive puncture, with $\dim B_i = n - 3 + \frac{1}{2}\dim S_i - \mu(S_i)$, so that $|S^+| = n - 3 + \frac{1}{2}\dim S^- + \mu(S^-)$, after

changing the sign and adding $2(n-3)$.
Next, for disjoint union, we use the identity

$$A \times (B \times_S A') = B \times_S (A \times A')(-1)^{(\dim S + \dim B)\dim A}$$

and apply it to $A = \mathcal{M}(\emptyset; S^+) \times_{S^+} \mathcal{M}(S^+; S^-), A' = \mathcal{M}(S'^+; S'^-) \times_{S'^-} \mathcal{M}(S'^-; \emptyset)$, $S = S'^+$ and $B = \mathcal{M}(\emptyset; S'^+)$. The corresponding exponent is given by $|S'^+|.|S^-|$, as desired. The case of multiple punctures is completely similar.
Finally, for gluing, we use the first identity to deduce that, if $S_{-2} = S'_{+1} = S$,

$$\begin{aligned} B'_{+2} & \times_{S'_{+2}} B_+ \times_{S_+} \mathcal{M}(S_+; S_{-1}, S_{-2}) \times_S (\mathcal{M}(S'_{+1}, S'_{+2}; S'_-) \times_{S'_-} B'_-) \times_{S_{-1}} B_{-1} \\ = \; & B'_{+2} \times_{S'_{+2}} B_+ \times_{S_+} \mathcal{M}(S_+; S_{-1}, S_{-2}) \times_{S_{-1}} B_{-1} \times_S (\mathcal{M}(S'_{+1}, S'_{+2}; S'_-) \times_{S'_-} B'_-) \\ & (-1)^{|S_{-1}|(|S'_{+1}|+|S'_{+2}|)} \\ = \; & B'_{+2} \times_{S'_{+2}} B_+ \times_{S_+} \mathcal{M}(S_+; S_{-2}, S_{-1}) \times_{S_{-1}} B_{-1} \times_S (\mathcal{M}(S'_{+1}, S'_{+2}; S'_-) \times_{S'_-} B'_-) \\ & (-1)^{|S_{-1}|.|S'_{+2}|}. \end{aligned}$$

In the last line, $B_+ \times_{S_+} \mathcal{M}(S_+; S_{-2}, S_{-1}) \times_{S_{-1}} B_{-1}$ has by definition the same orientation as B'_{+1}, therefore, the last line has the complex orientation up to sign $(-1)^{|S_{-1}|.|S'_{+2}|}$. This is the desired result, and the case of multiple punctures is again completely similar. □

7.3 Generalized holomorphic maps

First, we need to determine whether the perturbed orbits $\gamma^p_{kT'}$ are good or bad. This is one of the reasons we chose to extend the Morse functions f_T using a positive definite Hessian on the normal bundle of S_T.

Lemma 7.5. *Under the assumptions of lemma 7.4, all perturbed Reeb orbits $\gamma^p_{kT'}$ are good.*

Proof. The orbit $\gamma^p_{kT'}$ is bad if and only if k is even and $\mu_{CZ}(\gamma^p_{2T'}) - \mu_{CZ}(\gamma^p_{T'})$ is odd. By lemma 2.4, the last condition reads : $(\mu(S_{2T'}) - \frac{1}{2}\dim S_{2T'} + \mathrm{index}_p(f_{2T'})) -$

$(\mu(S_{T'}) - \frac{1}{2}\dim S_{T'} + \mathrm{index}_p(f_{T'}))$ is odd. But $\mathrm{index}_p(f_{2T'}) = \mathrm{index}_p(f_{T'})$, since the normal bundle to $S_{T'}$ in $S_{2T'}$ does not contribute to the Morse index of $f_{2T'}$. Hence, $\gamma^p_{kT'}$ is bad if and only if the non perturbed orbit $p \in S_{kT'}$ is bad. There are no such orbits under the assumptions of lemma 7.4. $\square$

Lemma 7.6. *A coherent set of orientations on $\mathcal{M}$, as in section 7.2, induces a coherent set of orientations for the moduli spaces with non-degenerate asymptotics by*

$$\begin{aligned}W^u(p_1^+) \times_{S_1^+} \dots \; W^u(p_{s^+}^+) \times_{S_{s^+}^+} \\ \mathcal{M}^{f_T}(S_1^+, \dots S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \times_{S_1^-} W^s(p_1^-) \dots \times_{S_{s^-}^-} W^s(p_{s^-}^-)\end{aligned}$$

multiplied with the sign $(-1)^{\delta^+ + \delta^-}$*, where*

$$\begin{aligned}\delta^+ &= \sum_{i=1}^{s^+} \Big((\mathrm{index}(p_i^+) + \dim S_i^+) \sum_{j=1}^{i-1} |S_j^+| \Big), \\ \delta^- &= \sum_{i=1}^{s^-} \Big(\mathrm{index}(p_i^-) \sum_{j=i+1}^{s^-} |S_j^-| \Big).\end{aligned}$$

Proof. In order to make the transition from degenerate asymptotics in S_i^+ to non-degenerate asymptotics at critical point $p_i^+ \in S_i^+$, we need to consider linear Cauchy-Riemann operators on cylinders corresponding to the moduli space $\mathcal{M}(\gamma_{p_i^+}; S_i^+)$. By proposition 5.14, these operators are surjective and their kernel is given by $W^u(p_i^+) \oplus \mathrm{span}(\frac{\partial}{\partial t}, R_\alpha)$. The second summand is canonically oriented by the complex orientation. Note that choosing an orientation for the first summand is equivalent to choosing an orientation for the operator on a sphere with one negative puncture corresponding to the asymptotics at p_i^+. Choosing such an orientation is equivalent to extend step 1 of the construction of coherent orientations to non-degenerate asymptotics.

For negative asymptotics, we consider similarly $\mathcal{M}(S_i^-; \gamma_{p_i^-})$. In this case, the corresponding operator is surjective and its kernel is $W^s(p_i^-) \oplus \mathrm{span}(\frac{\partial}{\partial t}, R_\alpha)$. This time, the orientation must be chosen so that the orientation on $\mathcal{M}(S; \gamma_p) \times \mathcal{M}(\gamma_p; S)$ induces the positive orientation on $\mathcal{M}(S; S) \simeq S$. In other words, we choose the orientation of $W^s(p)$ so that $T_pW^s(p) \oplus T_pW^u(p) \simeq T_pS$ as oriented vector spaces. This is the convention we use in Morse theory.

Gluing those cylinders at the positive punctures, we obtain

$$\begin{aligned}
W^u(p_1^+) \quad & \times_{S_1^+} \dots W^u(p_{s^+}^+) \times_{S_{s^+}^+} \mathcal{M}^{f_T}(S_1^+, \dots S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \\
= \quad & \mathcal{M}(\gamma_{p_1^+}; S_1^+) \times_{S_1^+} \dots \mathcal{M}(\gamma_{p_i^+}; S_i^+) \times_{S_{s^+}^+} \mathcal{M}^{f_T}(S_1^+, \dots S_{s^+}^+; S_1^-, \dots, S_{s^-}^-) \\
= \quad & \mathcal{M}(\gamma_{p_1^+}; S_1^+) \times_{S_1^+} \dots \mathcal{M}(\gamma_{p_{s^+}^+}; S_{s^+}^+) \times_{S_{s^+}^+} \\
& \mathcal{M}^{f_T}(S_{s^+}^+, S_1^+, \dots S_{s^+-1}^+; S_1^-, \dots, S_{s^-}^-)(-1)^{|S_{s^+}^+|(\sum_{j=1}^{s^+-1} |S_j^+|)} \\
= \quad & \mathcal{M}(\gamma_{p_1^+}; S_1^+) \times_{S_1^+} \dots \mathcal{M}(\gamma_{p_{s^+-1}^+}; S_{s^+-1}^+) \times_{S_{s^+-1}^+} \\
& \mathcal{M}^{f_T}(\gamma_{p_{s^+}^+}, S_1^+, \dots S_{s^+-1}^+; S_1^-, \dots, S_{s^-}^-)(-1)^{|S_{s^+}^+|(\sum_{j=1}^{s^+-1} |S_j^+|)} \\
= \quad & \mathcal{M}(\gamma_{p_1^+}; S_1^+) \times_{S_1^+} \dots \mathcal{M}(\gamma_{p_{s^+-1}^+}; S_{s^+-1}^+) \times_{S_{s^+-1}^+} \\
& \mathcal{M}^{f_T}(S_1^+, \dots S_{s^+-1}^+, \gamma_{p_{s^+}^+}; S_1^-, \dots, S_{s^-}^-)(-1)^{(\mathrm{index}(p_{s^+}^+) + \dim S_{s^+}^+)(\sum_{j=1}^{s^+-1} |S_j^+|)}
\end{aligned}$$

and we proceed similarly for the remaining punctures. The sign we finally obtain corresponds to the exponent $\sum_{i=1}^{s^+} \Big((\mathrm{index}(p_i^+) + \dim S_i^+) \sum_{j=1}^{i-1} |S_j^+| \Big) = \delta^+$ as desired. The case of negative punctures is completely similar. $\square$

Chapter 8

Proof of the main theorems

8.1 Cylindrical homology

We now want to use the results of the previous chapters to prove theorem 1.9.

The Morse-Bott chain complex $C_*^{\bar{a}}$ is the graded vector space generated by the nondegenerate periodic orbits γ_{kT}^p, in homotopy class $\bar{a}$ with grading $\mu(S_{kT}) - \frac{1}{2}\dim S_{kT} + \mathrm{index}(p) + n - 3$.

On the other hand, the chain complex $C_*'^{\bar{a}}$ for cylindrical homology, with contact form α_λ, generally contains additional generators, corresponding to closed Reeb orbits we might have created as a result of the perturbation. We need to check that these 'extra' orbits do not contribute to cylindrical homology, so that we can compute it using the Morse-Bott chain complex.

If $\bar{a} \neq 0$, the arguments of lemma 2.7 show that we can choose $\lambda > 0$ sufficiently small so that all 'extra' orbits have homotopy class very close to a large multiple of $\bar{a}$. Therefore, they are not contained in $C_*'^{\bar{a}}$.
If $\bar{a} = 0$, fix $k \in \mathbb{Z}$; by lemma 2.7, we can choose $\lambda > 0$ sufficiently small so that all 'extra' orbits have grading much larger than k. Hence the following part of the chain complex

$$C_{k+1}'^{\bar{a}} \xrightarrow{d'_{k+1}} C_k'^{\bar{a}} \xrightarrow{d'_k} C_{k-1}'^{\bar{a}}$$

does not contain any 'extra' generator and does not involve any J-holomorphic cylinder converging to an 'extra' orbit, when $t = 0$.

If $t \neq 0$, the multiplication by cohomological variables t can lower the grading of an 'extra' orbit. But there are only 2 types of t variables having negative grading : the variable t_0 with grading -2, corresponding to the positive generator of $H^0(M, \mathbb{Q})$, and the variables t_1^j with grading -1, for $j = 1, \dots, b_1(M)$, corresponding to the generators of $H^1(M, \mathbb{Q})$.
On one hand, note that the differential d does not involve variable t_0, because no non-constant holomorphic curve with a free marked point is rigid. Therefore, we can simply omit variable t_0 from the coefficient ring of cylindrical homology. Alternatively, since this variable cannot be destroyed by d, we can keep it and observe that cylindrical homology with t_0 is just the tensor product of cylindrical homology without t_0 and $\mathbb{Q}[t_0]$. In particular, we see that this variable does not bring new information.
On the other hand, the variables t_1^j are odd, hence each of them can appear at most once in a non-vanishing product. Therefore, these variables can lower the degree of an 'extra' orbit by at most $b_1(M)$. Then, we just need to choose $\lambda > 0$ sufficiently small so that the 'extra' orbits have grading much larger than $k + b_1(M)$.

From this we deduce that homology of the Morse-Bott chain complex in degree k agrees with $HF_k^{\bar{a}}(M, \xi)$. But since k was arbitrary, we deduce that homology of the Morse-Bott chain complex is isomorphic to contact homology.

Next, we want to rewrite the differential d for cylindrical homology with contact form α_λ using moduli spaces of generalized J-holomorphic curves instead.

In order to do this, let us consider the moduli space $\mathcal{M}_{0,1,1,k}^{(0,\lambda_0]}(\gamma^+; \gamma^-)$ of J_λ-holomorphic cylinders, $0 < \lambda \leq \lambda_0$, with the prescribed asymptotics. The restriction of this moduli space to $\lambda = \lambda_0$ is used to compute cylindrical homology with the perturbed contact form α_{λ_0} and almost complex structure J_{λ_0}. On the other hand, by the results of the previous chapters, the compactification of this moduli space at $\lambda = 0$ is given by the moduli space of generalized holomorphic cylinders $W^u(\gamma^+) \times_S \mathcal{M}^{f_T}(S; S') \times_{S'} W^s(\gamma^-)$

with the same asymptotics.
In general, the moduli space $\mathcal{M}^{(0,\lambda_0]}_{0,1,1,s^0}(\gamma^+;\gamma^-)$ can have additional boundary components at $\lambda \in (0,\lambda_0)$, if holomorphic cylinders split in several cylinders, with at least one cylinder of index 0. But if we choose $\lambda_0 > 0$ sufficiently small, the gluing estimates of section 5.3 show that the linearized Cauchy-Riemann operators will be surjective (after stabilization at $\lambda = 0$) and this cannot happen. Therefore, we obtain a cobordism between the moduli spaces at $\lambda = 0$ and $\lambda = \lambda_0$.
We deduce that the algebraic number of elements in

$$\mathcal{M}^{(0,\lambda_0]}_{0,1,1,s^0}(\gamma^+;\gamma^-) \times_M \bar{\theta} \ldots \times_M \bar{\theta}$$

agrees with the algebraic number of elements in

$$W^u(\gamma^+) \times_S \mathcal{M}^{f_T}(S;S') \times_{S'} W^s(\gamma^-) \times_M \bar{\theta} \ldots \times_M \bar{\theta}.$$

Hence, the differential d can be computed in terms of the Morse-Bott data.

In order to facilitate practical computations, it is convenient to consider separately the moduli spaces of J_λ-holomorphic cylinders converging to critical points p and q of f_T in the same orbit space S and with no marked points.
In that case, we obtain for $\lambda \to 0$ the moduli space

$$W^u(p) \times_S \mathcal{M}^{f_T}_{0,1,1,0}(S;S) \times_S W^s(q) = W^u(p) \cap W^s(q)$$

of gradient trajectories of the Morse function f_T on S, joining critical points p and q. Hence, the contribution of these trajectories to the differential d is exactly the Morse-Witten differential ∂ of the Morse functions f_T.

Therefore, the differential d for cylindrical homology can be expressed as

$$dp = \partial p + \sum_q n_{q,t}\, q$$

where $n_{q,t}$ is the sum over s^0 of the polynomials in t given by the 0-dimensional part of the fibered products

$$\left(W^u(p) \times_S \mathcal{M}^{f_T}_{0,1,1,s^0}(S;S') \times_{S'} W^s(q) \times_M\right) \times_M \bar{\theta} \ldots \times_M \bar{\theta}/\mathbb{R}$$

for $s^0 = 0, 1, 2, \ldots$ and $s^0 \neq 0$ if $S = S'$.

8.2 Contact homology

We now generalize the arguments of the previous section to compute contact homology and prove theorem 1.8.

The Morse-Bott chain complex C_* is the free supercommutative algebra with unit generated by the critical points p of all Morse functions f_T, or equivalently by the nondegenerate periodic orbits γ^p_{kT}, with grading $\mu(S_{kT}) - \frac{1}{2}\dim S_{kT} + \mathrm{index}(p) + n - 3$.

On the other hand, the chain complex C'_* for contact homology, with contact form α_λ, generally contains additional generators, corresponding to closed Reeb orbits we might have created as a result of the perturbation. We need to check that these 'extra' orbits do not contribute to contact homology, so that we can compute it using the Morse-Bott chain complex.

As in the previous section, we consider the chain complex at a fixed degree $k \in \mathbb{Z}$. By lemma 2.7, we can choose $\lambda > 0$ small to make the grading of the 'extra' orbits much larger than k.
This grading can be lowered by some of the t variables, and we can handle them as before : the variable t_0 contains no information, so we can omit it or include it using a slightly more refined argument as in the last section. The variables t_1 contribute to a bounded shift only.

However, the grading of the 'extra' orbits can also be lowered by multiplication with closed Reeb orbits with negative grading. Let γ be an 'extra' orbit, with grading $|\gamma|$ and action T, both very large, and assume that the product $\gamma\gamma_1 \ldots \gamma_l$ has grading less

or equal than $k+1$. Note that the grading of closed Reeb orbits is bounded below by $-\frac{1}{2}(\dim M + 1)$, hence l is bounded below by a multiple of $|\gamma|$. Since the action spectrum is bounded away from zero and $|\gamma| > \frac{c}{2}T$, the total action of the product $\gamma\gamma_1 \dots \gamma_l$ is bounded below by $(1+\epsilon)T$, where $0 < \epsilon < 1$ is a constant independent of $\lambda > 0$.
Hence, the action of each term in $d(\gamma\gamma_1 \dots \gamma_l)$ is bounded below by ϵT. Let us introduce some notation

$$\begin{aligned} C_*^T &= \{x \in C_* \,|\, \mathcal{A}_\lambda(x) < \epsilon T\}, \\ Z_*^T &= \{x \in C_*^T \,|\, dx = 0\}, \\ B_* &= \{x \in C_* \,|\, x = dy \text{ for some } y \in C_*\}, \\ H_*^T &= \frac{Z_*^T}{B_* \cap C_*^T}, \end{aligned}$$

and analogous definitions for the chain complex C'_*.
Then the above conclusions can be reformulated as $H_*^{'T} = H_*^T$, for $* \leq k$. On the other hand, note that

$$H_*^{'T} = \{x \in HC_k(M,\xi) \,|\, x \text{ has a representative in } C_*^{'T}\} = HC_*^T(M, \alpha_\lambda).$$

In particular, it follows that $\cup_{T>0} H_*^{'T} = HC_*(M,\xi)$. We would like to deduce from this that the homology of the Morse-Bott chain complex is isomorphic to contact homology, but we have to decrease $\lambda > 0$ in order to increase T, and the action filtering is not invariant under a general change of contact form for ξ.

Note that we can assume $f_T > 0$, after adding a large positive constant, that does not change the gradient dynamics. In particular, if $0 < \lambda' < \lambda$, then $1 + \lambda' f_T < 1 + \lambda f_T$. Hence, there exists a smooth function

$$g_{\lambda',\lambda} : [0,1] \times M \to \mathbb{R} : (t,p) \to g^t_{\lambda',\lambda}(p)$$

such that $g^t_{\lambda',\lambda} = 1 + \lambda' f_T$ if $t = 0$, $g^t_{\lambda',\lambda} = 1 + \lambda f_T$ if $t = 1$ and $\frac{\partial}{\partial t} g^t_{\lambda',\lambda} > 0$. According to [5], the symplectic cobordism $([0,1] \times M, d(g^t_{\lambda',\lambda}\alpha))$ induces an isomorphism $\phi_{\lambda',\lambda} : HC_*(M, \alpha_\lambda) \to HC_*(M, \alpha_{\lambda'})$ by counting holomorphic curves in the completed cobordism. Since $d(g^t_{\lambda',\lambda}\alpha) \geq 0$ on holomorphic curves, it follows that $\mathcal{A}_\lambda(x) \geq \mathcal{A}_{\lambda'}(\phi_{\lambda,\lambda'}(x))$, for all $x \in C'_*(M, \alpha_\lambda)$.
Hence, $\phi_{\lambda,\lambda'}(HC^T_*(M, \alpha_{\lambda'})) \supset HC^T_*(M, \alpha_\lambda)$. In particular, we can conclude that the homology of the Morse-Bott chain complex coincides with contact homology, as a vector space.

It is easy to check that the product structure is recovered as well : note that if $x, y \in C^T_*$, then $xy \in C^{2T}_*$. Hence, it follows that the isomorphism between H^T_* and HC^T_* preserves the product of elements in $H^{T/2}_*$.

Next, we rewrite the differential d for contact homology with contact form α_λ using moduli spaces of generalized J-holomorphic curves instead.

As in the previous section, we use the moduli space $\mathcal{M}^{(0,\lambda_0]}_{0,1,r,k}(\gamma^+; \gamma^-_1, \dots, \gamma^-_r)$ of J_λ-holomorphic curves, $0 < \lambda \leq \lambda_0$, with the prescribed topology and asymptotics. For $\lambda_0 > 0$ sufficiently small, this is a cobordism between the moduli spaces

$$\mathcal{M}_{0,1,r,k}(\gamma^+; \gamma^-_1, \dots, \gamma^-_r)$$

at $\lambda = \lambda_0$ and

$$W^u(p) \times_S \mathcal{M}^{f_T}(S; S_1, \dots, S_s) \times_{S_1} W^s(p_1) \dots \times_{S_s} W^s(p_s)$$

at $\lambda = 0$.

As before, we consider separately the holomorphic cylinders without marked points and with asymptotics in the same orbit space. Their contribution to d is the Morse-Witten differential ∂ for the Morse function f_T.

By the graded Leibniz rule, the differential d for contact homology is characterized by its value on a critical point $p \in S$; it is given by

$$dp = \partial p + \sum n_{(i_1,\dots,i_s),t} \frac{p_1^{i_1}}{i_1!} \cdots \frac{p_s^{i_s}}{i_s!}$$

where we sum over all unordered monomials $p_1^{i_1} \dots p_s^{i_s}$ and $n_{(i_1,\dots,i_s),t}$ is the sum over s^0 of the polynomials in t given by the 0-dimensional part of the fibered product

$$(-1)^{\delta^-} \Big(W^u(p) \times_S \mathcal{M}^{f_T}(S; S_1, \dots, S_s) \times_{S_1} W^s(p_1) \dots \times_{S_s} W^s(p_s) \Big) \times_M \bar{\theta} \dots \times_M \bar{\theta} / \mathbb{R}$$

for $s^0 = 0, 1, 2, \dots$ and $s^0 \neq 0$ if $s = 1$ and $S = S_1$.

Chapter 9

Examples

9.1 Case of a circle bundle

Let (M, ω) be a compact symplectic manifold of dimension $2n-2$, and assume that $[\omega] \in H^2(M, \mathbb{Z})$. Let $\pi : L \to M$ be the complex line bundle over M with $c_1(L) = [\omega]$.

For any choice of hermitian metric on L, the unit circle bundle $\pi : V \to M$ is a contact manifold. A contact form is obtained by choosing a connection form $i\alpha$ on V so that $\frac{1}{2\pi} d\alpha = \pi^*\omega$. For such a choice of α, the Reeb field R_α is tangent to the S^1 fibers of V. Therefore, every Reeb orbit is closed, and the space of Reeb orbits in every multiplicity $k = 1, 2, \ldots$ is naturally identified with M.

The symplectization of (V, α) is, as a manifold, the line bundle L with its zero section removed; we will denote it by L^*. An almost complex structure $\tilde{J}$ on $\xi = \ker \alpha$ compatible with $d\alpha$ induces an almost complex structure $\pi_* \tilde{J}$ on M compatible with ω. The extension J of $\tilde{J}$ on L^* is compatible with the standard complex structure on the fibers of L.

Let $\Delta_1, \ldots, \Delta_r$ be a basis of $H^*(M)$. Pick a basis of $H^*(V)$ of the form $\pi^*\Delta_{i_1}, \ldots,$ $\pi^*\Delta_{i_{r'}}, \widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_s$ where the elements $\pi^*\Delta_{i_1}, \ldots, \pi^*\Delta_{i_{r'}}$ span $\pi^*H^*(M)$. Introduce variables $t_{i_1}, \ldots, t_{i_{r'}}, \tilde{t}_1, \ldots, \tilde{t}_s$ corresponding to these basis elements of $H^*(V)$, and introduce variables $p_{k,i}$ and $q_{k,i}$ $(i = 1, \ldots, r)$ corresponding to the basis elements of

$H^*(M)$, for every positive integer k.

Let $\beta_1, \ldots, \beta_u$ be a basis of $H_2(M, \mathbb{Z})$, so that $\omega(\beta_2) = \ldots = \omega(\beta_u) = 0$ and $l = \omega(\beta_1) > 0$. Introduce variables $z_1, \ldots, z_u$ corresponding to these basis elements. Let $\tilde{z}_i$ $(i = 2, \ldots, u)$ be the variable corresponding to the image of β_i in $H_2(L)$ under the inclusion of M into L as the zero section. Those homology classes generate exactly $H_2(V)/\mathcal{R}$.

The grading of these variables is defined as follows :

$$\begin{array}{ll} |t_i| = \deg(\Delta_i) - 2, & |\tilde{t}_j| = \deg(\widetilde{\Delta}_j) - 2, \\ |p_{k,i}| = \deg(\Delta_i) - 2 - 2\frac{c}{l}k, & |q_{i,k}| = \deg(\Delta_i) - 2 + 2\frac{c}{l}k, \\ |\tilde{z}_i| = -2\langle c_1(TM), \beta_i\rangle, & \end{array}$$

where $c = \langle c_1(TM), \beta_1\rangle$. Note that this grading is fractional if $l \neq 1$, because in that case $H_1(L^*)$ contains torsion elements.
The grading of variables $q_{i,k}$ coincides with the grading of contact homology. Indeed, a tubular neighborhood of the closed orbit of multiplicity l above $p \in M$ has the form $U \times S^1$, where U is a neighborhood of p in M. With the product framing, the Maslov index vanishes. On the other hand, in order to obtain a capping disk for that orbit, consider a sphere C homologous to β_1 passing through p. The disk is realized by a section of L over C with a zero of order l at p and no pole. The Maslov index in that trivialization will be $2c$, so we obtain $\frac{2c}{l}k$ for an orbit of multiplicity k.

Define

$$\bar{u} = \sum_{j=1}^{r'} t_{i_j}\Delta_{i_j} + \epsilon \sum_{i=1}^{s} \tilde{t}_i \pi_* \widetilde{\Delta}_i + \sum_{k=1}^{\infty} \left(\bar{p}_k e^{ikx} + \bar{q}_k e^{-ikx}\right)$$

where ϵ is an odd variable, π_* is the integration along the fiber of V, $\bar{p}_k = \sum_{i=1}^{r} p_{k,i}\Delta_i$ and $\bar{q}_k = \sum_{i=1}^{r} q_{k,i}\Delta_i$.

Let

$$F(\bar{v}, z) = \sum_d \sum_{n=0}^{\infty} \frac{z_1^{d_1} \ldots z_u^{d_u}}{n!} \langle \bar{v}, \ldots, \bar{v}\rangle_{0,n,d}$$

be the Gromov-Witten potential (for genus 0) of (M, ω).

Proposition 9.1. *Let (M, ω) be a symplectic manifold with $[\omega] \in H^2(M, \mathbb{Z})$ and satisfying $c_1(TM) = \tau[\omega]$ for some $\tau \in \mathbb{R}$. Assume that M admits a perfect Morse function and that only one of the $\tilde{t}$ variables is nonzero and has odd parity. Then contact homology $HC_*(V, \xi)$ is the homology of the chain complex generated by infinitely many copies of $H_*(M, \mathbb{R})$, with degree shifts $2\frac{c}{l}k - 2$, $k = 1, 2, \ldots$ and with differential d, given by*

$$dq_{k,i} = k \sum_{j=1}^{r} (g^{-1})_{ij} \frac{\partial}{\partial p_{k,j}} H(p, q, t, \tilde{t}, \tilde{z})|_{p=0}$$

where

$$H(p, q, t, \tilde{t}, \tilde{z}) = \int d\epsilon \, \frac{1}{2\pi} \oint dx F(\bar{u}(x), \tilde{z} e^{-i\langle c_1(L), \beta\rangle x})$$

and where $g_{ij} = \int_M \Delta_i \cup \Delta_j$.

Recall that the effect of integrating with respect to an odd variable ϵ is to pick the coefficient B of ϵ in the integrand $A + B\epsilon$.

Proof. First observe that $c_1(\xi) = p^* c_1(TM) = \tau p^* \omega = 0$. Next, since M admits a perfect Morse function, the chain complex for contact homology involves directly homology of the orbit spaces, all diffeomorphic to M.

Since the projection p is holomorphic, it is clear that holomorphic curves in L^* are equivalent to the data of a closed holomorphic sphere C in M, with a holomorphic section of L over C. The zeroes and poles of that section correspond to the positive and negative punctures in L^* respectively, and their multiplicities match.

Note that, once the position and multiplicities of zeroes and poles of a section have been chosen, the only remaining degree of freedom is the phase of the section. Then, pulling back a single class $\widetilde{\Delta}_j$ to the moduli space amounts to fix the phase of the section, and then require an extra constraint at that marked point, corresponding to the class $\pi_* \widetilde{\Delta}_j$.

After the phase is fixed, by proposition 6.6, the virtual neighborhood of level 1 holomorphic curves in L^* is isomorphic to an open set in the virtual neighborhood of holomorphic spheres in M. Hence, we can compactify the moduli space of holomorphic curves in L^* by adding strata of codimension at least 2, the result being

isomorphic to the virtual cycle of holomorphic spheres in M. This explains the relationship between d and F.
Moreover, generalized holomorphic curves (including pieces of gradient flow trajectories between several components) do not contribute to the differential, because the unique $\tilde{t}$ variable can kill the S^1 degree of freedom for only one component of the generalized holomorphic curve. Therefore, the differential of the Morse-Bott chain complex is given by the above formula. □

9.2 Standard 3-sphere

We can apply the results of the previous section to compute explicitly contact homology of the standard contact 3-sphere. In this case, the base M is $\mathbb{C}P^1$, and we obtain variables $q_{k,0}$ and $q_{k,1}$ $(k = 1, 2, \dots)$ corresponding to the generators of $H^0(\mathbb{C}P^1)$ and $H^2(\mathbb{C}P^1)$ respectively. It is convenient to reindex these variables in the following way :

$$\begin{cases} q_{2i} &= q_{i,1}, \\ q_{2i-1} &= q_{i,0}, \end{cases} \quad \text{and} \quad \begin{cases} p_{2i} &= p_{i,0}, \\ p_{2i-1} &= p_{i,1}. \end{cases}$$

Proposition 9.2. *Contact homology $HC_*(S^3, \xi_0)$ of the standard contact 3-sphere is isomorphic to the free supercommutative algebra with unit generated by t_0 and q_i, $(i = 2, 3, \dots)$, where $|t_0| = -2$ and $|q_i| = 2i$.*

Proof. In this case, $M = \mathbb{C}P^1$ with its standard Kähler structure. Its Gromov-Witten potential is given by

$$F(v, z) = \frac{1}{2} v_0^2 v_1 + z \sum_{n=0}^{\infty} \frac{v_1^n}{n!}$$

where v_0 generates $H^0(\mathbb{C}P^1)$, v_1 generates $H^2(\mathbb{C}P^1)$ and z generates $H_2(\mathbb{C}P^1)$ so that $\omega(z) = 1$. Using proposition 9.1, we obtain

$$H = \frac{1}{2} t_0^2 \tilde{t} + \tilde{t} \sum_{i=1}^{\infty} p_{2i} q_{2i-1} + \tilde{t} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sum_{l=1}^{n} j_l = i} p_{2i+1} q_{2j_1} \dots q_{2j_n} + O(p^2)$$

where $\tilde{t}$ is the variable corresponding to the volume form on S^3. From this we deduce the formula for the differential :

$$dq_{2i} = i\tilde{t}q_{2i-1}$$

and

$$dq_{2i+1} = (i+1)\tilde{t}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\sum_{l=1}^{n} j_l = i} q_{2j_1}\dots q_{2j_n}.$$

Claim 1. Every expression containing a $\tilde{t}$ factor is exact.

Let us prove this by induction on the largest index of the q present in the expression, and on the exponent of that variable. First note that $\tilde{t}q_1^k = \frac{1}{k+1}d(q_1^{k+1})$. Then, let us assume that the expression has the form $\tilde{t}q_n^k F$, where F involves only variables t_0 and q_i $(i = 1, \dots, n-1)$. By the induction hypothesis, $\tilde{t}F$ has a primitive B involving variables t_0 and q_i $(i = 1, \dots, n-1)$ only as well. We have

$$d(q_n^k B) = \tilde{t}q_n^k F + kq_n^{k-1}dq_n B.$$

Since dq_n is an expression containing only variables with index lower than n, by the induction hypothesis $kq_n^{k-1}dq_n B$ is exact.

Claim 2. For every monomial q_n^k $(n \geq 2)$, there exists an expression C containing only variables t_0 and q_i $(i = 1, \dots, n-1)$ and q_n up to power $k-1$, such that $q_n^k + C$ is closed.

Such a C would have to satisfy

$$dC = -kq_n^{k-1}dq_n.$$

But since dq_n contains a factor $\tilde{t}$, the right hand size is exact and we can find a solution C.

Note that the above claim is not true for $n = 1$, since $dq_1 = \tilde{t}$. Moreover, an expression without a $\tilde{t}$ factor cannot be exact. The proposition now clearly follows. $\square$

9.3 Brieskorn spheres

We now turn to a more general example involving the Brieskorn spheres. Let $V(a) = V(a_0, \dots, a_n) = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid z_0^{a_0} + \dots + z_n^{a_n} = 0\}$ and $\Sigma(a) = V(a) \cap S^{2n+1}$.

Theorem 9.3. (Brieskorn) *When $n = 2m+1$ and $p = \pm 1 (mod\, 8)$, $a_0 = p, a_1 = 2, \dots, a_n = 2$, then $\Sigma(a)$ is diffeomorphic to S^{4m+1}.*

On $\mathbb{C}^{n+1}$, consider the 1-form $\alpha_p = \frac{i}{8}\sum_{j=0}^{n} a_j(z_j d\overline{z}_j - \overline{z}_j dz_j)$. Its restriction to $\Sigma(a)$ is a contact form, with Reeb field $R_{\alpha_p} = 4i(\frac{z_0}{a_0}, \dots, \frac{z_n}{a_n})$. Denote the corresponding contact structure by ξ_p. These are distinguished by contact homology. This result is originally due to Ustilovsky [35], and was proved by perturbing contact form α_p in order to have non-degenerate closed Reeb orbits.

Note that the quotient of $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$ by the flow of R_{α_p} is a weighted projective space $\mathbb{C}P^n_w$, i.e. an orbifold. The quotient of $\Sigma(a) = S^{2n-1}$ by this Reeb flow is the zero locus of the polynomial $z_0^p + z_1^2 + \dots + z_n^2$ in $\mathbb{C}P^n_w$, i.e. a complete intersection in a toric orbifold. $\Sigma(a)$ is a principal circle orbi-bundle over this orbifold.

Theorem 9.4. (Ustilovsky) *Under the assumptions of theorem 9.3, the contact homology for cylindrical curves $HF_k(\Sigma, \xi_p) = \mathbb{Q}^{c_k}$ where*

$$c_k = \begin{cases} 0 & \text{if } k \text{ is odd or } k < 2n-4, \\ 2 & \text{if } k = 2\lfloor \frac{2N}{p} \rfloor + 2(N+1)(n-2), \text{ for } N \in \mathbb{Z}, N \geq 1, 2N+1 \notin p\mathbb{Z}, \\ 1 & \text{otherwise.} \end{cases}$$

Here, we will prove this theorem using the contact form α_p and the Morse-Bott formalism, instead of perturbing α_p to obtain nondegenerate Reeb orbits.

Let us first study the periodic orbits of R_{α_p} and their Maslov indices. The Reeb flow is given by

$$\varphi_t(z_0, \dots, z_n) = (e^{4it/p} z_0, e^{2it} z_1, \dots, e^{2it} z_n).$$

Hence, all Reeb orbits are periodic, and there are 2 values of the action for simple orbits :

(i) Action $= \pi$ (when $z_0 = 0$).
In that case, the orbit space is

$$S_\pi = \{[z_1, \dots, z_n] \in \mathbb{C}P^{n-1} \mid z_1^2 + \dots + z_n^2 = 0\},$$

i.e. the nondegenerate quadric Q_{n-2} in $\mathbb{C}P^{n-1}$.

Lemma 9.5. *If n is odd, then $H_*(Q_{n-2}) \simeq H_*(\mathbb{C}P^{n-2})$.*

Proof. Note that Q_{n-2} is the Grassmannian of oriented 2-planes in $\mathbb{R}^n$. Indeed, the manifold $N_\pi = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1^2 + \dots, z_n^2 = 0\}$ is the unit tangent bundle of the sphere S^{n-1}, and the Reeb flow coincides with the geodesic flow on S^{n-1}. The computation of the homology is then standard and gives, for n odd, the announced result. □

Let us compute the Maslov index of these periodic orbits. The linearized Reeb flow splits into the tangential and normal bundles to S_π. For the tangential part, the linearized flow is $e^{2it}I_{n-2}$ for $0 \leq t \leq \pi$, so we obtain contribution $2(n-2)N$, where N is the multiplicity of the orbit, and for the normal part, the linearized flow is just multiplication by $e^{4it/p}$, so we obtain contribution $1 + 2\lfloor \frac{2N}{p} \rfloor$. Hence, the Maslov index is :

$$\mu = 2N(n-2) + 1 + 2\lfloor \frac{2N}{p} \rfloor.$$

(ii) Action $= p\pi$ (when $z_0 \neq 0$).
In that case, the orbit space incorporates the p-covered orbits of case (i) as a singularity with group $\mathbb{Z}_p$.

Lemma 9.6. *$S_{p\pi}$ is homeomorphic to $\mathbb{C}P^{n-1}$.*

Proof. We follow the arguments of [33]. Consider the projection $\phi : \Sigma(a) \to S^{2n-1} : (z_0, \dots, z_n) \to \frac{(z_1, \dots, z_n)}{\|(z_1, \dots, z_n)\|}$. Clearly, this map is surjective, and equivariant with respect to the Reeb flow on $\Sigma(a)$ and multiplication by a phase on

S^{2n-1}. Moreover, any two points in $\Sigma(a)$ projecting to the same point in S^{2n-1} lie on the same Reeb orbit. Hence, the orbit spaces are homeomorphic. But the one of S^{2n-1} is clearly $\mathbb{C}P^{n-1}$. □

The Maslov index is very easy to compute, since the Reeb flow is now completely periodic. For the tangential part to S_π, we obtain p times the previous result, and for the normal part, we obtain 2 (one complete turn). Hence

$$\mu = 2N((n-2)p + 2).$$

Proof of Theorem 9.4. Note that all holomorphic cylinders come in S^1 families, since they can be pushed along the Reeb field. Therefore, the differential coincides with the Morse-Witten differential of the orbit spaces. Hence, cylindrical homology is just the direct sum of the homology of all orbit spaces, with the appropriate degree shiftings.

The grading corresponding to the homology classes in $S_{N\pi}$, for $N \notin p\mathbb{Z}$, is given by :

$$2N(n-2) + 2\lfloor\frac{2N}{p}\rfloor + 2k \qquad k = 0, \dots n-2.$$

Hence, we obtain one generator in each even degree, starting at degree $2n-4$ corresponding to $N = 1$ and $k = 0$. Moreover, there is an overlap between N ($k = n-2$) and $N+1$ ($k = 0$) at

$$2(N+1)(n-2) + 2\lfloor\frac{2N}{p}\rfloor$$

when the integral part of $\frac{2N}{p}$ does not jump between N and $N+1$. We get exactly two generators for these degrees. However, there will be a jump when $N+1 \in p\mathbb{Z}$ or $2N+1 \in p\mathbb{Z}$. In the first case, $N+1 = mp$, and we actually have to use the generators of case (ii) above. The degrees of the generators corresponding to the homology classes in $S_{mp\pi}$ are given by

$$2mp(n-2) + 4m - 2 + 2k \qquad k = 0, \dots, n-1.$$

For $N = mp - 1$ and $k = n - 2$, we obtain a generator in degree

$$2(mp-1)(n-2) + 2(2m-1).$$

But the generator for mp and $k = 0$ has degree

$$2mp(n-2) + 4m - 2 - 2(n-2).$$

So we still have 2 generators in that degree, despite the jump. However, when $2N+1 \in p\mathbb{Z}$, there is nothing to compensate for the jump, and we do not have an overlap. Therefore, we obtain exactly the ranks given in theorem 9.4. □

9.4 Unit cotangent bundle of torus

Let $M = ST^*T^n$ be the unit cotangent bundle of T^n, with respect to the standard flat metric. M is equipped with a natural contact form α, which is obtained by restricting the Liouville 1-form $\theta = \sum_{i=1}^n p_i dq_i$ on M.

The Reeb flow on M coincides with the geodesic flow on T^n, so we obtain closed Reeb orbits when the coordinates p_i $(i = 1, \ldots, n)$ are rationally dependent 2 by 2. Each connected component of N_T corresponds to a nonzero element $\bar{a} = (a_1, \ldots, a_n) \in \pi_1(T^n) = \mathbb{Z}^n$, where $T = \sqrt{a_1^2 + \ldots + a_n^2} = \|\bar{a}\|$, and is a copy of the torus T^n.

The symplectization of M is isomorphic to T^*T^n minus its zero section. The symplectic form is the standard $\omega = \sum_{i=1}^n dp_i \wedge dq_i$ if we substitute $e^t = r = \sqrt{\sum_{i=1}^n p_i^2}$. We can equip the symplectization with almost complex structure J, preserving ξ, defined by $J\frac{\partial}{\partial p_i} = \frac{1}{r}\frac{\partial}{\partial q_i}$. Note that this almost complex structure is not integrable.

The linearized Reeb flow along a periodic Reeb orbit is given, in the q_i, p_j coordinates, by

$$\Psi(t) = \begin{pmatrix} I & tI \\ 0 & I \end{pmatrix}.$$

Therefore, all the periodic orbits have Maslov index $\frac{n-1}{2}$, since we have to restrict ourselves to the unit cotangent bundle. Subtracting half the dimension of the orbit space and adding $n-3$, we get grading $n-3$.

Since there are no contractible periodic orbits, cylindrical homology is well defined, and we can even restrict ourselves to a fixed homotopy class $\bar{a}$ of closed Reeb orbits. On the other hand, holomorphic cylinders have zero energy, since the period of a Reeb orbit depends only on its homotopy class. Hence, all holomorphic cylinders are vertical cylinders over a Reeb orbit. Therefore, the differential d of our chain complex coincides exactly with the Morse-Witten complex of the orbit space T^{n-1} in homotopy class $\bar{a}$. Gathering our results, we have shown

Proposition 9.7. *Cylindrical homology $HF_*^{\bar{a}}(ST^*T^n, \xi)$ in homotopy class $\bar{a}$ is isomorphic to the standard homology $H_{*+n-3}(T^{n-1})$ of T^{n-1}, shifted by degree $n-3$.*

As a corollary of this result, we can reprove a theorem originally due to Giroux [10]. On T^3, let $\alpha_k = \cos 2\pi kz\, dx + \sin 2\pi kz\, dy$ and denote the corresponding contact structure by ξ_k. Then ξ_1 is the contact structure considered above, when $n = 2$. The contact structure ξ_k is obtained from ξ_1 by a k-fold covering of T^3.

Corollary 9.8. (Giroux) *Contact structures ξ_k on T^3 are pairwise non isomorphic.*

Proof. The computation of $HF_*^{\bar{a}}(T^3, \xi_k)$ is analogous to the above computation, except that we now have k copies of the orbit space S^1 in homotopy class $\bar{a}$. Therefore, cylindrical homology is the direct sum of k copies of $H_{*-1}(S^1)$. In particular, we obtain different results for different values of k. ☐

The proof of corollary 9.8 using cylindrical homology was already mentioned in [5], but its proof relies on the techniques developed in this thesis.

9.5 Unit cotangent bundle of Klein bottle

This last example is a little more exotic. It will turn out that theorem 1.9 does not apply to this case. However, we will see that our Morse-Bott techniques still allow us to compute cylindrical homology without working out an explicit perturbation of the contact form.

As in our previous example, the contact form is the Liouville 1-form restricted to the cotangent bundle of K^2. The Reeb flow on ST^*K^2 coincides with the geodesic flow. We choose of course to work with the flat metric of K^2.

We see the Klein bottle K^2 as the quotient of $\mathbb{R}^2$ under the discrete group generated by $(x, y) \to (x + 1, 1 - y)$ and $(x, y) \to (x, y + 1)$. The homotopy class $\bar{a} = (a_1, a_2)$ of loops in K^2 contains the projection of the line $y = \frac{a_2}{a_1}x$ in $\mathbb{R}^2$.

Let us determine the orbit spaces in homotopy class (a_1, a_2) :

(i) $a_1 \neq 0$, a_2 odd.
There are no periodic orbits, because the projection of the line $y = \frac{a_2}{a_1}x$ to K^2 closes with an angle.

(ii) $a_1 \neq 0$, a_2 even.
This time, the projection of the line $y = \frac{a_2}{a_1}x$ to K^2 closes smoothly. Therefore, the closed orbits foliate a torus, and the orbit space is S^1.

(iii) $a_1 = 0$, a_2 odd.
The projection of the line $y = y_0$ to K^2 is closed if and only if $y_0 \in \frac{1}{2}\mathbb{Z}$. Therefore, there are exactly 2 closed orbits.

(iv) $a_1 = 0$, $a_2 \neq 0$ even.
This time, the projection of the line $y = y_0$ is always closed. Therefore, the closed orbits foliate K^2, and the orbit space is a closed interval. The endpoints are a_2-covers of the 2 simple orbits in homotopy class $(0, 1)$.

As in the previous example, the period of a closed Reeb orbit in homotopy class $\bar{a} = (a_1, a_2)$ is given by $T = \sqrt{a_1^2 + a_2^2}$.

When a_2 is even, the Reeb dynamics are identical to the case of ST^*T^2, therefore the corresponding closed Reeb orbits have grading $n - 3 = -1$.
When $a_1 = 0$ and a_2 is odd, the pull-back of the contact distribution to a closed Reeb

orbit is not trivial. Therefore, we have to trivialize ξ along a double cover of that orbit and use fractional grading as explained in [5].

On the other hand, for $a_1 = 0$, $a_2 \neq 0$ even, the submanifold N_T is not orientable, so lemma 7.5 does not apply and we have to check for bad orbits. Use a Morse function f_T on the closed interval with 2 minima at the endpoints and a maximum in the middle. Clearly, the perturbed Reeb orbits at the maximum is good, because its index is independent of the multiplicity.

Claim. The perturbed Reeb orbits corresponding to the endpoints are bad.

Consider a linear Cauchy-Riemann operator on a rank 2 vector bundle E over a 1-punctured sphere, with the asymptotics of those perturbed Reeb orbits. Since the asymptotics of that operator will be invariant under rotation, we can choose the linear operator to be invariant under rotation as well. The change of trivialization of such a double Reeb orbit corresponds to a $\mathbb{Z}_2$ action on $E : (z, x_1, x_2) \to (-z, -x_1, -x_2)$. Clearly, $(z, x_1, x_2) \to (-z, x_1, x_2)$ induces the identity on the kernel and cokernel, because that induced map is homotopic to the identity via $(z, x_1, x_2) \to (e^{\mathrm{i}\theta} z, x_1, x_2)$, $0 \leq \theta \leq \pi$.
On the other hand, $(z, x_1, x_2) \to (z, -x_1, -x_2)$ clearly induces $-I$ on the kernel and cokernel. Since the index of the Cauchy-Riemann operator is odd (it is -1, see above), this action reverses the orientation of the determinant line. Therefore, the corresponding perturbed Reeb orbits are bad.

Summing up, we have

Proposition 9.9. *Cylindrical homology $HF_k^{\bar{a}}(ST^*K^2, \xi) = \mathbb{Q}^{c_k}$, where the non-zero ranks c_k are given by*

$$\begin{cases} c_{-1} = 1, c_0 = 1 & \text{if } a_1 \neq 0, a_2 \text{ even}, \\ c_{-1} = 2 & \text{if } a_1 = 0, a_2 \text{ odd}, \\ c_0 = 1 & \text{if } a_1 = 0, a_2 \neq 0 \text{ even}. \end{cases}$$

Chapter 10

Applications

10.1 Invariant contact structures

Lutz [22] studied contact structures that are invariant under the action of a Lie group G. In particular, he obtained a general construction method for T^k-invariant contact structures on manifolds $T^k \times B^{k+1}$ such that B admits a knotted fibration $B \setminus \Sigma \to S^{k-1}$.

Definition 10.1. *A knotted fibration along the knot N, over the sphere S^{k-1}, is a triplet (E, π, N) such that*

(i) E is a connected, compact, orientable manifold.

(ii) N is either empty or a closed submanifold of codimension k in E.

(iii) $\pi : E \setminus N \to S^{k-1}$ is a locally trivial fibration.

(iv) If N is non-empty, there is an open neighborhood W of N and a diffeomorphism $h : N \times D^k \to W$ such that $h(z, 0) = z$ and the following diagram commutes :

$$\begin{array}{ccc} N \times (D^k \setminus \{0\}) & \xrightarrow{h} & W \setminus N \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi} \\ D^k \setminus \{0\} & \xrightarrow[n]{} & S^{k-1} \end{array}$$

where p_2 is the projection on the second factor and n is the normalization map.

A knotted fibration can also be described by certain maps $\varphi : E \to \mathbb{R}^k$. The correspondence is given by $N = \varphi^{-1}(0)$ and $\pi = n \circ \varphi$.
The existence result for invariant contact structures can be stated as follows.

Proposition 10.2. (Lutz) *Let $M = T^k \times B^{k+1}$, where B is a closed, orientable manifold. Let (B, π, Σ) be a knotted fibration over S^{k-1}. Then there exists a T^k-invariant contact structure on M, corresponding to the given knotted fibration.*

We now describe the contact structure corresponding to a knotted fibration given by a map $\varphi : B \to \mathbb{R}^k$, following Lutz [22].
We construct a contact form α on M of the form

$$\alpha = \sum_{i=1}^{k} \varphi_i d\theta_i + p_2^* \beta$$

where $(\theta_1, \dots, \theta_k)$ are the coordinates on T^k and $p_2 : M \to B$ is the natural projection.
The 1-form β on B is obtained as follows. In a small open neighborhood W of $\Sigma \subset B$, pick a closed 1-form β_0 such that $\beta_0 \wedge d\varphi_1 \wedge \ldots \wedge d\varphi_k > 0$.
Choose trivialization charts $U_j \times F$ for $\pi : B \setminus \Sigma \to S^{k-1}$ such that the open sets U_j cover S^{k-1}. On the Riemann surface with boundary F, choose a symplectic form ω and a primitive α_j. Let g_j be a partition of unity on S^{k-1} with respect to covering U_j. We define $\beta_1 = \sum_j g_j \alpha_j$.
Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a smooth decreasing function such that $f(t) = 1$ if $t < \frac{\epsilon}{2}$ and $f(t) = 0$ if $t > \epsilon$. Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a smooth, C^1 small, decreasing function with support slightly larger than $f^{-1}(1)$ and $\lim_{t\to 0} \frac{g'(t)}{t} = 0$.
Then, we define $\beta = (f(\|\varphi\|) + g(\|\varphi\|))\beta_0 + (1 - f(\|\varphi\|))\beta_1$.
It turns out that the 1-form α is a contact form, for any such choice of β. The isomorphism class of the corresponding contact structure $\xi = \ker \alpha$ is independent of the choices made in that construction.
In what follows, it will be convenient to multiply the function φ by a small positive constant δ, in order to obtain simpler Reeb dynamics.

10.2 Computation of cylindrical homology

We first want to find the closed Reeb orbits of the contact form constructed in the previous section. Let $p_1 : T^k \times B \to T^k$ be the natural projection. If γ is a closed Reeb orbit in M, then we define its reduced homotopy class to be the homotopy class $\bar{a} = (a_1, \dots, a_k)$ of $p_1(\gamma)$ in $\pi_1(T^k)$.

We will denote the connected components of ∂F by $\partial_j F$, $j = 1, \dots, f$. For each component, we introduce a rotation number $l_j = l(\partial_j F)$: there are 2 natural trivializations of the tangent bundle along $\partial_j F$. The first one is the fixed trivialization of that closed Reeb orbit, coming from a capping disk or a homotopy to a representative with fixed trivialization. The second one corresponds to the basis formed by the vectors $\frac{\partial}{\partial \theta_i}$, $i = 1, \dots, k$, followed by the vectors $J\frac{\partial}{\partial \theta_i}$, $i = 1, \dots, k$. The second trivialization is related to the first one by a loop of unitary matrices. We define l_i to be the corresponding element in $\pi_1(U(k)) = \mathbb{Z}$.

Lemma 10.3. *The closed Reeb orbits in the reduced homotopy class $\bar{a} = (a_1, \dots, a_k)$ lie all on the fibre $F_{\bar{a}} = \pi^{-1}(\frac{\bar{a}}{\|\bar{a}\|})$; there are 2 families of orbit spaces :*

(i) in the jth connected component of $f^{-1}(1)$, orbit spaces $S^1_{i,j} = T^k$ with $\mu(S^1_{i,j}) - \frac{1}{2}\dim S^1_{i,j} = 2il_j$, for $i = 1, 2, \dots$ and $j = 1, \dots, f$.

(ii) in $f^{-1}(0)$, orbit spaces $S^2_p = T^{k-1}$ with $\mu(S^2_p) - \frac{1}{2}\dim S^2_p = \mathrm{index}(p) - 1$, for each critical point p of $\|\varphi\|^2$ on $F_{\bar{a}}$.

Proof. First note that the Reeb field satisfies $d\varphi(R_\alpha) = 0$ everywhere. Otherwise, $\imath(R_\alpha)d\alpha$ would contain some terms in $d\theta$ that cannot be canceled. Therefore, the Reeb field has the form

$$R_\alpha = \sum_{i=1}^{k} c_i \frac{\partial}{\partial \theta_i} + cX$$

where X is tangent to the level curves of $\|\varphi\|$ on each fiber F.

Hence, in $f^{-1}(0)$, we can choose X to be the Hamiltonian vector field on F of the function $\|\varphi\|^2$ with respect to the symplectic form $\sum_i g_i d\alpha_i$. We then have

$$\begin{aligned} \imath(R_\alpha)d\alpha &= -\sum_{i=1}^{k} c_i d\varphi_i + \imath(X)\sum_i g_i d\alpha_i + c\sum_i \alpha_i(X)dg_i \\ &= -\sum_{i=1}^{k} c_i d\varphi_i + cd\|\varphi\|^2 + c\sum_i \alpha_i(X)dg_i. \end{aligned}$$

Therefore, we deduce that $c_i = 2c(\varphi_i + \sum_j \alpha_j(X)\frac{\partial g_j}{\partial \varphi_i})$. The value of c is then computed using the condition $\alpha(R_\alpha) = 1$.

Note that $\sum_{i=1}^k \varphi_i \frac{\partial}{\partial \theta_i}$ scales like δ and X scales like δ^2. Hence, for $\delta > 0$ small enough, the only closed orbits in reduced homotopy class $\bar{a}$ are obtained when $X = 0$, i.e. at critical points of $\|\varphi\|^2$. There, the Reeb field is given by $R_\alpha = 2c\sum_{i=1}^k \varphi_i \frac{\partial}{\partial \theta_i}$, so we must be in $F_{\bar{a}}$ to obtain an orbit in the reduced homotopy class $\bar{a}$. These orbits foliate the T^k factor in M, hence form orbit spaces T^{k-1}.

The linearized Reeb flow splits in 2 summands : on the complex subspace generated by the complementary of $\sum_{i=1}^k \varphi_i \frac{\partial}{\partial \theta_i}$ on the T^k factor, we obtain

$$\begin{pmatrix} I_{k-1} & 0 \\ tI_{k-1} & I_{k-1} \end{pmatrix}$$

with a single crossing at $t = 0$ giving contribution $\frac{k-1}{2}$. On $TF_{\bar{a}}$, if the Morse index of p is 0 or 2, we obtain a rotation by a very small angle, positive for index 2 and negative for index 0. Hence, we get contribution index$(p) - 1$. If the Morse index of p is 1, the linearized flow is hyperbolic, giving contribution 0.

Next, we can always choose $\delta > 0$ small enough so that there are no closed Reeb orbits when $0 < f(\|\varphi\|) < 1$, for the same reason as above.

Finally, in $f^{-1}(1)$, we can choose X so that $\beta_0(X) = 1$. Since β_0 is closed, we have

$$\imath(R_\alpha)d\alpha = -\sum_{i=1}^{k} c_i d\varphi_i - cg'(\|\varphi\|)d\|\varphi\|.$$

Therefore, we must have $c_i = -c\frac{g'}{\|\varphi\|}\varphi_i$. All integral curves of $-c\frac{g'}{\|\varphi\|}\sum_{i=1}^k \varphi_i \frac{\partial}{\partial\theta_i}$ are closed in T^k, and the period to attain reduced homotopy class $\bar{a}$ is diverging to $+\infty$ as $\|\varphi\| \to 0$. On the other hand, all integral curves of X on F are closed and their period is bounded when $\|\varphi\| \to 0$. Hence, in each connected component of $f^{-1}(1)$, we obtain an orbit space $S^1_{i,j} = T^k$ when the ratio of the 2 periods is an integer $i = 1, 2, \ldots$ This time, the linearized flow is given by

$$\begin{pmatrix} I_k & 0 \\ tI_k & I_k \end{pmatrix}$$

with a single crossing at $t = 0$ giving contribution $\frac{k}{2}$. However, the trivialization we used for this index computation does not necessarily agree with the fixed trivialization of $\partial_j F$. Hence, we must add correction term $2l_j$ to take this into account. $\square$

Using the orbit spaces and the T^k symmetry, we can now compute cylindrical homology for a non-zero reduced homotopy class $\bar{a}$.

Proposition 10.4. *The cylindrical homology $HF_*^{\bar{a}}(M, \xi)$ is isomorphic to*

$$(H_{*-1}(F) \otimes H_{*+n-3}(T^{k-1})) \oplus ((\bigoplus_{j=1}^{f} u_j \mathbb{Q}[u_j]) \otimes H_{*+n-3}(T^k))$$

where u_j is a variable of degree $2l_j$.

Proof. First, let us find all holomorphic cylinders that are invariant under the T^k action. Note that, in this case, we must have $\frac{\partial u}{\partial t}(s,t) = \sum_{i=1}^k a_i \frac{\partial}{\partial\theta_i}$. Therefore, we have

$$\begin{aligned} \frac{\partial u}{\partial s}(s,t) &= -J\frac{\partial u}{\partial t}(s,t) \\ &= -J\sum_{i=1}^k a_i \frac{\partial}{\partial\theta_i} \\ &= -J(\frac{1}{c}R_\alpha - X) \\ &= \frac{1}{c}\frac{\partial}{\partial t} + JX. \end{aligned}$$

Moreover, we have $d\alpha(X, JX) = \sum_i g_i d\alpha_i(X, JX) = d\|\varphi\|^2(JX) \geq 0$. Therefore, JX is a gradient-like vector field for $\|\varphi\|^2$, and counting its trajectories [26] gives the Morse-Witten differential for the homology of F.

Note that the linearized Cauchy-Riemann operator at these holomorphic cylinders is invariant under $t \to t + c$, and if $\delta > 0$ is small, it depends very slowly on s. Hence, using the arguments of proposition 5.14, this operator is surjective. Therefore, these holomorphic cylinders are generic and isolated elements in the set of all holomorphic cylinders.

Therefore, if there are holomorphic cylinders that are not invariant under the T^k action, then T^k acts freely on their virtual neighborhood. Their index is given by $k - 1 + \Delta\text{index} - 1$, which is strictly less than k for Morse index difference 1. Hence, by proposition 6.6, these cylinders do not contribute to the differential d. For a Morse index difference of 2, it is not possible to obtain a rigid cylinder using the evaluation maps $ev^{\pm}$.

On the other hand, there are no rigid cylinders joining two orbits in the orbit spaces S^2, since their gradings differ by at least 2. Indeed, the index of such cylinders is then at least $k + 2 - 1$, and the use of evaluation maps $ev^{\pm}$ can reduce this dimension by k only. Hence, the differential of cylindrical homology coincides with the Morse-Witten differential of $\|\varphi\|$ on $F_{\bar{a}}$, for each generator of $H_*(T^{k-1})$, and we obtain the announced result. $\square$

10.3 Contact structures on $T^2 \times S^3$

In this section, we consider T^2-invariant contact structures $\xi_{p,q}$ on $T^2 \times S^3$, corresponding to the knotted fibrations given by

$$\varphi : S^3 \subset \mathbb{C}^2 \to \mathbb{C} : (z_1, z_2) \to z_1^p + z_2^q$$

where the integers $p, q \geq 2$ are relatively prime. Such a function clearly defines a knotted fibration, by standard results on singularities of complex hypersurfaces [27].

Classical methods cannot distinguish this infinite family of contact structures.

Proposition 10.5. *The formal homotopy class of $\xi_{p,q}$ is trivial, for all $p,q \geq 2$ relatively prime.*

Proof. We can equip $\xi_{p,q}$ with a T^2-invariant CR structure J. Therefore, the complex vector bundle $(\xi_{p,q}, J)$ is the pull-back of a complex vector bundle over S^3. These bundles are classified by the homotopy class of their clutching function, that is $\pi_2(U(2)) = 0$. Therefore, the complex bundle is trivial and the formal homotopy class of $\xi_{p,q}$ is the reduction of the structure group to the trivial group. Any two such reductions are homotopic. □

However, contact homology can distinguish infinitely many contact structures in this family.

Corollary 10.6. *There are infinitely many pairwise non-isomorphic contact structures on $T^2 \times S^3$ in the trivial formal homotopy class.*

Proof. Let us compute the genus g of F. Since F has exactly one boundary component, corresponding to the single component of Σ, we have $F \simeq \bigvee_{i=1}^{2g} S^1$. Hence, the genus of F can be computed using the Milnor number [27] : $\mu = 2g$. This number can be computed as the dimension of the vector space $\mathbb{C}[z_1, z_2]/(\frac{\partial}{\partial z_1}\varphi, \frac{\partial}{\partial z_2}\varphi) = \mathbb{C}[z_1, z_2]/(z_1^{p-1}, z_2^{q-1})$. Hence, we obtain $g = \frac{(p-1)(q-1)}{2}$. Since $HF_1^{\bar{a}}(T^2 \times S^3, \xi_{p,q})$ depends on this value, we obtain infinitely many different results. □

10.4 Contact structures on T^5

In this section, we consider T^2-invariant contact structures $\xi_{a,b,c}$ on T^5. The contact structure $\xi_{1,1,1}$ was constructed explicitly by Lutz [22]. It corresponds to the knotted fibration given by $\varphi : T^3 \to \mathbb{R}^2$ where

$$\begin{aligned} \varphi_1 &= \sin\theta_1 \cos\theta_3 - \sin\theta_2 \sin\theta_3, \\ \varphi_2 &= \sin\theta_1 \sin\theta_3 + \sin\theta_2 \cos\theta_3, \end{aligned}$$

and it admits the following invariant contact form :

$$\alpha = \varphi_1 d\theta_4 + \varphi_2 d\theta_5 + \sin\theta_2 \cos\theta_2 d\theta_1 - \sin\theta_1 \cos\theta_1 d\theta_2 + \cos\theta_1 \cos\theta_2 d\theta_3.$$

Then, we define $\xi_{a,b,c}$, for integers $a, b, c \geq 1$, as $\pi^*_{a,b,c}\xi_{1,1,1}$, where $\pi_{a,b,c} : T^5 \to T^5 : (\theta_1, \dots, \theta_5) \to (a\theta_1, b\theta_2, c\theta_3, \theta_4, \theta_5)$.

Again, classical methods cannot distinguish this infinite family of contact structures.

Proposition 10.7. *The formal homotopy class of $\xi_{a,b,c}$ is trivial, for all $a, b, c \geq 1$.*

Proof. According to a result of Geiges [8], the formal homotopy class of a contact structure in dimension 5 is determined by its first Chern class. Therefore, we just have to show that $c_1(\xi_{1,1,1}) = 0$, because it will follow that $c_1(\xi_{a,b,c}) = \pi^*_{a,b,c}c_1(\xi_{1,1,1}) = 0$. We rather compute $c_1(T(\mathbb{R} \times M), J) = c_1(\xi \oplus \mathbb{C}) = c_1(\xi)$, where J is an almost complex structure on the symplectization of M, compatible with the symplectic form $d(e^t\alpha)$. An explicit choice of such a J is given by

$$\begin{aligned} J\frac{\partial}{\partial\theta_4} &= -\nabla\varphi_1 - \varphi_1\frac{\partial}{\partial t}, \\ J\frac{\partial}{\partial\theta_5} &= -\nabla\varphi_2 - \varphi_2\frac{\partial}{\partial t}, \\ JX &= \frac{1}{2}\nabla\|\varphi\|^2 - \beta(X)\frac{\partial}{\partial t}. \end{aligned}$$

The first two equations give two trivial summands in $(T(\mathbb{R}\times M), J)$. The last equation shows that the last summand is isomorphic to (TT^2, j), which is trivial as well. Hence $c_1 = 0$. □

However, contact homology can distinguish infinitely many contact structures in this family.

Corollary 10.8. *There are infinitely many pairwise non-isomorphic contact structures on T^5 in the trivial formal homotopy class.*

Proof. Let us determine the homotopy type of the fibre F for the contact structure $\xi_{1,1,1}$. Since we have

$$\varphi = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 \end{pmatrix} \begin{pmatrix} \sin\theta_1 \\ \sin\theta_2 \end{pmatrix},$$

it is clear that the projection $T^3 \to T^2 : (\theta_1, \theta_2, \theta_3) \to (\theta_1, \theta_2)$ induces a homeomorphism between F and $T^2 \backslash \{(0,0), (0,\pi), (\pi,0), (\pi,\pi)\}$. Hence, F is a torus with 4 disks removed. After the finite cover $\pi_{a,b,c}$, the projection $T^3 \to T^2 : (\theta_1, \theta_2, \theta_3) \to (\theta_1, \theta_2)$ induces a branched cover with $4ab$ completely branched points and c branches. Hence, there are $4ab$ components and the genus is given by the Riemann-Hurwitz formula : $2 - 2g = c.0 - (c-1)4ab$. Hence, $g = 2ab(c-1) + 1$.
On the other hand, the natural trivialization along Σ is simply given by (θ_1, θ_2). Since the corresponding branch points have multiplicity c, we deduce that $l(\partial\Sigma) = c$.
Therefore, $HF_i^{\bar{a}}(M, \xi_{a,b,c})$ depends on ab and on c. □

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