SPECIAL CUBE COMPLEXES

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Abstract. We introduce and examine a "special" class of cube complexes. We show that special cube-complexes virtually admit local-isometries to the standard 2-complexes of naturally associated right-angled Artin groups. Consequently, special cube-complexes have linear fundamental groups. In the word-hyperbolic case, we prove the separability of quasiconvex subgroups of fundamental groups of special cube-complexes. Finally, we give a linear variant of Rips's short exact sequence.

Contents

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1. Introduction

The purpose of this paper is to study nonpositively curved cube complexes whose fundamental groups embed in right-angled Artin groups. Our central result is an explicit criterion on a nonpositively curved cube complex X, that implies that X admits a local-isometry to the standard cube complex of a right-angled Artin group. Our criterion imposes various natural restrictions on the interaction between hyperplanes of X as follows:

Theorem 1.1. *Let X be a compact nonpositively curved cube-complex and suppose the following hold:*

1. *Each hyperplane embeds.*
2. *Each hyperplane is 2-sided.*
3. *No hyperplane directly self-osculates.*
4. *No two hyperplanes inter-osculate.*

Then there is a local-isometry $X \to A$ from X to the standard cube complex of a finitely generated right-angled Artin group. Consequently, there is an embedding $\pi_1 X \hookrightarrow \pi_1 A$, and hence $\pi_1 X \subset SL_n(\mathbb{Z})$ for some n.

The first two conditions are well-known conditions to researchers studying nonpositively curved cube complexes, and we refer the reader to Figure 1 on page 8 for pictures of hyperplanes failing the various conditions (the fourth picture describes an indirect self-osculation, which is here allowed).

A complex satisfying the conditions in the theorem is *A-special*. In fact we consider analogous conditions yielding *C-special* complexes whose fundamental groups embed in right-angled Coxeter group, but there is virtually no difference between A-special and C-special cube complexes, and so we will just discuss special cube complexes in this introduction.

In [**?**], "virtual cleanliness" was used to prove that certain groups are residually finite. The results of this paper strengthen those results by showing that the fundamental groups of compact clean $\mathcal{VH}$-complexes are not only residually finite but are actually subgroups of $SL_n(\mathbb{Z})$. For instance we can conclude:

Theorem 1.2. *Let P be a negatively curved n-gon of finite groups. Suppose that $n \geq 6$ and each Gersten-Stallings angle of P is $\leq \frac{\pi}{2}$. Then $\pi_1 P$ is linear.*

As in [**?**], a similar statement holds when $n = 4$ and at least two acute angles, or that $n = 5$ with at least one acute angle.

In [**?**], Misha Kapovich has independently proven a similar statement to Theorem 1.2. He showed that $\pi_1 P$ is actually a discrete subgroup of $\text{Isom}(\mathbb{H}^m)$ for some m, provided that $n \geq 6$ is even, and that all angles are acute.

A central notion in [**?**, **?**] is a "clean" $\mathcal{VH}$-complex, which is a certain type of nonpositively curved square complex. In this paper, we have succeeded in providing a generalization of cleanliness that works for arbitrary dimensions. We hope this work can be considered as the next step in a program seeking to prove that fundamental groups of certain nonpositively curved cube complexes are linear.

In addition to proving linearity by embedding our groups in right-angled Artin groups, we establish the separability of quasiconvex subgroups, thus generalizing results in [**?**], [**?**] and [**?**]. For instance as in Theorem 7.3 we have:

Theorem 1.3. *Let X be a compact special cube complex. If $\pi_1 X$ is word-hyperbolic, then every quasiconvex subgroup is separable.*

In fact, we are able to prove a partial converse to this:

Theorem 1.4. *Suppose X is a compact nonpositively curved cube complex with $\pi_1 X$ word-hyperbolic. If each quasiconvex subgroup of X is separable then X is virtually special.*

The previous results show that fundamental groups of special cube complexes behave nicely as far as linearity and finite index subgroups are concerned. Another important property follows from unpublished work of Droms [**?**] (see also [**?**]) who proved that: Every right-angled Artin group is residually torsion-free nilpotent. In particular it is locally indicable, meaning that every nontrivial finitely generated subgroup admits a morphism onto $\mathbb{Z}$. Combining this result with Theorem 1.1 we get:

Theorem 1.5. *Let X be a compact special cube complex. Then $\pi_1 X$ is residually torsion-free nilpotent. In particular, if $\pi_1 X$ is nontrivial, then it admits a morphism onto $\mathbb{Z}$.*

(A simple geometric proof of this last fact is also available.)

Finally, as an application of our technique we give a linear version of Rips's construction. The following can be deduced easily from Theorem 10.1:

Theorem 1.6. *Let Q be a finitely presented group. There exists a group G which is a finitely presented subgroup of $SL_n(\mathbb{Z})$ for some n, and a finitely generated normal subgroup $N \subset G$ such that there is a short exact sequence $1 \to N \to G \to Q \to 1$.*

After completing this work, we learned from Robert Ghrist about the class of "State complexes" which are a class of particularly nice special cube complexes. We refer the reader to [**?**] for more about State complexes, many beautiful examples, and for the relationship between state and special cube complexes.

We conclude this introduction with a brief section-by-section account of the paper.

In Section 2 we present some background material about nonpositively curved cube complexes.

In Section 3 we describe several intersection pathologies of hyperplanes in a cube complex. We define special cube complexes as those cube complexes with a finite cover whose hyperplanes do not admit any of these pathologies.

In Section 4 we characterize A-special cube complexes as those cube complexes admitting a local isometry to the standard cube complex of a right-angled Artin group.

In Section 5 we show that various 2-dimensional square complexes are virtually A-special. This results in a seemingly effortless proof that their fundamental groups are linear.

In Section 6 we describe a construction which, given a local isometry $Y \to A$ where Y is compact and A is the cube complex of a right-angled Artin group, produces a finite covering $\widehat{A} \to A$ such that $Y \to A$ lifts to an embedding $Y \to \widehat{A}$ and Y is a retract of $\widehat{A}$.

In Section 7 we show that when X is special and $\pi_1 X$ is word-hyperbolic, all quasiconvex subgroups of $\pi_1 X$ are separable. The construction in Section 6 plays an essential role here. We also give an alternative proof where no hyperbolicity assumption is made. This rather uses an isometric embedding in a right-angled Coxeter group.

In Section 8 we prove that when $\pi_1 X$ is word-hyperbolic, a compact nonpositively curved cube complex X is (virtually) special provided that its quasiconvex subgroups are separable. In fact, we only require that the hyperplane subgroups are separable, and that for each pair of crossing hyperplanes, the subgroup generated by a sufficiently small pair of finite index subgroups of their fundamental groups is separable.

In Section 9 we show that a nonpositively curved cube complex is X is (virtually) special if and only if single and double cosets arising from the various hyperplane subgroups are separable.

In Section 10 we give a special (and hence linear) version of Rips's short exact sequence described in Theorem 1.6.

In Section 11 we collect a number of problems about special cube complexes. A major question is whether there exist Gromov-hyperbolic $CAT(0)$ cube complexes which are not special. The situation is well-understood in the (non-hyperbolic) case when X is the product of two locally finite regular trees. Then a uniform lattice of X is special if and only if it is reducible (virtually a product lattice). Examples of irreducible uniform lattices were given in [**?**], [**?**].

CAT(0) cube complexes can be studied combinatorially through their 2-skeletons or even 1-skeletons. We discuss properties of the transition between the 2-skeleton and the entire cube complex in the appendix comprising Section 12.

Section 13 is an appendix where we establish several general results about the combinatorial distance on the set of vertices in a $CAT(0)$ cube complex.

2. Background on nonpositively curved cube complexes

Definition 2.1 (Nonpositively curved cube complex)**.** Let $I = [-1, 1] \subset \mathbb{R}$. A *cube complex* X is a CW-complex such that the attaching map of each k-cell is defined on the boundary of $I^k \subset \mathbb{R}^k$ and its restriction to each $(k-1)$-face of ∂I^k into X^{k-1} is an isometry onto I^{k-1} postcomposed with some $(k-1)$-cell of X^{k-1}. The image of a k-cell of X is a *k-cube*, but we use the usual terminology *vertex*, *edge*, *square* for $k = 0, 1, 2$. In all the paper, we denote by $\overrightarrow{a}, \overrightarrow{b}, \ldots$ oriented edges of X, by $\overleftarrow{a}, \overleftarrow{b}, \ldots$ their opposites and by $a, b, \ldots$ the associated geometric edges. (We will usually think of $\overrightarrow{a}$ and $\overleftarrow{a}$ as points in a vertex link.) A *square complex* is a 2-dimensional cube complex. A combinatorial map $f : X \to Y$ between cube complexes is a map such that for each k-cell $\varphi : I^k \to X$ the map $f \circ \varphi$ is a k-cell of Y precomposed by an isometry of I^k.

Most of the time, all our cube complexes are *simple*, in the sense that the link of each vertex is a simplicial complex. In particular, two distinct squares cannot meet along two consecutive edges as this would give a double edge in the link of the intermediate vertex.

A *flag complex* is a simplicial complex such that any collection of $(i + 1)$ pairwise adjacent vertices span an i-simplex. A cube complex is *nonpositively curved* if the link of each vertex is a flag complex (in particular the cube complex is simple). If furthermore the cube complex is simply-connected, it is said to be $CAT(0)$. Nonpositively curved cube complexes were introduced in [?] and we refer, for instance to [?] for details concerning the CAT(0) property.

Examples.

1) Any 2-dimensional simplicial complex K has a subdivision K' which is a square complex. The vertices of K' are the vertices of the first barycentric subdivision of K, each triangle is subdivided into three squares. In particular any finitely presented group is the fundamental group of a compact square complex.

2) A fundamental result is Sageev's characterization of groups with codimension-1 subgroups as precisely the groups which act essentially on a CAT(0) cube complex [?]. Sageev's construction has stimulated much recent work in this subject. For instance, in [?] his construction is used to show that word-hyperbolic Coxeter groups act properly discontinuously and cocompactly on CAT(0) cube complexes, and the same is shown for finitely presented $C'(\frac{1}{6})$ small-cancellation groups in [?]. Subsequently, it was shown in [?, ?], that any group acting on a space with walls (see [?]) gives rise to a group action on a $CAT(0)$-cube complex.

Definition 2.2 (Hyperplanes)**.** A *midcube* in I^n is the subset obtained by restricting one of the coordinates to 0, so the midcube is parallel to two $(n-1)$-faces of I^n. The edges of I^n *dual* to this midcube are the edges perpendicular to it. For instance, $I \times I \times \{0\} \times I$ is one of the four midcubes of C^4, and its eight dual edges are $\{\pm 1\} \times \{\pm 1\} \times I \times \{\pm 1\}$. The *center of a k-cube* in a cube complex is the image of $(0, \cdots, 0)$ from the corresponding k-cell. The center of an edge is its *midpoint*.

Given a cube complex X, we form a new cube complex Y, whose cubes are the midcubes of cubes of X. The vertices of Y are the midpoints of edges of X. The restriction of a $(k+1)$-cell of X to a midcube of I^{k+1} defines the attaching map of a k-cell in Y. Each component of Y is a *hyperplane* of X. An edge of X is *dual* to some hyperplane H if its midpoint is a vertex of H. Each edge a is dual to a unique hyperplane, which we will denote by $H(a)$.

Definition 2.3 (Walls)**.** Declare two edges a, b of X to be *elementary parallel* if they appear as opposite edges of some square of X. *Parallelism* on edges of X is the equivalence relation generated by elementary parallelisms. A *wall of* X is a parallelism class of edges. For any edge a we will denote by $W(a)$ the wall through a, that is the paralleslism class of a.

We obtain an exact identification between hyperplanes and walls when we associate to a hyperplane the set of edges of X dual to it. We will use hyperplanes for the geometric intuition and the more combinatorial notion of walls for many proofs.

2.1. **The 2-skeleton of a nonpositively curved cube complex.**

Here we explain the relationship between a nonpositively curved cube complex and its 2-skeleton. The complete proof of the main lemma is given in Appendix 12.

Definition 2.4 (completable)**.** A cube complex is *completable* whenever it is simple and there is an isomorphism between its 2-skeleton and the 2-skeleton of some nonpositively curved cube complex.

Clearly nonpositively curved cube complexes are completable, and a simple cube complex is completable if and only if its 2-skeleton is completable.

The following result is proved in Appendix 12 at the end of the paper:

Lemma 2.5. *Let X denote any simple cube complex and Y a nonpositively curved cube complex. Then any combinatorial map $X^2 \to Y$ extends to a unique combinatorial map $X \to Y$.*

Corollary 2.6 (existence of cube-completion)**.** *Any completable cube complex combinatorially embeds in a nonpositively curved cube complex by a map inducing an isomorphism at the level of 2-skeleta.*

In the sequel, such an embedding will be called a *cube-completion* of the completable complex.

Proof. Consider an isomorphism $f : X^2 \to Y^2$ between the 2-skeleton of a completable cube complex X and the 2-skeleton of a nonpositively curved cube complex Y. Now apply Lemma 2.5 above. We get a combinatorial map $\overline{f} : X \to Y$ extending f. It remains to show that $\overline{f}$ is injective.

But this follows because f is injective, and X is simple so $\overline{f}$ itself is injective.

□

Note that a cube-completion of a completable complex restricts to a cube-completion of its 2-skeleton.

Corollary 2.7 (property of cube-completions)**.** *Let X, Y denote completable cube complexes, and let $j_X : X \to \overline{X}, j_Y : Y \to \overline{Y}$ denote cube-completions of X, Y.*

Then for any combinatorial map $f : X \to Y$ there exists a unique combinatorial map $\overline{f} : \overline{X} \to \overline{Y}$ such that $\overline{f} \circ j_X = j_Y \circ f$.

Proof. Apply Lemma 2.5 to the map $j_Y \circ f \circ j_X^{-1} : \overline{X}^2 \to \overline{Y}$ ($\overline{X}$ is completable because nonpositively curved). We thus obtain a (unique) combinatorial map $\overline{f} : \overline{X} \to \overline{Y}$ such that the relation $\overline{f} \circ j_X = j_Y \circ f$ holds on X^2. By the uniqueness in Lemma 2.5 the relation holds on the whole of X.

□

The uniqueness property of the extension above implies as usual for universal objects:

Corollary 2.8 (uniqueness of cube-completion)**.** *Any two cube-completions $j_1 : X \to X_1, j_2 : X \to X_2$ are isomorphic, in the sense that there is an isomorphism $f : X_1 \to X_2$ such that $f \circ j_1 = j_2$.*

2.2. **Local-isometries.**

In this subsection all the cube complexes we consider are simple.

Definition 2.9. Let $\phi \colon A \to B$ be a combinatorial map between cube complexes. The map ϕ is an *immersion* if the induced map $\operatorname{link}(v) \to \operatorname{link}(w)$ is an embedding for each $v \in A^0$ mapping to $w \in B^0$. Assume now that A, B are simple. The map ϕ is a *local-isometry* if it is an immersion, and moreover $\phi(\operatorname{link}(v))$ is a full subcomplex of $\operatorname{link}(w)$.

Recall that a subcomplex $C \subset D$ of a simplicial complex is *full* if any simplex of D whose vertices are in C is in fact entirely contained in C.

We generalize this notion for combinatorial maps between arbitrary (that is not necessarily simple) square complexes. We say that a combinatorial map $\phi\colon A \to B$ is a *local-isometry* if it is an immersion, and moreover for any two oriented edges $\overrightarrow{a_1}, \overrightarrow{a_2}$ of A with the same origin v not adjacent in $\text{link}(v, A)$, the images $\varphi(\overrightarrow{a_1}), \varphi(\overrightarrow{a_2})$ are not adjacent in $\text{link}(\varphi(v), B)$.

A subcomplex $A \subset B$ is *locally-convex* if the embedding $A \to B$ is a local-isometry.

Remark 2.10. Note that if in the definition above A is nonpositively curved, then in order to have a local-isometry it is sufficient to require that for two vertices a_1, a_2 of $\text{link}(v)$, if $\phi(a_1)$ and $\phi(a_2)$ are adjacent, then a_1 and a_2 are adjacent (in other words, the induced map between the 2-skeleta $\phi : A^2 \to B^2$ is a local-isometry).

Suppose that $f : A \to B$ is a combinatorial map between completable cube complexes and let $\overline{f} : \overline{A} \to \overline{B}$ denote its natural extension (see Corollary 2.7). Then $\overline{f}$ is a local-isometry if and only if the map induced by f between the 2-skeleta is a local-isometry. In particular any local-isometry between completable square complexes extends to a local-isometry between their nonpositively curved cube completions.

We also note that, when X is nonpositively curved, so are each of its hyperplanes, and (after subdividing) they map to X by local-isometries.

Lemma 2.11. *Let $\phi\colon A \to B$ be a local-isometry of connected simple cube complexes. Suppose B is nonpositively curved and finite dimensional. Then:*

(1) *A is nonpositively curved and finite dimensional.*
(2) *The map $\widetilde{\phi}\colon \widetilde{A} \to \widetilde{B}$ between universal covers is an isometry between the CAT(0) space $\widetilde{A}$ and a convex subspace of the CAT(0) space $\widetilde{B}$.*
(3) *Consequently, $\widetilde{\phi}$ is an embedding, and so $\phi_\star\colon \pi_1 A \to \pi_1 B$ is injective.*

Proof.

(1) For a vertex $\tilde{v} \in \tilde{A}^0$, a complete graph K in $\text{link}(\tilde{v})$ maps to a complete graph in $\text{link}(\tilde{\phi}(\tilde{v}))$. As $\tilde{B}$ is nonpositively curved $\tilde{\phi}(K)$ is the 1-skeleton of a simplex τ. As $\tilde{\phi}$ is a local-isometry there is a simplex σ in $\text{link}(\tilde{v})$ whose 1-skeleton is K.
(2) Classical (see [**?**, II.4.14] - here we use that a finite dimensional nonpositively curved cube complex is a complete length space of nonpositive curvature in the $CAT(0)$ sense, see [**?**, I.7.33 and II.5.20]).
(3) If $\phi_\star(\gamma) = 1$ then for all $x \in \tilde{A}$ we have $\tilde{\phi}(x) = \phi_\star(\gamma)\tilde{\phi}(x) = \tilde{\phi}(\gamma.x)$, hence $\gamma.x = x$ by injectivity of $\tilde{\phi}$.

□

2.3. Right-angled Artin Groups and Coxeter Groups.

Definition 2.12. Let Γ be a simplicial graph. The *right-angled Artin group* or *graph group* associated to Γ is the group presented by

$$A(\Gamma) = \left\langle x_i : i \in \text{Vertices}(\Gamma) \;\middle|\; [x_i, x_j] : (i, j) \in \text{Edges}(\Gamma) \right\rangle \tag{1}$$

The *right-angled Coxeter group* associated to Γ is the group presented by

$$C(\Gamma) = \left\langle x_i : i \in \text{Vertices}(\Gamma) \mid \; x_i^2 : i \in \text{Vertices}(\Gamma),\; [x_i, x_j] : (i,j) \in \text{Edges}(\Gamma) \right\rangle \tag{2}$$

Note that we do not require that Γ be finite here.

Free groups arise from graphs with no edges, and free abelian groups arise from complete graphs. While the class of graph groups appears to be rather limited, it turns out that the groups appearing as subgroups of graph groups form a surprisingly rich class. This paper provides further interesting examples of such subgroups.

An easy argument given in [**?**] shows that each finitely generated right-angled Artin group is a subgroup of a finitely generated right-angled Coxeter group and hence a subgroup of $SL_n(\mathbb{Z})$ for some n. Therefore, embedding a group as a subgroup of a right-angled Artin group gives an easy route towards showing the group is linear.

Note that unless Γ is edgeless, the corresponding graph group $A(\Gamma)$ is never Gromov-hyperbolic. The right-angled Coxeter group $C(\Gamma)$ is Gromov-hyperbolic whenever Γ is finite and there is no full circuit of length four in Γ. There are plenty of such graphs, even allowing word-hyperbolic right-angled Coxeter groups of arbitrary virtual cohomological dimension [**?**, **?**].

The standard 2-complex X of Presentation (1) extends to a nonpositively curved cube complex $ART(\Gamma)$ by adding an n-cube (in the form of an n-torus) for each set of n pairwise commuting generators. In the terminology of Section 2.1, the standard 2-complex is a completable square complex, whose cube-completion is obtained by adding these tori.

Let us also associate to Presentation (2) a square complex $COX(\Gamma)$. The complex $COX(\Gamma)$ has two vertices v^+, v^-. For each $i \in \text{Vertices}(\Gamma)$ there is an edge a_i between v^+ and v^-. We then glue a square along $a_i a_j a_i a_j$ whenever $(i,j) \in \text{Edges}(\Gamma)$. Note that the resulting square complex is not simple hence not completable. The universal cover of $COX(\Gamma)$ is isomorphic to the universal cover of the standard 2-complex of Presentation (2), in which all disks bounding a loop of label x_i^2 have been shrunk to an unoriented edge with label x_i. Then $C(\Gamma)$ is the group of automorphisms of the universal cover of $COX(\Gamma)$ which project either to the identity or to the natural symmetry exchanging v^+ and v^-, and preserving all other faces. So $\pi_1(COX(\Gamma))$ corresponds to the index 2 subgroup of elements of $C(\Gamma)$ with even length.

Now $COX(\Gamma)$ resembles the 2-skeleton of a nonpositively curved cube complex. Indeed consider the Davis-Moussong realization of the Coxeter group $C(\Gamma)$ (see [**?**]). This is the unique $CAT(0)$ cube complex $DM(\Gamma)$ whose 2-skeleton is as follows. The vertices of $DM(\Gamma)$ are the elements of $C(\Gamma)$. There is an edge between w_1 and w_2 if and only if $w_2 = w_1 x_i$ for some generator x_i. And there is a square with vertices w_1, w_2, w_3, w_4 whenever there are generators x_i, x_j with i, j distinct adjacent vertices of Γ, and $w_2 = w_1 x_i, w_3 = w_2 x_j, w_4 = w_3 x_i$. Clearly for any base vertex $\tilde{v}$ in $\widetilde{COX}(\Gamma)$ there is one and only one isomorphism between the 1-skeleton of $\widetilde{COX}(\Gamma)$ and $DM(\Gamma)^1$ sending $\tilde{v}$ to 1, and preserving the labelling of edges into $\{x_i\}_{i\in I}$. Now this uniquely extends to a combinatorial map $j_{\tilde{v}} : \widetilde{COX}(\Gamma) \to DM(\Gamma)$. This map is not an isomorphism. For any

FIGURE 1. From left to right, the diagrams above correspond to the pathologies enumerated in Definition 3.1

square of $\widetilde{COX}(\Gamma)$ there exists exactly one other square with the same boundary, and these two squares are identified under $j_{\tilde{v}}$. No other identification occurs.

3. Special Cube Complexes

The hyperplanes in a special cube complex (which will be defined in Definition 3.2) interact in an organized fashion. The following definition describes pathologies which are not allowed to occur.

Definition 3.1 (Five Pathologies)**.**

(1) *self-intersecting* hyperplane
(2) *one-sided* hyperplane
(3) *directly self-osculating* hyperplane
(4) *indirectly self-osculating* hyperplane
(5) a pair of *inter-osculating* hyperplanes both intersect and osculate.

A hyperplane H in X *self-intersects* if it contains more than one midcube from the same cube. Equivalently, H self-intersects if the map $H \to X$ is not injective. Or H self-intersects if it has two dual edges that are consecutive in some square. To be precise suppose that v is a vertex, and e_1, e_2 are two edges containing v, and consecutive in a square of X. We will say that two hyperplanes H_1, H_2 (or the corresponding walls W_1, W_2) intersect at $(v; e_1, e_2)$ if e_1, e_2 are dual to H_1, H_2 (if e_1, e_2 belong to W_1, W_2). And we say that a hyperplane H (or the corresponding wall W) self-intersects at $(v; e_1, e_2)$ if e_1, e_2 are dual to H (if $e_1, e_2 \in W$). Two self-intersections are illustrated in the first diagram of Figure 1. A hyperplane is *embedded* if it does not self-intersect.

For a subspace $S \subset X$, let $N(S)$ equal the *open cubical neighborhood* consisting of the union of all open cubes of X intersecting S. An embedded hyperplane H in X is *two-sided* if $N(H)$ is homeomorphic to the product $H \times (-1, 1)$, and specifically, there is a combinatorial map $H \times [-1, 1] \to X$ mapping $H \times \{0\}$ identically to H. More generally, an arbitrary hyperplane H is *two-sided* if the map $H \to X$ extends to $H \times [-1, 1] \to X$. The usual pathology we are avoiding is the generalization of the Möebius strip suggested by the second diagram in Figure 1.

The hyperplane is *one-sided* if it is not two-sided. Let us express these properties in terms of walls. Declare two oriented edges $\overrightarrow{a}, \overrightarrow{b}$ of X to be *elementary parallel* whenever there is a square of X containing $\overrightarrow{a}$ and $\overrightarrow{b}$, and such that the attaching map sends two opposite edges of I^2 with the same orientation to $\overrightarrow{a}$ and $\overrightarrow{b}$ respectively. Define the *parallelism* on oriented edges of X as the equivalence relation generated by elementary parallelism. *An oriented wall of* X is then a parallelism class of oriented edges. We will denote by $W(\overrightarrow{a})$ the oriented wall through an oriented edge $\overrightarrow{a}$. Any oriented wall defines a wall, by forgetting orientation. A wall $W(a)$ is *transversally orientable* if the two oriented edges defined by a belong to distinct oriented walls (in which case the wall corresponds to two distinct oriented walls). This is equivalent to requiring that the corresponding hyperplane is two-sided.

Let $v \in X^0$ be a vertex and let $\overrightarrow{e_1}, \overrightarrow{e_2}$ be distinct oriented edges such that $\iota(\overrightarrow{e_1}) = v = \iota(\overrightarrow{e_2})$, but $\overrightarrow{e_1}$ and $\overleftarrow{e_2}$ are not consecutive in some square containing v. The hyperplanes H_1 and H_2 *osculate* at $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$ if e_1 is dual to H_1 and e_2 is dual to H_2. The hyperplane H *self-osculates* at $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$ if e_1 and e_2 are dual to H. It is equivalent to say that the corresponding wall W *self-osculates at* $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$, in the sense that $W = W(e_1) = W(e_2)$.

Suppose H is 2-sided, so there is a consistent choice of orientation on its dual edges. Then we say H *directly self-osculates at* $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$ if H self-osculates at $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$, and there is a consistent choice of orientation on its dual edges inducing on e_1, e_2 the orientation $\overrightarrow{e_1}, \overrightarrow{e_2}$. Equivalently, the wall W corresponding to H self-osculates at $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$ and $\overrightarrow{e_1} \parallel \overrightarrow{e_2}$ (in which case we say that the oriented wall $W(\overrightarrow{e_1})$ *self-osculates* at $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$). A directly self-osculating hyperplane is illustrated in the third diagram in Figure 1.

We say a 2-sided hyperplane H *indirectly self-osculates at* $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$ if H self-osculates at $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$, and there is a consistent choice of orientation on its dual edges inducing on e_1, e_2 the orientation $\overrightarrow{e_1}, \overleftarrow{e_2}$ (or $\overleftarrow{e_1}, \overrightarrow{e_2}$). Equivalently, the wall corresponding to H self-osculates at $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$ and $\overrightarrow{e_1} \parallel \overleftarrow{e_2}$ (in which case we say that the oriented wall $W(\overrightarrow{e_1})$ indirectly self-osculates at $(v; \overrightarrow{e_1}, \overrightarrow{e_2})$). An indirectly self-osculating hyperplane is illustrated in the fourth diagram in Figure 1. Indirectly self-osculating hyperplanes are of less interest since they cannot appear in the first subdivision.

Finally H_1 and H_2 *inter-osculate* if they both intersect and osculate.

Specifically, they contain two distinct midcubes of some cube, and they have dual edges e_1 and e_2 which are adjacent to a vertex v, but do not lie in a square. A pair of inter-osculating hyperplanes are illustrated in the last diagram in Figure 1.

Definition 3.2. Let X denote any simple cube complex. X is an A-*special* cube complex if none of the pathologies (1), (2), (3), (5) of Definition 3.1 occur in X. That is:

(1) Each hyperplane embeds.
(2) Each hyperplane is 2-sided.
(3) No hyperplane directly self-osculates.
(4) No two hyperplanes inter-osculate.

X is a C-*special* cube complex if:

(1) Each hyperplane embeds.
(2) X^1 is a bipartite graph (multiple edges are allowed, but not loops).
(3) No hyperplane self-osculates.
(4) No two hyperplanes inter-osculate.

X is a *special* cube complex if:

(1) Each hyperplane embeds.
(2) No hyperplane directly self-osculates.
(3) No two hyperplanes inter-osculate.

Example 3.3.

(1) Any graph is A-special and any bipartite graph is C-special.

(2) The cube complex of any right-angled Artin group is A-special, and likewise, the square complex of any right-angled Coxeter group has all the properties of a C-special complex, except that it is not simple.

To verify this we adopt the wall viewpoint. No wall self intersects (in either complex), because the edges are labelled in such a way that two parallel edges have the same label, and two consecutive edges in a square have distinct labels. In $ART(\Gamma)$ the parallelism class of an oriented edge is reduced to this oriented edge: in particular the walls are two-sided and the oriented walls do not directly self-osculate (yet all walls of $ART(\Gamma)$ are indirectly self-osculating). In $COX(\Gamma)$ the parallelism class of an edge is reduced to this edge: in particular the walls do not self-osculate (but all walls are one-sided). The fact that walls are reduced to one edge also easily implies in both complexes that no two walls inter-osculate.

In particular, any cartesian product of loops is A-special. (A *loop* is a graph with one vertex and one edge.)

(3) Any $CAT(0)$ cube complex Y is A-special.

To see that no pathology appears for walls in Y we first use the fact that the natural combinatorial map of any hyperplane of a nonpositively curved complex into the first subdivision is a local isometry (see Remark 2.10). So by Lemma 2.11 each hyperplane of Y embeds, and is a totally geodesic subspace of Y.

In a nonpositively curved cube complex the union of two edges a, b with a common vertex v is a local geodesic, as soon as there is no square containing both of them. This remark and the unicity of geodesic between two points of Y implies that no wall of Y self-osculates. Using the same remark again and the non-existence of right-angled geodesic triangles in a $CAT(0)$ space we see that Y has no inter-osculating pair of walls.

It is easy to check that in fact Y is also C-special because it is simply connected (see Proposition 3.10 below).

Remark 3.4 (X is special $\iff$ X^2 is special)**.** The definitions of A-special, C-special and special may be expressed in terms of (oriented) walls, that is, in terms of parallelism of (oriented) edges. Hence each of the properties we are dealing with only depends on the 2-skeleton of the cube complex.

Now we give some procedures to build new special complexes out of old ones.

Lemma 3.5 (stability under cartesian product)**.** *Suppose that X_1, X_2 are cube complexes satisfying one of the following properties. Then $X_1 \times X_2$ has that property.*

(1) *Each hyperplane embeds*
(2) *Each hyperplane is two-sided*
(3) *The 1-skeleta are bipartite*
(4) *Each hyperplane embeds and no hyperplane directly self-osculates*
(5) *Each hyperplane embeds and no hyperplane self-osculates*
(6) *No two hyperplanes inter-osculate.*

Proof. It suffices to prove the lemma when X_1, X_2 are connected. We work with walls.

First of all, an oriented edge of $X_1 \times X_2$ is of the form $\overrightarrow{a_1} \times v_2$ or $v_1 \times \overrightarrow{a_2}$ (with $\overrightarrow{a_i}$ an oriented edge of X_i and v_j a vertex of X_j). Its parallelism class is of the same type: in fact the oriented wall of $X_1 \times X_2$ through $\overrightarrow{a_1} \times v_2$ is precisely $W(\overrightarrow{a_1}, X_1) \times X_2^0$ (here we use that X_2 is connected, and as usual we denote by X_2^0 the set of vertices of X_2). The wall through $a_1 \times v_2$ is then $W(a_1, X_1) \times X_2^0$

(1) Suppose that such a wall self-intersects. Then there are edges a_1, b_1 with $a_1 \parallel b_1$ in X_1, and vertices v_2, w_2 of X_2 such that the edges $a_1 \times v_2$ and $b_1 \times w_2$ are consecutive in some square of $X_1 \times X_2$. Clearly this square cannot be of type $e_1 \times e_2$ with e_i an edge of X_i. Hence $v_2 = w_2$ and a_1, b_1 are consecutive in some square of X_1. So X_1 contains a self-intersecting wall.

(2) Suppose that the wall $W(a_1, X_1) \times X_2^0$ is one sided. Fix a base vertex v_2 in X_2. Then $\overrightarrow{a_1} \times v_2$ is parallel in $X_1 \times X_2$ to $\overleftarrow{a_1} \times v_2$.

In a sequence σ of elementary parallelisms beginning with $\overrightarrow{a_1} \times v_2$, all edges are of type $\overrightarrow{a} \times v$ and we suppress all elementary parallelism of type $\overrightarrow{a_1'} \times w_2 \parallel \overrightarrow{a_1'} \times w_2'$ with w_2, w_2' joined by an edge of X_2. Then forgetting the second factor we obtain a sequence of elementary parallelism in X_1 (which we will denote by $p_1(\sigma)$ in the sequel). So $\overrightarrow{a_1} \times v_2 \parallel \overleftarrow{a_1} \times v_2$ leads to $\overrightarrow{a_1} \parallel \overleftarrow{a_1}$, and $W(\overrightarrow{a_1}, X_1)$ is one-sided.

(3) If $\tau_1 : X_1^0 \to \{-1, 1\}, \tau_2 : X_2^0 \to \{-1, 1\}$ are 2-colorings, then the map $(v_1, v_2) \mapsto \tau_1(v_1)\tau_2(v_2)$ defines a 2-coloring of $(X_1 \times X_2)^1$.

(4) Suppose that the oriented wall $W(\overrightarrow{a} \times v)$ of $X_1 \times X_2$ directly self-osculates. Then there is a sequence σ of elementary parallelisms between oriented edges $\overrightarrow{a_1} \times v_2$ and $\overrightarrow{b_1} \times w_2$, with $v_2 = w_2$, $\iota(\overrightarrow{a_1}) = \iota(\overrightarrow{b_1}) = v_1$, $\overrightarrow{a_1} \parallel \overrightarrow{a} \parallel \overrightarrow{b_1}$, and there is no square in $X_1 \times X_2$ in which $\overrightarrow{a_1} \times v_2$ and $\overrightarrow{b_1} \times w_2$ are consecutive. As above we may project σ onto a sequence $p_1(\sigma)$ of elementary parallelisms between $\overrightarrow{a_1}$ and $\overrightarrow{b_1}$. So either $W(\overrightarrow{a_1})$ self-intersects or it directly self-osculates at $(v_1, \overrightarrow{a_1}, \overrightarrow{b_1})$.

(5) The same argument shows that if $W(a \times v)$ self-osculates in $X_1 \times X_2$, then either $W(a)$ self-intersects or it self-osculates in X_1.

(6) Suppose that W, V are distinct intersecting walls of $X_1 \times X_2$.

Then we may have W, V of the same type, for example $W = W(a_1 \times v_2)$ and $V = W(b_1 \times w_2)$. Then a square where W, V intersect is not of the form $e_1 \times e_2$ (else $W = V$). So the square is $C \times \{u_2\}$ with C a square of X_1. Hence $W(a_1)$ and $W(b_1)$ intersect in this square.

If furthermore W, V osculate in $X_1 \times X_2$ then the projection argument shows that $W(a_1)$ and $W(b_1)$ also osculate in X_1.

The other possibility for intersecting walls of $X_1 \times X_2$ is that W, V are of different type: say $W = W(a_1 \times v_2)$ and $V = W(v_1 \times a_2)$. But such walls never osculate in $X_1 \times X_2$. Because if w_i is a vertex of an edge b_i, then the product edges $b_1 \times w_2$ and $w_1 \times b_2$ belong to the square $b_1 \times b_2$.

□

Corollary 3.6. *Any product of A-special (resp. C-special, special) cube complexes is still A-special (resp. C-special, special).*

Proof. By Lemma 3.5, the corollary is true for a finite product. But if one of the pathologies excluded in the definition of special complexes appears in an infinite product, it must occur in a finite subcomplex, and hence in the product of finitely many special complexes. This is impossible. □

Lemma 3.7. *Let $f : X \to Y$ be a combinatorial map of cube complexes. If one of the following holds for Y then the corresponding property also hold for X.*

1. *Each hyperplane of Y embeds*
2. *Each hyperplane of Y is two-sided*
3. *Y^1 is bipartite*
4. *No hyperplane of Y self-osculates (or no hyperplane of Y directly self-osculates) and one of the following holds: either each hyperplane of Y embeds and f is an immersion, or $f_{|X^2} : X^2 \to Y^2$ is a local-isometry*
5. *No two hyperplanes of Y inter-osculate, each hyperplane of Y embeds, and $f_{|X^2} : X^2 \to Y^2$ is a local-isometry.*

Proof. Any sequence of elementary parallelisms between (oriented) edges of X is sent by f to such a sequence (f sends edges to edges and squares to squares). Hence f maps walls of X into walls of Y.

Suppose that X has a wall W which self intersects or is one-sided: then $f(W)$ is contained in a wall of Y that either self-intersects or is one sided.

Clearly a 2-coloring of Y^1 precomposed with f gives a 2-coloring of X^1.

Suppose an oriented wall W of X self-osculates at $(v; \overrightarrow{a}, \overrightarrow{b})$. Then $W(f(\overrightarrow{a})) = W(f(\overrightarrow{b})) = V$ in Y. Because f is an immersion we also have $f(a) \neq f(b)$. So either V self-intersects at $(f(v); f(a), f(b))$ (impossible if f is a local-isometry), or V self-osculates at $(f(v); f(\overrightarrow{a}), f(\overrightarrow{b}))$.

The same argument remains valid if we have a self-osculating wall in X.

To conclude, suppose that two walls W, V of X intersect and osculate at $(v; \overrightarrow{a}, \overrightarrow{b})$. If $W(f(a)) = W(f(b))$ then Y has a self-intersecting wall. Otherwise $W(f(a)) \neq W(f(b))$ and the walls $W(f(a)), W(f(b))$ intersect in Y. In this case $f(\overrightarrow{a}), f(\overrightarrow{b})$ cannot be adjacent in $\text{link}(f(v))$, because this would imply by the local-isometry hypothesis that $\overrightarrow{a}, \overrightarrow{b}$ are adjacent in $\text{link}(v)$ which is impossible. Thus $W(f(a)), W(f(b))$ osculate at $(f(v); f(\overrightarrow{a}), f(\overrightarrow{b}))$ and Y has an inter-osculating pair of walls. □

Corollary 3.8 (coverings of special cube complexes)**.** *Any covering space of an A-special, a C-special or a special cube complex is A-special, C-special or special (respectively).*

Corollary 3.9 (subcomplexes of special cube complexes)**.** *The following properties are preserved under subcomplexes of simple cube complexes:*

1. *Each hyperplane embeds*
2. *Each hyperplane is two-sided*
3. *The 1-skeleton is bipartite*
4. *Each hyperplane embeds and no hyperplane directly self-osculates*
5. *Each hyperplane embeds and no hyperplane self-osculates*

If Y has no self-intersecting hyperplanes and no inter-osculating hyperplanes, and X is a locally-convex subcomplex of Y, then X has no inter-osculating hyperplanes. In particular a locally convex subcomplex of an A-special, C-special or a special cube complex is A-special, C-special or special (respectively).

The three variants of a special cube complex are virtually equivalent:

Proposition 3.10. *Let X be a special cube complex. Assume X has a finite number of walls (for examples X is compact). Then X has a finite cover which is A-special and C-special.*

Proof. In a cube complex X, the combinatorial length modulo 2 of an edge path is preserved by homotopies with fixed extremities. This is because all elementary homotopies occur inside a square. Hence there is a morphism $\ell : \pi_1(X) \to \mathbb{Z}_2$ giving the parity of the lengths of closed paths. If the morphism is trivial then the distance modulo 2 to some fixed vertex of X gives a 2-coloring on X^1. Otherwise the 2-sheeted covering of X defined by the kernel of the morphism has a bipartite 1-skeleton.

Suppose furthermore that X has a finite number of walls, say $W_1, \cdots, W_n$. Fix some wall W_i. The parity of the intersection of an edge path with W_i is preserved by homotopies with fixed extremities. Hence there is a morphism $\ell_i : \pi_1(X) \to \mathbb{Z}_2$ counting the parity of the number of intersections of a closed path with the wall W_i. (Note that $\ell = \sum_i \ell_i$).

Suppose that no wall of X self-intersects. Take the finite covering $p : X' \to X$ corresponding to $\cap_i \ker(\ell_i)$. We claim that all walls of X' are 2-sided. For suppose that there is an oriented edge $\overrightarrow{a'}$ parallel in X to its opposite edge. Choose a sequence of elementary parallelisms from $\overrightarrow{a'}$ to $\overleftarrow{a'}$. In the k-th square of this sequence let $\overrightarrow{a'_k}$ be the edge joining the endpoints of the elementary parallel oriented edges (for example $\iota(\overrightarrow{a'_1}) = \tau(\overrightarrow{a'}))$. Then the last edge $\overrightarrow{a_m}$ satisfies $\tau(\overrightarrow{a'_m}) = \iota(\overrightarrow{a'})$, so $\gamma' = (\overrightarrow{a'}, \overrightarrow{a'_1}, \cdots, \overrightarrow{a'_m})$ is a closed path of X. By construction, the closed path $p(\gamma')$ must satisfy $\ell_i(p(\gamma')) = 0$ for all i, and especially for the index i such that $W(p(a')) = W_i$. This means that the cardinality of $\{p(a'), p(a'_1), \cdots, p(a'_m)\} \cap W_i$ is even. By construction, the walls $W(p(a'_k))$ and $W(p(a'))$ intersect. But there are no self-intersecting walls in X, so $p(a'_k) \notin W(p(a'))$. Finally, $\{p(a'), p(a'_1), \cdots, p(a'_m)\} \cap W_i = \{p(a')\}$ which is a contradiction.

The same reasoning shows that X' has no indirectly self-osculating wall.

At this point we have proved the following:

Lemma 3.11. *Any cube complex X has a finite cover X' whose 1-skeleton is bipartite. If furthermore X has finitely many walls and X has no self-intersecting wall, then we may assume that all hyperplanes of X' are two-sided, and that no oriented wall of X' indirectly self-osculates.*

In the previous Lemma, if X was special with finitely many walls then by Lemma 3.8 the finite cover X' is both A- and C-special.

□

Since all A-special and C-special cube complexes are special the previous proposition shows that it is equivalent for a cube complex X with finitely many walls to have a finite covering $X' \to X$ where X' is A-special, C-special or special.

Remark 3.12 (Subdivisions)**.** The first subdivision X' of X does not contain any 1-sided hyperplanes. However, a 1-sided hyperplane in X leads to a directly self-osculating hyperplane in X'.

Indirectly self-osculating hyperplanes cannot exist in the first subdivision.

The reader can verify that if X is A-special, then its first subdivision X' is C-special. In particular, any cartesian product of double edges is C-special. (A *double edge* is a graph with two edges glued together along two vertices.)

The following is proved in Appendix A:

Lemma 3.13 (special $\Rightarrow$ nonpositively curved)**.** *Let X be a special cube complex. Then X is completable, hence it is contained in a unique smallest nonpositively curved cube complex with the same 2-skeleton as X.*

It is a well-known problem to decide whether X has a finite cover $\widehat{X}$ such that each hyperplane in $\widehat{X}$ embeds. This is especially interesting when X is also a 3-manifold, as it is related to the virtual Haken problem. There are counterexamples when X is a compact nonpositively curved 2-complex. On the other hand, it is currently unknown whether X always has such a cover when X is compact and $\pi_1 X$ is word-hyperbolic.

Definition 3.14 (typing maps)**.** Let B be any simple square complex in which each hyperplane embeds. Let Γ_B be the simplicial graph whose vertices are hyperplanes of B, and whose edges connect distinct intersecting hyperplanes. We wish to map B to the cube complex $\text{ART}(\Gamma_B)$ or $\text{COX}(\Gamma_B)$.

All hyperplanes of B are two-sided whenever there is a combinatorial map $B^1 \to \text{ART}(\Gamma_B)$ sending parallel oriented edges of some wall W to the loop in $\text{ART}(\Gamma_B)$ that is labelled by W and with the same orientation. Such a map immediately extends to a combinatorial map $\tau_A : B \to \text{ART}(\Gamma_B)$ which we call an *A-typing of B*.

Similarly B^1 is bipartite whenever there is a combinatorial map $B^1 \to \text{COX}(\Gamma_B)$ sending parallel edges of some wall W to the edge labelled by W. Such a map also extends to a combinatorial map $\tau_C : B \to \text{COX}(\Gamma_B)$ which we call a *C-typing of B*. Indeed there are exactly two possible extensions of $B^1 \to \text{COX}(\Gamma_B)$ at any square of B.

As we have seen in Lemma 3.11, up to a finite covering, a compact square complex without self-intersecting wall admits an A-typing or a C-typing.

Note that, as a map, a typing is not unique in general (think of a bouquet of two circles). Nevertheless, any two typings differ by a wall-preserving automorphism of the target cube complex.

Lemma 3.15. *Let B denote a square complex in which each hyperplane embeds.*
If hyperplanes of B are two-sided then an A-typing map is an immersion if and only if no hyperplane of B directly self-osculates.
If B^1 is bipartite then a C-typing map is an immersion if and only if no hyperplane of B self-osculates.

Proof. The conditions are necessary because of Lemma 3.7 and Example 3.3.

Conversely, the fact that hyperplanes embed implies that adjacent vertices of $\text{link}(v, B)$ are sent by τ to distinct vertices. And the fact that no hyperplane (directly) self-osculates implies that nonadjacent vertices of $\text{link}(v, B)$ are sent by τ to distinct vertices. □

Corollary 3.16. *A square complex B admits a combinatorial immersion in a cartesian product of copies of S^1 if and only if the following hold:*

(1) *Each hyperplane of B embeds,*
(2) *Each hyperplane of B is two-sided,*
(3) *No hyperplane of B directly self-osculates.*

Proof. These conditions are necessary because of Lemma 3.7 and Example 3.3.

Conversely each hyperplane of B is two-sided and we may apply Lemma 3.15: An A-typing gives an immersion of B in $\mathrm{ART}(\Gamma_B)$. We may compose this with the inclusion of $\mathrm{ART}(\Gamma_B)$ in the Artin complex of the complete graph on the set of walls of B.

□

4. Special cube complexes and Right-angled Artin or Coxeter groups

In this section we show that each A-special or C-special, nonpositively curved cube complex immerses by a local-isometry into the cube complex of a right-angled Artin or Coxeter group. While the proof is quite simple, it is essentially the main theorem in the paper.

Lemma 4.1. *Suppose that B is some A-special or C-special square complex. Then any A-typing or C-typing on B is a local-isometry into the 2-skeleton of $ART(\Gamma_B)$ or $COX(\Gamma_B)$.*

Proof. By Lemma 3.15 we know that τ_A (or τ_C) is an immersion.

Fix two oriented edges $\overrightarrow{a}, \overrightarrow{b}$ of B with origin v. Suppose that $\tau(\overrightarrow{a}), \tau(\overrightarrow{b})$ are adjacent in $\mathrm{link}(\tau(v))$. By definition of $\mathrm{ART}(\Gamma_B)$ or $\mathrm{COX}(\Gamma_B)$ this means that the walls $W(a), W(b)$ intersect. As B is special these walls cannot osculate at $(v; \overrightarrow{a}, \overrightarrow{b})$. Hence $\overrightarrow{a}, \overrightarrow{b}$ are adjacent in $\mathrm{link}(v, B)$ and τ is a local-isometry.

□

Theorem 4.2. *Let B be any cube complex. Then B is A-special (resp. C-special) if and only if there exists a graph Γ and there is an immersion $B \to ART(\Gamma)$ that is a local-isometry at the level of the 2-skeleta (resp. and there is a local-isometry $B^2 \to COX(\Gamma)$).*

Proof. If B is A-special or C-special then so is the 2-skeleton B^2 (see Remark 3.4). Note that B^2 is completable by Lemma 3.13.

Lemma 4.1 shows that B^2 admits a local-isometry to the 2-skeleton of a right-angled Artin or Coxeter complex. In the A-special case, by the nonpositive curvature of right-angled Artin complexes and Lemma 2.5, we extend τ to a combinatorial map from B into the right-angled Artin complex. This extension is a local-isometry by Remark 2.10.

Conversely suppose there exists a local-isometry from B^2 to the 2-skeleton of the cube complex of a right-angled Artin group (resp. Coxeter group). Then by Lemma 3.7 we know that B^2 is special. Hence B is A-special or C-special as in Remark 3.4. □

Lemma 4.3. *Let B denote a compact C-special connected cube complex. Let $\tilde{v}$ denote a base vertex in the universal cover $\widetilde{B}$ and let $\tau : B^2 \to COX(\Gamma_B)$ denote some C-typing map.*

Then $\tau_* : \pi_1 B \to \pi_1(COX(\Gamma_B)) \subset C(\Gamma_B)$ is an embedding and the composition $j_{\widetilde{\tau}(\tilde{v})} \circ \widetilde{\tau} : \widetilde{B}^2 \to DM(\Gamma_B)$ extends to an equivariant isometric embedding of $CAT(0)$ cube complexes $\overline{\tau}_{\tilde{v}} : \overline{\widetilde{B}} \to DM(\Gamma_B)$ (here we denote by $\overline{B}$ the $CAT(0)$ completion of B).

Proof. We equip B (hence B^2) with the base point corresponding to $\tilde{v}$ under the universal covering $\widetilde{B} \to B$, and $COX(\Gamma_B)$ with the base point $\tau(v)$. Recall that we have already defined the combinatorial map $j_{\widetilde{\tau}(\tilde{v})} : \widetilde{COX(\Gamma_B)} \to DM(\Gamma_B)$ (see Definition 2.12).

Let us first check that the equivariant combinatorial map $j_{\widetilde{\tau}(\tilde{v})} \circ \widetilde{\tau}$ is a local-isometry. To prove that it is an immersion of simple square complexes it is enough to show that distinct oriented edges with the same origin are sent to distinct oriented edges. But this follows because $j_{\widetilde{\tau}(\tilde{v})}$ is an isomorphism between the 1-skeleta and τ is a local-isometry (see Lemma 4.1).

To conclude it suffices to note that if two distinct oriented edges with the same origin x in $\widetilde{COX(\Gamma_B)}$ are not adjacent in link(x), then their images under $j_{\widetilde{\tau}(\tilde{v})}$ are not adjacent either. This follows from the well-known fact that the order of $x_i x_j$ is 2 if and only if i, j are distinct adjacent vertices of Γ_B.

The C-special complex B is completable by Lemma 3.13. The cube completion $B \to \overline{B}$ restricts to a cube completion $B^2 \to \overline{B}$, and the induced map $\widetilde{B^2} \to \widetilde{\overline{B}}$ is still a cube completion. On the other hand $DM(\Gamma_B)$ is $CAT(0)$, hence it is equal to its completion. By Remark 2.10 the local-isometry of square complexes $j_{\widetilde{\tau}(\tilde{v})} \circ \widetilde{\tau}$ extends to a unique local-isometry $\overline{\tau}_{\tilde{v}} : \widetilde{\overline{B}} \to DM(\Gamma_B)$. Uniqueness implies equivariance. By Lemma 2.11 the map $\overline{\tau}_{\tilde{v}}$ is an isometry into the $CAT(0)$ cube complexe $DM(\Gamma_B)$.

By equivariance of $\overline{\tau}_{\tilde{v}}$ we see that τ_* is an embedding (same argument as in the proof of Lemma 2.11).

□

Theorem 4.4. *Let B be a compact connected cube complex. If B is virtually special, then $\pi_1 B$ is linear.*

As it is well-known that finitely generated linear groups are residually finite, Theorem 4.4 implies in particular that $\pi_1 B$ is residually finite.

Proof. Let $p : B' \to B$ be a finite cover such that B' is C-special. Choose a C-typing map $\tau : B'^2 \to COX(\Gamma_{B'})$. By Lemma 4.3 this induces an injection $\tau_* : \pi_1(B'^2) = \pi_1(B') \to C(\Gamma_{B'})$. But the right-angled Coxeter group $C(\Gamma_{B'})$ is linear because B' is compact, hence $C(\Gamma_{B'})$ is finitely generated.

Thus the theorem follows from the following well-known fact (see [**?**] for a proof):

Lemma 4.5. *Let $\Gamma' \subset \Gamma$ be a finite index subgroup. If Γ' is linear over some field then Γ is linear over the same field.*

□

5. Applications to virtually clean $\mathcal{VH}$-complexes

Definition 5.1. A hyperplane Y of a cube complex X is *clean* whenever it has no self-intersection and no direct self-osculation. It is *fully clean* whenever it has no self-intersection and no self-osculation at all.

Definition 5.2. A simple square complex is a *$\mathcal{VH}$-complex* if the edges are divided into two classes *vertical* and *horizontal* such that the attaching map of each square is of the form $v_1 h_1 v_2 h_2$ where v_1, v_2 are vertical, and h_1, h_2 are horizontal.

Observe that parallelism of edges perserves the horizontal or vertical nature. Thus hyperplane are either horizontal (dual to vertical edges) or vertical (dual to horizontal edges).

Definition 5.3. A $\mathcal{VH}$-complex X is *horizontally clean* if each horizontal hyperplane is clean.

Note that any horizontally clean $\mathcal{VH}$-complex X is a nonpositively curved square complex.

Definition 5.4. A nonpositively curved $\mathcal{VH}$-complex is *thin* if the fundamental group of each horizontal hyperplane maps to a malnormal subgroup.

For instance, this occurs if for some n, X does not admit a $\mathcal{VH}$-immersion of $I_n \times I_2$, where $I_m \cong [0,m]$ is a graph with $m+1$ edges. We regard $I_n \times I_2$ to be a $\mathcal{VH}$-complex whose horizontal edges are parallel to I_n, and whose vertical edges are parallel to I_2. A *$\mathcal{VH}$-immersion* takes horizontal edges to horizontal edges, and vertical to vertical.

Note that in a $\mathcal{VH}$-complex X the walls cannot self-intersect, and the fundamental group of a hyperplane embeds as the stabilizer in $\pi_1(X)$ of a hyperplane of the universal cover of X.

The following was proven in [**?**]:

Theorem 5.5. *Any compact thin $\mathcal{VH}$-complex is virtually horizontally clean.*

The following was proven in [**?**]:

Proposition 5.6. *Let X be a compact horizontally clean $\mathcal{VH}$ complex. Then X has a finite cover which is a subcomplex of a product $A \times B$ of graphs.*

Theorem 5.7. *Let X be a compact, virtually horizontally clean $\mathcal{VH}$-complex. Then X is virtually special.*

Proof. By Proposition 5.6, the complex X has a finite cover X' contained in a product of graphs. The product of two graphs is A-special by Corollary 3.6 and Example 3.3. Now any subcomplex of a product $A \times B$ of graphs has no pair of inter-osculating hyperplanes. Indeed, two intersecting walls must correspond to edges of distinct factors, and if $v = \iota(\overrightarrow{a})$ and $w = \iota(\overrightarrow{b})$, then the walls of $A \times B$ dual to $\overrightarrow{a} \times w$ and $v \times \overrightarrow{b}$ intersect only in the square $a \times b$. Finally X' is A-special by Corollary 3.9. □

So in this context Theorem 4.4 gives:

Theorem 5.8. *Let G be the fundamental group of a compact virtually clean $\mathcal{VH}$-complex. Then G is commensurable with a subgroup of a right-angled Artin group, and hence G is linear.*

Combining Theorem 5.5 and Theorem 5.8 we obtain:

Theorem 5.9. *Let G be the fundamental group of a compact thin $\mathcal{VH}$-complex. Then G is commensurable with a subgroup of a right-angled Artin group, and hence G is linear.*

Remark 5.10. The class of virtually clean $\mathcal{VH}$-complexes is more general than one might expect. It includes most negatively curved polygons of finite groups, and most hyperbolic buildings whose chambers have at least four sides [**?**]. It is conjectured that the Dehn complex of every prime alternating link projection lies in this class [**?**].

The amalgamated free product of two free groups amalgamating a cyclic subgroup was shown to be linear in [**?**] under the mild assumption that the edge group is a maximal cyclic subgroup of each factor. This was proven in general in [**?**].

Theorem 5.11. *Consider a group G presented by*

$$\left\langle a_1, \ldots, a_m, t_1, \ldots t_k \mid U_i^{t_i} = V_i \ (1 \le i \le k) \right\rangle$$

where U_i and V_i are cyclically reduced words in the $a_j^{\pm 1}$ letters, and $|U_i| = |V_i|$ for each i. Then G is linear.

Proof. The standard 2-complex of such a presentation is a $\mathcal{VH}$-complex. It was shown to be virtually clean in [**?**]. The theorem therefore follows from Theorem 5.8. □

6. Canonical Completion and Retraction

In this section we explain how to factor some "special" immersions of square complexes as the composition of an inclusion (the "completion") and a covering map. The procedure will be canonical enough to allow the existence of a retraction to the completion. The material presented here is a generalization of the method in [**?**].

Definition 6.1 (clean map)**.** Let A be any cube complex and $f : A \to B$ be a combinatorial map. For a vertex v of A and distinct oriented edges $\overrightarrow{a_1}, \overrightarrow{a_2}$ of A satisfying $\iota(\overrightarrow{a_1}) = \iota(\overrightarrow{a_2}) = v$, we say that f *is clean at* $(v; \overrightarrow{a_1}, \overrightarrow{a_2})$ whenever $f(\overrightarrow{a}_1)$ and $f(\overrightarrow{a}_2)$ are not parallel. We say that f is *clean* if it is clean everywhere.

Similarly we say that f *is fully clean at* $(v; \overrightarrow{a_1}, \overrightarrow{a_2})$ whenever $f(a_1)$ and $f(a_2)$ are not parallel. And we say that f is *fully clean* if it is fully clean everywhere.

A cube complex A is *clean* (resp. *fully clean*) $\iff$ $1_A : A \to A$ is clean (resp. fully clean). It amounts to ask that each hyperplane of A is clean (resp. fully clean) in the sense of Definition 5.1.

Note that the cleanliness of $f : A \to B$ implies that $f : A^1 \to B^1$ is an immersion, and since we work with simple cube complexes, it follows that $f : A \to B$ is an immersion. The cleanliness of f has the following reformulation: if $\overrightarrow{b}$ is an oriented edge of B then for each $v \in A^0$, there is at most one oriented edge $\overrightarrow{a}$ with initial vertex v and such that $f(\overrightarrow{a}) \parallel \overrightarrow{b}$.

The full cleanliness is equivalent to the following condition : if b is an edge of B then for each $v \in A^0$, there is at most one oriented edge $\overrightarrow{a}$ with initial vertex v and such that $f(a) \parallel b$. If such an oriented edge $\overrightarrow{a}$ exists we denote by $b \cdot v$ the terminal vertex of $\overrightarrow{a}$, and otherwise we let $b \cdot v = v$.

Definition 6.2. Let A be any cube complex and $f : A \to B$ be a combinatorial map. For a vertex v of A and distinct oriented edges $\overrightarrow{a_1}, \overrightarrow{a_2}$ of A satisfying $\iota(\overrightarrow{a_1}) = \iota(\overrightarrow{a_2}) = v$, we say that f *inter-osculates at* $(v; \overrightarrow{a_1}, \overrightarrow{a_2})$ if the vertices $\overrightarrow{a_1}, \overrightarrow{a_2}$ of $\text{link}(v, A)$ are not adjacent and the walls associated to $f(a_1), f(a_2)$ intersect.

f is *special* (resp: *fully special*) if f is clean and inter-osculates nowhere (resp: if f is fully clean and inter-osculates nowhere).

The following criterion ensures that a map is special:

Lemma 6.3. *Let $f : A \to B$ be a local-isometry of cube complexes where B is special. Then f is special. If furthermore B is fully clean then f is fully special.*

Proof. Let v be a vertex of A, and let $\overrightarrow{a_1}, \overrightarrow{a_2}$ denote distinct oriented edges of A satisfying $\iota(\overrightarrow{a_1}) = \iota(\overrightarrow{a_2}) = v$. Then $f(\overrightarrow{a_1}), f(\overrightarrow{a_2})$ are distinct oriented edges satisfying $\iota(f(\overrightarrow{a_1})) = \iota(f(\overrightarrow{a_2})) = f(v)$. Since B is clean the oriented walls $M(f(\overrightarrow{a}_1)), M(f(\overrightarrow{a}_2))$ are distinct. This proves that f is clean. If we assume that B is fully clean then the walls $M(f(a_1)), M(f(a_2))$ are distinct, thus f is fully clean.

If we assume that $M(f(a_1))$ intersects $M(f(a_2))$, then since no two walls of B inter-osculate, there must exist a square of B in which $f(\overleftarrow{a}_1)$ and $f(\overrightarrow{a}_2)$ are consecutive. Hence $\overrightarrow{a_1}, \overrightarrow{a_2}$ are adjacent in $\text{link}(v, A)$ since f is a local-isometry. □

Remark 6.4. Assume that $f : A \to B$ is a special map of cube complexes and that B is special. Then A is special.

A cube complex X is special if and only if $1_X : X \to X$ is special.

To conclude note that the classes of clean maps, fully clean maps, special maps and fully special maps are stable under composition.

The object of this section is the following:

Proposition 6.5. *Let A, B be square complexes such that A^1 and B^1 are simplicial graphs. Let $f : A \to B$ be a fully special map of square complexes. Then there exists a covering map $p : \mathsf{C}(A, B) \to B$ (of finite degree if A is finite), an injection $j : A \to \mathsf{C}(A, B)$ and a cellular map $r : \mathsf{C}(A, B) \to A$ such that $f = pj$ and $rj = 1_A$.*

Furthermore distinct walls of $j(A)$ define distinct walls of $\mathsf{C}(A, B)$. And non intersecting walls of $j(A)$ define non intersecting walls of $\mathsf{C}(A, B)$.

Note that in the 1-dimensional case, where A and B are simplicial graphs, the cleanliness condition on f amounts to requiring that f be an immersion.

Proof. We begin by assuming that $f : A \to B$ is a combinatorial immersion of square complexes where B^1 is assumed to be a simplicial graph (no loops or double edges). We will add the additional hypotheses as they are needed.

Our goal is to produce a square complex $\mathsf{C}(A, B)$, an injection $j : A \to \mathsf{C}(A, B)$ together with a covering map $p : \mathsf{C}(A, B) \to B$ such that $f = pj$. We also need a cellular map $r : \mathsf{C}(A, B) \to A$ such that rp is the identity on A (i.e. r is a retraction to A).

Let $G_0 = A^0 \times B^1$. The right projection $A \times B \to B$ restricts to a covering map $p_0 : G_0 \to B^1$. There is also an obvious injection $j_0 : A^0 \to A^0 \times B^0$ sending v to $(v, f(v))$. Clearly $f = p_0 j_0$ on A^0. Moreover, the first projection r_0 on $A^0 \times B^0$ satisfies $r_0 j_0 = 1_{A^0}$.

By changing some edges in G_0 we are going to define a new graph G_1 such that:

- G_1 has the same set of vertices as G_0 (i.e. $A^0 \times B^0$);

- there is a new covering map $p_1 : G_1 \to B^1$, an injection $j_1 : A^1 \to G_1$, and a cellular retraction map $r_1 : G_1 \to A^1$ such that $f = p_1 j_1$ on A^1, and $r_1 j_1 = 1_{A^1}$.

In what follows, we assume that f is fully clean.

Since A^1 and B^1 are simplicial, we may identify edges of A and B with 2-subsets of A^0 and B^0.

The edges of G_0 are the 2-subsets of $A^0 \times B^0$ of the form $\{(v,x),(v,y)\}$, with $\{x,y\}$ an edge of B^1. We define the edges of G_1 to be the 2-subsets $e = \{(v,x),(v',y)\}$ with $\{x,y\}$ an edge b of B^1 and:

- either $v' = v$ and no edge a containing v is sent to an edge parallel to b (in which case we say that e is *horizontal*);
- or $\{v,v'\}$ is an edge a of A such that $f(a) \parallel b$ (in which case we say that e is *diagonal*).

In particular, for any edge $a = \{v,w\}$ of A, the 2-subset $\{(v,f(v)),(w,f(w))\}$ is a diagonal edge of G_1, and so the map j_0 extends to an injective graph morphism $j_1 : A_1 \to G_1$. The left projection $A^0 \times B^0 \to A^0$ sends a horizontal edge to a single vertex, and sends a diagonal edge to an edge of A. Thus r_0 extends to a simplicial map $r_1 : G_1 \to A_1$.

Similarly, the right projection $A^0 \times B^0 \to B^0$ extends to a graph morphism $p_1 : G_1 \to B^1$. We now check that p_1 is a covering map, i.e. a local isomorphism. Fix a vertex (v,x) of G_1 and an edge $b = \{x,y\}$ of B. Then $\{(v,x),(b \cdot v,y)\}$ is an edge of G_1 containing (v,x) and projecting onto b, and moreover it is the only such edge.

The relations $p_1 j_1 = f$ and $r_1 j_1 = 1_{A^1}$ are straightforward.

We now attach squares to G_1, producing the square complex $\mathsf{C}(A,B)$, so that j_1, p_1, r_1 extend to cellular maps with the required properties. We want p_1 to extend to a covering, hence we must define the boundaries of the squares of $\mathsf{C}(A,B)$ as the lifts of the boundaries of the squares of B^1. We first check that these lifts are closed paths.

Let $(\overrightarrow{b_1}, \overrightarrow{b_2}, \overrightarrow{b_3}, \overrightarrow{b_4})$ be a 4-circuit in B^1 bounding a square of B. Let $\overrightarrow{b_i} = (x_i, x_{i+1})$ where the indices vary modulo 4. Fix a vertex (v,x_1) of G_1 projecting to x_1.

As above, the lift of $\overrightarrow{b_1}$ at (v,x_1) is $((v,x_1),(b_1 \cdot v, x_2))$. Similarly, the lift of $\overrightarrow{b_2}$ at $(b_1 \cdot v, x_2)$ is $((b_1 \cdot v, x_2),(b_2 \cdot (b_1 \cdot v), x_3))$, and so on. Thus the endpoint of the lift of the path $(\overrightarrow{b_1}, \overrightarrow{b_2}, \overrightarrow{b_3}, \overrightarrow{b_4})$ is $(b_4 \cdot (b_3 \cdot (b_2 \cdot (b_1 \cdot v))), x_1)$, and this lift is closed provided $b_4 \cdot (b_3 \cdot (b_2 \cdot (b_1 \cdot v))) = v$. *We are going to establish this in all possible cases provided f is (fully) special.*

(1) Assume first that $b_1 \cdot v = v$. Note that in this case we must also have $b_3 \cdot v = v$ (because $b_1 \parallel b_3$).

1-a) If $b_4 \cdot v = v$ then $b_2 \cdot v = v$ and thus $b_4 \cdot (b_3 \cdot (b_2 \cdot (b_1 \cdot v))) = v$.

1-b) Suppose $b_4 \cdot v = w \neq v$. Then $a = \{v,w\}$ is an edge of A sent to an edge parallel to b_4, and hence to b_2. So we also have $b_2 \cdot v = w$. Thus $b_4 \cdot (b_3 \cdot (b_2 \cdot (b_1 \cdot v))) = b_4 \cdot (b_3 \cdot w)$, and we are done if we can prove that $b_3 \cdot w = w$.

Suppose $b_3 \cdot w \neq w$, so there is an edge a' containing w such that $f(a') \parallel b_3$. We cannot have $a' = a$ or else $b_1 \cdot v \neq v$. As f is special there is a square in A containing a, a'. In this latter case let a'' be the edge of this square parallel to a' and containing v. Observe that f sends it to an edge parallel to $f(a')$, hence to b_3, hence to b_1, which contradicts $b_1 \cdot v = v$.

(2) Suppose that $b_1 \cdot v = w \neq v$.

2-a) If $b_4 \cdot v = v$ we have $b_1 \cdot (b_2 \cdot (b_3 \cdot (b_4 \cdot v))) = v$ by case 1-b), hence also $b_4 \cdot (b_3 \cdot (b_2 \cdot (b_1 \cdot v))) = v$.

2-b) Otherwise $b_4 \cdot v = u \neq v$. If the edges $a = \{v, w\}$ and $a' = \{v, u\}$ are not distinct, then one verifies that $b_1 \cdot v = w,\ b_2 \cdot w = v,\ b_3 \cdot v = w,\ b_4 \cdot w = v$ and so we are done. Otherwise, since f is special there is a square in A containing a, a'. Let t denote the fourth vertex of this square: then it is easy to check that $b_1 \cdot v = w$, $b_2 \cdot w = t$, $b_3 \cdot t = u$, and $b_4 \cdot u = v$.

We have shown: if $f : A \to B$ is a fully special map of square complexes, then the boundary of any square of B lifts to a closed path.

Let $\mathsf{C}(A, B)$ denote the square complex obtained by attaching squares to G_1 along the lifts of boundaries of squares of B. By construction, $p_1 : G_1 \to B^1$ extends to a covering map $p : \mathsf{C}(A, B) \to B$. Note that the degree of p is bounded by the number of vertices in A, and that $\mathsf{C}(A, B)$ is not connected in general.

The image under j_1 of the boundary of any square C of A is the lift of $\partial f(C)$. Thus j_1 extends to a clearly injective morphism $j : A \to \mathsf{C}(A, B)$.

Using our analysis of the lifts above, we see that r_1 sends the lift of a square Q of B either to a point (when all edges of the lift are horizontal), or to an edge (when two opposite edges of the lift are horizontal, and the two others are diagonal), or to the boundary of a square (when the four edges of the lift are diagonal). This shows that r_1 extends to a cellular map $r : \mathsf{C}(A, B) \to A$ (satisfying $rj = 1_A$).

We note that if e is a diagonal edge of $G_1 = \mathsf{C}(A, B)^1$, then the wall $W(e, \mathsf{C}(A, B))$ consists of diagonal edges. And r sends a sequence of elementary parallelisms in $\mathsf{C}(A, B)$ onto a sequence of elementary parallelisms in A. Hence if two edges of $j(A)$ define the same wall in $\mathsf{C}(A, B)$ then in fact they are parallel in A. And if the two edges define intersecting walls in $\mathsf{C}(A, B)$ then in fact their walls in $j(A)$ also intersect (r maps a diagonal square to a square of A).

□

Remark 6.6.

(1) We note that in the previous construction for any vertex $v \in A^0$ if $b' \parallel b$ then $b \cdot v = b' \cdot v$, and $b \cdot (b \cdot v)) = v$. Thus the maps $v \mapsto b \cdot v$ define an action of the free product of cyclic groups of order two, one for each wall of B. So showing that squares lift to squares amounts to proving that this action is also defined on the quotient group $C(\Gamma_B)$.

(2) The walls of $\mathsf{C}(A, B)$ can be described completely (we will not need this description in this article).

The horizontal walls of $\mathsf{C}(A, B)$ correspond to pairs (v, W), where v is a vertex of A and W is a wall of B containing no edge of the form $f(a)$, with a an edge adjacent to v.

Any diagonal edge e of $\mathsf{C}(A, B)$ is $\{(v, x), (w, y)\}$ with $\{x, y\}$ some edge b of B, and $\{v, w\}$ an edge a of A such that $f(a) \parallel b$. We set $\overrightarrow{a} = (v, w)$ and $\overrightarrow{b} = (x, y)$. Either $f(\overrightarrow{a}) \parallel \overrightarrow{b}$, we say that the sign of e is $+1$. Then e is parallel in $\mathsf{C}(A, B)$ to $j(a)$. Else $f(\overrightarrow{a}) \parallel \overleftarrow{b}$ (the sign of e is -1), in which case

e is parallel in $\mathsf{C}(A,B)$ to $\{(v,f(w)),(w,f(v))\}$. Furthermore $j(\{v,w\})$ is not parallel to $\{(v,f(w)),(w,f(v))\}$ (because the sign of diagonal edges is preserved by parallelism).

In particular we see that the preimage of a wall W of A under retraction consists in two walls of $\mathsf{C}(A,B)$ (one of which contains $j(W)$).

Corollary 6.7. *Let B denote some nonpositively curved cube complex whose 1-skeleton is simplicial. Assume B is fully clean and special. Let $f : A \to B$ denote a local isometry. Then there exists a covering $p : \mathsf{C}(A,B) \to B$, an embedding $j : A \to B$ and a cellular map $r : B \to A$ such that $f = pj$ and $rj = 1_A$.*

Proof. Note that A has the same propertie as B because it admits a local isometry to B. Using Lemma 6.3, we see that we may apply Proposition 6.5 to the restriction of $f : A \to B$ to the 2-skeleta.

B^2 is special, because so is B. Using the covering $p^2 : \mathsf{C}(A^2,B^2) \to B^2$ and Corollary 3.8 we see that $\mathsf{C}(A^2,B^2)$ is special. We then denote by $\mathsf{C}(A,B)$ the nonpositively curved completion of $\mathsf{C}(A^2,B^2)$ (in view of Lemma 3.13). Using Lemma 2.5 we extend the maps p^2, j^2 from the 2-skeleta to the nonpositively curved cube complexes and denote by p, j the resulting maps.

Then p is a covering and j is an embedding. Furthermore the relation pj extends the restriction of f to 2-skeleta, so $pj = f$ by uniqueness in Lemma 2.5.

Now we extend $r^2 : \mathsf{C}(A^2,B^2) \to A^2$ to a cellular map $r : \mathsf{C}(A,B) \to A$. It is easy to check that r sends cellularly the 2-skeleton of a cube of $\mathsf{C}(A,B)$ to the 2-skeleton of a cube of A (the image cube may be of lower dimension). Even if r^2 is not combinatorial, on each 2-skeleton of a cube of $\mathsf{C}(A^2,B^2)$ it is the composition of a projection onto some face together with a combinatorial map. This latter combinatorial map extends by Lemma 2.5, and the extension to the full cube of the projection is straightforward. Again the property $rj = 1_A$ follows by uniqueness. □

Remark 6.8. The hypothesis that the 1-skeleton be simplicial is just a technicality.

Note that the second cubical subdivision of X always has a simplicial 1-skeleton. Observe also that if X has a simplicial 1-skeleton, then so has any cover $Y \to X$.

If X is a compact completable cube complex with residually finite fundamental group (e.g. if X is special) then X admits a compact cover $X' \to X$ whose 1-skeleton is simplicial.

Indeed the universal cover of X embedds in a $CAT(0)$ cube complex. Hence $\widetilde{X}$ has a simplicial 1-skeleton. On the other hand by the residual finiteness assumption the universal cover $\widetilde{X} \to X$ factors through a finite cover $X' \to X$ in such a way that $\widetilde{X} \to X'$ is injective on the union of cubes containing a given vertex of $\widetilde{X}$. This implies in particular that ${X'}^1$ is simplicial.

7. Separability of quasiconvex subgroups

7.1. **Word-hyperbolic fundamental groups of special cube complexes.**

We begin by recalling some standard definitions.

Definition 7.1 (quasiconvexity)**.** A subset S of a geodesic metric space X is K*-quasiconvex* if for every geodesic γ in X whose endpoints lie in S, the K-neighborhood of S contains γ. We say that S is *convex* if it is 0-quasiconvex.

A group H acting on a geodesic metric space X is *quasiconvex* if the orbit Hx is a K-quasiconvex subspace of X for some $K > 0$ and some $x \in X$. If H preserves a convex closed subset C and is cocompact on C we say that H is *convex*. Convexity clearly implies quasiconvexity.

We will use the previous notions in the following context: either X is a $CAT(0)$ cube complex equipped with its $CAT(0)$ metric, or X is the set of vertices of a cube complex equipped with the combinatorial distance (here a geodesic is the sequence of vertices of a combinatorial geodesic of the 1-skeleton).

In the first case we say that H is $CAT(0)$ *quasiconvex*, in the second that H is *combinatorially quasiconvex*. It is easily seen that $CAT(0)$ quasiconvexity does not depend on the choice of $x \in X$ (see [**?**]). Sometimes we will explicitly write: H is combinatorially (K, v)-quasiconvex.

Let B denote a compact connected cube complex. Fix a basepoint v and choose a basepoint $\tilde{v}$ in the universal cover $\tilde{B}$ that projects onto v. We regard $\pi_1(B, v)$ as the deck transformation group of $\widetilde{B}$, and let its subgroups act accordingly.

If B is nonpositively curved then according to the previous remarks the $CAT(0)$ quasiconvexity of a subgroup $H \subset \pi_1(B, v)$ is equivalent to the $CAT(0)$ quasiconvexity of the orbit $H\tilde{v}$ and is independent of the choice of the basepoint v.

We will say that a subgroup $H \subset \pi_1(B, v)$ is combinatorial (quasi)convex if it is combinatorial $(K, \tilde{v})$-quasiconvex. This is clearly independent of the choice of $\tilde{v}$, and Corollary 7.8 explains the independence of the choice of v.

When $\widetilde{X}$ is Gromov-hyperbolic, all these notions of quasiconvexity are equivalent (see [**?**]).

In [**?**, **?**] it was shown that:

Proposition 7.2. *Let X be a δ-hyperbolic CAT(0)-cube complex, and let H be a quasiconvex subgroup of a group G acting properly-discontinuously on X. Then for any compact subset $U \subset X$ and any $x \in X$, there is a convex subcomplex $Y \subset X$ such that Y is invariant under H , $U \subset Y$ and Y is contained in a K-neighborhood of Hx.*

Consequently, if G is torsion-free, then any compact subspace of $H\backslash X$ is contained in a locally convex compact core $H\backslash Y$.

A subgroup H of G is *separable* if H is the intersection of finite index subgroups of G.

Theorem 7.3. *Let B be a compact connected C-special cube complex such that B^1 is simplicial. If the group $\pi_1 B$ is word-hyperbolic, then every quasiconvex subgroup is separable.*

Proof. Given the quasiconvex subgroup H of the hyperbolic group $\pi_1 B$, we apply Proposition 7.2 to obtain a compact cube complex A and a local-isometry $f : A \to B$ such that f_* maps $\pi_1 A$ isomorphically onto H.

As only fundamental groups are involved we may replace A, B by their 2-skeleta, and f by its restriction. Since B^1 is simplicial and f is an immersion, A^1 is simplicial too. Furthermore, $f : A \to B$ is a fully special map of square complexes by Lemma 6.3.

Now we may apply Proposition 6.5. We find a finite cover $p : \mathsf{C}(A,B) \to B$, an injection $j : A \to \mathsf{C}(A,B)$ and a cellular map $r : \mathsf{C}(A,B) \to A$ such that $f = pj$ and $rj = 1_A$.

The map p_* identifies $\Gamma' = \pi_1\mathsf{C}(A,B)$ with a finite index subgroup of $\Gamma = \pi_1 B$. Thus $H = f_*(\pi_1 A)$ is identified with the subgroup $j_*(\pi_1 A)$. The morphism $j_* r_*$ is then a retraction of Γ' onto H.

As B is compact and C-special the group Γ is linear and finitely generated, hence residually finite. So is the finite index subgroup Γ'. But any retract of a residually finite group is a separable subgroup. Indeed, given a homomorphism $\rho : \Gamma' \to \Gamma'$ with image H and satisfying $\rho(h) = h$ for any $h \in H$, we may write $H = f^{-1}(\{1\})$ where $f : \Gamma' \to \Gamma'$ denotes the map $f(g) = g^{-1}\rho(g)$. Since f is continuous in the profinite topology H is closed and thus separable (see for instance [?] for more details). As Γ' is of finite index in Γ we see that H is also separable in Γ. □

Corollary 7.4. *Let B be a compact cube complex such that $\pi_1 B$ is word-hyperbolic. If B is virtually special then every quasiconvex subgroup of $\pi_1 B$ is separable.*

Proof. By assumption B has a finite cover B' which is C-special. This cover has a finite cover B'' whose 1-skeleton is simplicial (see Remark 6.8). Observe that B'' is still C-special by Corollary 3.8. Applying Theorem 7.3, we see that every quasiconvex subgroup of $\pi_1 B''$ is separable.

Now the corollary follows from the easy assertion of Lemma 7.5. □

Lemma 7.5. *Let Γ be a word-hyperbolic group. Assume that $\Gamma' \subset \Gamma$ is a finite index subgroup such that every quasiconvex subgroup of Γ' is separable (in Γ'). Then every quasiconvex subgroup of Γ is separable (in Γ).*

Proof. First we may assume that Γ' is normal in Γ.

Let H be a quasiconvex subgroup of Γ. Set $H' = H \cap \Gamma'$. Then H' is quasiconvex in Γ' (see [?]).

Let $\gamma \in \Gamma - H$. We must find a finite index subgroup of Γ containing H but not γ. If γ is not in the subgroup $H\Gamma'$ then we are done.

So assume $\gamma = \lambda\gamma'$ for some $\lambda \in H, \gamma' \in \Gamma'$. Clearly $\gamma' \notin H'$. Thus there is a finite index subgroup $\Gamma'' \subset \Gamma'$ containing H', but not γ'.

There are only finitely many conjugates of Γ'' under H, because H' is of finite index in H and conjugation by an element of H' preserves Γ''. Consider the intersection Γ''' of all conjugates $\lambda\Gamma''\lambda^{-1}$ (with $\lambda \in H$).

The subgroup Γ''' is still of finite index, and it still contains H'. Now the subgroup $H.\Gamma'''$ cannot contain γ', else $\gamma' = \lambda.\gamma'''$ ($\lambda \in H, \gamma''' \in \Gamma'''$) thus $\lambda \in H'$ and finally $\gamma' \in \Gamma'''$. So this finite index subgroup containing H cannot contain γ and we are done. □

Combining Theorem 5.5, Theorem 5.7, and Corollary 7.4, we also get a new proof of the following result which follows by combining the main results of [?] and [?]:

Corollary 7.6. *Let X be a compact thin $\mathcal{VH}$-complex. Then every quasiconvex subgroup of $\pi_1 X$ is separable.*

In [?], some positive results were obtained on separability of quasiconvex subgroups of right-angled Artin groups determined by a graph Γ which is a tree. The details are substantially more technical, primarily because the locally-convex core result for quasiconvex subgroups does not hold.

7.2. **Combinatorial quasiconvex subgroups of C-special cube complex groups.** In this subsection X denotes a compact connected cube complex, with a base vertex v_0. We set $\Gamma = \pi_1(X, v_0)$ and let $p : \widetilde{X} \to X$ denote the universal cover of X. We choose a vertex $\tilde{v}_0$ in $\widetilde{X}$ such that $p(\tilde{v}_0) = v_0$. We give an alternative proof of Corollary 7.4 in a slightly more general context.

Lemma 7.7. *Suppose X is C-special. Let $\tau : X^2 \to COX(\Gamma_X)$ denote some C-typing map, and let $\tilde{v}$ denote some base vertex in $\widetilde{X}$. Then the combinatorial embedding $\overline{\tau}_{\tilde{v}} : \widetilde{\widetilde{X}} \to DM(\Gamma_X)$ of Lemma 4.3 induces a $\pi_1 X$-equivariant isometry from $\widetilde{X}^0 = \widetilde{\widetilde{X}}^0$ onto a convex subset of $DM(\Gamma_X)^0 = C(\Gamma_X)$.*

Here $\widetilde{X}^0$ is equipped with the combinatorial distance induced by $\widetilde{X}^1$, and $C(\Gamma_X)$ is equipped with the word metric relative to the Coxeter generating set.

Proof. By Lemma 4.3 the map $\overline{\tau}_{\tilde{v}} : \widetilde{\widetilde{X}} \to DM(\Gamma_X)$ is a $CAT(0)$ isometry. So its image is a $CAT(0)$ convex subcomplex of $DM(\Gamma_X)$. By Proposition 13.7 we know that the set of vertices of this subcomplex is also combinatorially convex.

□

Corollary 7.8. *Let X be a compact connected nonpositively curved C-special cube complex.*

Then combinatorial-quasiconvexity is independent of the choice of the base vertex. It implies $CAT(0)$-convexity.

A combinatorially quasiconvex subgroup of $\pi_1 X$ is virtually the fundamental group of a compact C-special cube complex.

Proof. Let $H_1 \subset \pi_1(X, v_1)$ be a combinatorial-quasiconvex subgroup. Choose some other vertex v_2 and a combinatorial path σ joining v_1 to v_2. Let L denote the length of σ. We get an isomorphism $\sigma_* : \pi_1(X, v_1) \to \pi_1(X, v_2)$ and we must show that $H_2 = \sigma_*(H_1)$ is a combinatorial-quasiconvex subgroup.

Fix a preimage $\tilde{v}_1$, let $\tilde{\sigma}$ denote the lift of σ with origin $\tilde{v}_1$, and set $\tilde{v}_2 = \tau(\tilde{\sigma})$. The orbit $H_2.\tilde{v}_2$ is the set of endpoints of those lift of σ emanating from a vertex of $H_1.\tilde{v}_1$.

Fix a combinatorial geodesic $\tilde{\sigma}_2$ between two vertices of $H_2.\tilde{v}_2$. Join the endpoints of $\tilde{\sigma}_2$ to $H_1.\tilde{v}_1$ by lifts $\tilde{\sigma}^-, \tilde{\sigma}^+$: thus we get a path $\tilde{\sigma}_1 = \tilde{\sigma}^- \tilde{\sigma}_2 \overline{\tilde{\sigma}^+}$ between two vertices of $H_1.\tilde{v}_1$.

Embed isometrically all this situation in $DM(\Gamma_X)^1$ by $\overline{\tau}_{\tilde{v}_1}$. In [?][Lemma 5.1.3] it was shown that there is a geodesic $\tilde{\gamma}_1$ of $DM(\Gamma_X)^1$ with the same endpoints as $\overline{\tau}_{\tilde{v}_1}(\tilde{\sigma}_1)$, and at distance $\leq 2L$ of $\overline{\tau}_{\tilde{v}_1}(\tilde{\sigma}_2)$.

By Lemma 7.7 the image of $\overline{\tau}_{\tilde{v}_1}$ is combinatorially convex. Thus we have a geodesic $\tilde{\sigma}_1'$ in $\widetilde{X}$, such that $\tilde{\tau}_1(\tilde{\sigma}_1') = \tilde{\gamma}_1$, and still $\tilde{\sigma}_1'$ is at distance $\leq 2L$ of $\tilde{\sigma}_2$. By R-combinatorial-quasiconvexity of $H_1.\tilde{v}_1$ the path $\tilde{\sigma}_1'$ is at distance $\leq R$ of $H_1.\tilde{v}_1$, hence the path $\tilde{\sigma}_2$ is at distance $\leq R + 3L$ of $H_2.\tilde{v}_2$.

In the concluding remarks on quasiconvexity in [?], it was also proven that any combinatorial-quasiconvex subset $E \subset DM(\Gamma_X)^0$ is contained in a $CAT(0)$-convex subcomplex $C \subset DM(\Gamma_X)$ such that E is at finite Hausdorff distance of C and any $w \in C(\Gamma_X)$ preserving E also preserves C. Taking preimage under the equivariant embedding $\overline{\tau}_{\tilde{v}}$ (which is simultaneously a $CAT(0)$-isometry and a combinatorial-isometry), one easily deduces that a combinatorial-quasiconvex subgroup H of $\pi_1(X, \tilde{v})$ is cocompact on a $CAT(0)$-convex subcomplex $\tilde{Y}$ of $\widetilde{X}$, hence is $CAT(0)$-(quasi)convex.

The compact cube complex $Y = H\backslash\tilde{Y}$ has $\pi_1 Y = H$. The map $Y \to X$ is a local-isometry. Hence Y is C-special by Lemma 3.7. □

Corollary 7.9. *Let X be a compact connected virtually special cube complex. Then every combinatorial-quasiconvex subgroup of $\pi_1(X)$ is separable in $\pi_1(X)$.*

Proof. By Lemma 7.5 it suffices to prove the corollary when X is a compact connected C-special cube complex.

By Lemma 7.7 the map $\overline{\tau}_{\tilde{v}} : \widetilde{X}^1 \to DM(\Gamma_X)^1$ is a combinatorial isometry with convex image. Hence a combinatorially quasiconvex subgroup H of $\pi_1(X, v)$ is mapped in $C(\Gamma_X)$ by τ_* onto a combinatorially quasiconvex subgroup. By [?, Thm 2] we know that $\tau_*(H)$ is separable in $C(\Gamma_X)$, and consequently separable in $\tau_*(\pi_1(X))$. □

Remark 7.10. Note that in Corollary 7.9 we did not assume the word-hyperbolicity of $\pi_1(X)$.

The separability result of [?] that we need in its proof relies on a convex hull lemma in right-angled Coxeter groups, a variation on the theme of Proposition 7.2.

Remark 7.11. In Lemma 7.7 we have seen that the image of the universal cover of a C-special cube complex inside the Davis-Moussong complex of the associated right-angled Coxeter group has a combinatorially convex set of vertices. The same is true when we lift any A-typing from a A-special cube complex X to its Artin complex $ART(\Gamma_X)$.

To see this note first that the following analogue of Lemma 4.3 holds: if B is a compact nonpositively curved A-special complex then any A-typing map $\tau : B^2 \to ART(\Gamma_B)$ extends to a local-isometry $\tau : B \to ART(\Gamma_B)$, and the lifts $\widetilde{\tau} : \widetilde{B} \to \widetilde{ART(\Gamma_B)}$ are $CAT(0)$-isometries onto $CAT(0)$-convex subcomplexes. But any $CAT(0)$-convex subcomplex of a $CAT(0)$ cube complex has a combinatorially convex set of vertices (see Proposition 13.7 in Appendix 13).

8. Enough separable subgroups implies special

In this section we prove a converse to Corollary 7.4.

The significance of separability is contained in the following well-known lemma (see for instance [?]). As first observed by Scott, it is a geometric characterization of separability.

Lemma 8.1. *Let $\widehat{X} \to X$ be a based connected covering space of a connected complex X. Suppose $\pi_1\widehat{X}$ is a separable subgroup of $\pi_1 X$. Then for each compact subspace $D \subset \widehat{X}$, there is a finite intermediate covering space $\bar{X}$ satisfying $\widehat{X} \to \bar{X} \to X$ so that D embeds in $\bar{X}$.*

Lemma 8.2 (regular neighborhood)**.** *Let X be a cube complex. For each hyperplane $Y \to X$ there is a cube complex N and a cellular "I-bundle" map $p : N \to Y$, together with a combinatorial map $j : N \to X$ with the following property:*

The preimage under p of any k-cube Q of Y is a $(k+1)$-cube of N mapped by j onto the unique $(k+1)$-cube of X containing the midcube Q.

Such a triple (N, p, j) will be called a (closed) regular neighborhood *of the hyperplane $Y \to X$. It is unique up to isomorphism.*

Proof. For each k-cube Q of Y there is a combinatorial map $I^{k+1} \to X$ whose image is the unique $(k+1)$-cube of X containing Q. Thus there is a combinatorial map $\varphi_Q : Q \times I \to X$ sending $Q \times \{0\}$ to Q. When Q_1 is a face of Q_2 there is a map $\varphi_{Q_1Q_2} : Q_1 \times I \to Q_2 \times I$ mapping (m, t) to $(m, \varepsilon_{12}t)$ (with $\varepsilon_{12} = \pm 1$) such that $\varphi_{Q_2} \circ \varphi_{Q_1Q_2} = \varphi_{Q_1}$.

We may glue together all the $Q \times I$ using the maps $\varphi_{Q_1Q_2}$. This provides the cube complex N, on which the maps p and j are obviously defined, and satisfy the desired properties.

Assume (N', p', j') is an other closed regular neighborhood of $Y \to X$. Consider a k-cube Q' of N' on which p' is not injective. Then $p'(Q')$ is a $(k-1)$-cube of Y. Let Q_1 denote the only k-cube of X with the same center as $p'(Q')$: the cube Q_1 appears in the previous construction of N. There is a unique isomorphism $\varphi_{Q'} : Q' \to Q_1$ such that $p \circ \varphi_{Q'} = p', j \circ \varphi_{Q'} = j'$ on Q'. When Q'' is a face of Q' on which p' is not injective either, then $\varphi_{Q'}$ restricts to $\varphi_{Q''}$. For each cube C' of N' we have $C' \subset {p'}^{-1}(p'(C'))$ and $Q' = {p'}^{-1}(p'(C'))$ is one of the cubes of N' on which p' is not injective. Thus N' is covered by all such cubes, and the maps $\varphi_{Q'}$ are the restrictions of a combinatorial map $\varphi : N' \to N$ such that $p \circ \varphi = p', j \circ \varphi = j'$. Similarly there is a combinatorial map $N \to N'$ and it is the inverse of φ.

□

Remark 8.3.

(1) Note that Y is 2-sided if and only if N is isomorphic to $Y \times I$. In any case N is locally isomorphic to $Y \times I$. In particular N is nonpositively curved if X is nonpositively curved (see the end of Remark 2.10).

(2) Given a vertex v in N there is one and only one edge containing v and shrunk by p to a vertex of Y. The map p is injective on all other edges. The set of edges of N mapped by p onto vertices of Y consists of a wall of N. Note that this wall does not self-intersect or self-osculate in N. The corresponding hyperplane is identified by p with Y, so from now on we will consider Y to be contained in N.

The *interior of* N_Y is the union of open cubes containing an edge meeting Y. The *boundary of* N_Y (denoted by ∂N_Y) is the union of cubes disjoint from Y.

(3) Assume that $X' \to X$ is a covering of cube complexes and $Y' \to X'$ is some hyperplane projecting to some hyperplane $Y \to X$. Let $N \to X, N' \to X'$ denote regular neighborhoods of Y, Y'. Then there is a covering map $N' \to N$ such that $N' \to N \to X = N' \to X' \to X$.

(4) If $\widetilde{X}$ is a CAT(0) cube complex, then $N(\widetilde{H}) \simeq \widetilde{H} \times I \subset \widetilde{X}$ is the smallest subcomplex containing $\widetilde{H}$.

The various kinds of cleanliness may be interpreted using regular neighborhoods.

Lemma 8.4. *Let Y denote a hyperplane of a cube complex X, and let $N_Y \to X$ denote its regular neighborhood.*

(1) *Y is clean whenever $N_Y \to X$ is an embedding on the interior of N_Y and on each component of its boundary.*
(2) *Y is fully clean whenever $N_Y \to X$ is an embedding.*
(3) *Y is fully clean and 2-sided if and only if $N_Y \to X$ is an isomorphism onto a subcomplex isomorphic to $Y \times I$ (under an isomorphism commuting with $p : N \to H$ and the projection $H \times I \to H$).*

Proof. A self-intersection of Y is equivalent to the non-injectivity of $N_Y \to X$ in the interior of N_Y.

A direct self-osculation of Y is equivalent to the non-injectivity of $N_Y \to X$ on one connected component of ∂N_Y.

A self-osculation of Y is equivalent to the non-injectivity of $N_Y \to X$ on ∂N_Y.

□

Definition 8.5. Let X be a nonpositively curved cube complex. Let H denote a hyperplane of X. The *self-interaction radius of H* is the minimal combinatorial distance r_H in the universal cover $\widetilde{X}$ between regular neighborhoods $N(\widetilde{H}_1), N(\widetilde{H}_2)$, where $\widetilde{H}_1$ and $\widetilde{H}_2$ are distinct hyperplanes projecting to H.

Remark 8.6.

(1) A hyperplane H has $r_H = 0$ if and only if it is self-intersecting or self-osculating.
(2) If $X' \to X$ is a covering and if H' is a hyperplane of X' mapping to a hyperplane H of X, then $r_{H'} \geq r_H$.
(3) Let $j : N \to X$ be the regular neighborhood of H. The self-interaction radius of H is the infimum of the integers $r \geq 0$ such that there exists a length r combinatorial path g in X, such that the endpoints of g are in $j(N)$, but g is not path-homotopic to $j(\sigma)$ for any path σ in N.

Definition 8.7 (hyperplane subgroup)**.** Let X be a simple cube complex. Let $\overrightarrow{a}$ denote an oriented edge of X with origin v, and let H_a denote the hyperplane dual to a. Lift the edge $\overrightarrow{a}$ to the regular neighborhood $N(H_a)$ and let v_a be the origin of the lift. This basepoint maps to v in X under $j_a : N(H_a) \to X$. The *hyperplane subgroup at* $(v; \overrightarrow{a})$ is the image of $\pi_1(H_a, v_a)$ under $(j_a)_*$. We will denote it by $K_{v;\overrightarrow{a}}$.

Given some hyperplane H of a cube complex X we will say that $\pi_1 H$ is a separable subgroup of $\pi_1 X$ if there is a vertex v, an oriented edge $\overrightarrow{a}$ with origin v and a dual to H such that the hyperplane subgroup $K_{v;\overrightarrow{a}}$ is a separable subgroup of $\pi_1(X, v)$. This is independent of the choices of v and $\overrightarrow{a}$.

Lemma 8.8. *Let X be a compact, connected, nonpositively curved cube complex. Let H be a hyperplane of X, and suppose that $\pi_1(H)$ is a separable subgroup.*

For each $n \geq 0$, there exist a finite cover $X' \to X$ such that $r_{H'} > n$ for any hyperplane H' of X' projecting to H.

Proof. Using the monotonicity of the self-interaction radius (Remark 8.6.2) and by taking a regular finite covering if necessary, we see that it is enough to prove a weaker statement: there exists a finite cover $X' \to X$ and a hyperplane H' of X' projecting to H such that $r_{H'} > n$.

Fix an edge a dual to H and choose a base vertex x in a. We let $\Gamma = \pi_1(X, x)$ act as the automorphism group of the universal cover $\widetilde{X} \to X$. We also choose a vertex $\tilde{x}$ projecting onto x and denote by $\widetilde{H}$ the hyperplane of $\widetilde{X}$ through the lift $\tilde{a}$ of a at $\tilde{x}$. The stabilizer Λ of $N(\widetilde{H})$ in Γ is $\pi_1(H, x)$.

Consider the set $B = B(\widetilde{H}, \Gamma) = \{\gamma \in \pi_1(X, x) - \Lambda \text{ such that } d(N(\widetilde{H}), \gamma N(\widetilde{H})) \leq n\}$. Then $r_H > n \iff B = \emptyset$.

Clearly B is invariant under left and right multiplication by elements of Λ. Observe that Λ is cocompact on $N(\widetilde{H})$, and hence on the ball $\beta^n(N(\widetilde{H}))$. Thus there are finitely many elements $b_1, \cdots, b_m \in \pi_1(X, x) - \Lambda$ such that B is the disjoint union of the double cosets $\Lambda b_i \Lambda$.

By assumption Λ is separable, and so there is a finite index subgroup $\Gamma' \subset \Gamma$ such that $\Lambda \subset \Gamma'$ and $\Gamma' \cap \{b_1, \cdots, b_m\} = \emptyset$.

Consider the based covering $(X', x') \to (X, x)$ corresponding to Γ'. Let H' denote the hyperplane of X' dual to the edge a', where $\overrightarrow{a'}$ is the lift of $\overrightarrow{a}$ at x'. We are done if we show that $B' = B(\widetilde{H}, \Gamma') = \emptyset$. But $B(\widetilde{H}, \Gamma') \subset B \cap \Gamma'$ and $B \cap \Gamma' = \emptyset$. □

Corollary 8.9. *Let X be a compact connected nonpositively curved cube complex such that $\pi_1 Y$ is a separable subgroup of $\pi_1 X$ for each hyperplane Y of X. Then there is a finite connected cover $\bar{X} \to X$ such that:*

1. *Each hyperplane of $\bar{X}$ embeds*
2. *No hyperplane of $\bar{X}$ self-osculates*

Proof. By Lemma 8.8, for each hyperplane Y of X there is a finite cover $X_Y \to X$ in which all hyperplanes mapping to Y have positive self-interaction radius. Thus all hyperplanes mapping to Y are embedded and do not self-osculate.

Let $X' \to X$ denote a finite cover factoring through each $X_Y \to X$. Then any hyperplane Y' of X' maps to some hyperplane Y in X, hence to some hyperplane $\bar{Y}$ in X_Y. So Y' does not self-intersect or self-osculate, for such a pathology would project to the same pathology for $\bar{Y} \subset X_Y$ (see the proof of Lemma 3.7).

□

Lemma 8.10. *Let Z be a nonpositively curved cube complex. Let Y_1, Y_2 be intersecting hyperplanes of Z such that each Y_i is fully clean.*

If the intersection $N_1 \cap N_2$ of the regular neighborhoods of Y_1, Y_2 is connected, then Y_1, Y_2 do not osculate. More generally Y_1, Y_2 do not osculate at any vertex of the connected component in $N_1 \cap N_2$ of a square of $Y_1 \cap Y_2$.

Proof. Let v be a vertex of Z and let $\overrightarrow{e_1}, \overrightarrow{e_2}$ be oriented edges of Z that originate at v, and are dual to Y_1 and Y_2. Under our assumptions the regular neighborhoods N_1, N_2 embed (see Lemma 8.4).

Let Q' be a square of Z such that Y_1, Y_2 intersect in Q'. Then $Q' \subset N_1 \cap N_2$. Assume v and Q' are in the same connected component of $N_1 \cap N_2$. We must show that there is a square of Z containing e_1, e_2.

By assumption there is a combinatorial path in $N_1 \cap N_2$ whose initial point is in Q' and whose endpoint is v: we consider a shortest such path, and assume its length is > 0. Then the first edge e' of this path is outside Q' (by minimality), but it is still in $N_1 \cap N_2$. Hence for $k = 1$ and $k = 2$ there is a square Q_k containing e' and an edge parallel to e_k. Since Y_k embeds and does not self-osculate, Q_k contains the edge of Q' dual to Y_k and containing the origin of e'. Since Z is nonpositively curved there is a 3-cube Q'' in X containing e' and Q'. Note that $Q'' \subset N_1 \cap N_2$. The square in Q'' opposite to Q' is still a square of intersection of Y_1 and Y_2 contained in $N_1 \cap N_2$, and it is nearer to v. In this way, we have found a shorter path from an intersection square to v. If we proceed in this way, we see that there is a square of $N_1 \cap N_2$ containing v. But this square has to contain e_1, e_2. Again, this is because Y_1, Y_2 embed and do not self-osculate. □

Lemma 8.11. *Let X denote a connected compact nonpositively curved cube complex. Let $\overrightarrow{a}, \overrightarrow{b}$ denote oriented edges of X with the same origin x, adjacent in $\operatorname{link}(x, X)$. Assume that $\pi_1(X, x)$ is Gromov-hyperbolic and that the hyperplane subgroups $K_{x;\overrightarrow{a}}, K_{x;\overrightarrow{b}}$ are separable.*

Then there is a finite connected based covering $(X', x') \to (X, x)$ with the following properties:

1. *The subgroup $A = \langle K_{x';\overrightarrow{a'}}, K_{x';\overrightarrow{b'}} \rangle$ is quasiconvex.*
2. *In the based covering $(\bar{X}, \bar{x}) \to (X', x')$ corresponding to the subgroup $A \subset \pi_1(X', x')$ the hyperplanes dual to $\bar{a}, \bar{b}$ do not inter-osculate.*

Here we denote by $\overrightarrow{a'}$ the lift of $\overrightarrow{a}$ at the base point x', and we define similarly $\overrightarrow{b'}, \overrightarrow{\bar{a}}, \overrightarrow{\bar{b}}$.

Proof. We will use definitions and results of Appendix 13.

Let $(\widetilde{X}, \tilde{x}) \to (X, x)$ denote the universal cover. Let δ be an integer hyperbolicity constant. Note that X is compact so $\dim(\widetilde{X}) = \dim(X) < \infty$. (See Definition 13.12.)

By Lemma 8.8, there is a finite cover $X_1 \to X$ in which each hyperplane H_1 projecting to $H(a)$ satisfies $r(H_1) > 2\delta \dim(X)$. Similarly there is a finite cover $X_2 \to X$ in which each hyperplane H_2 projecting to $H(b)$ satisfies $r(H_2) > 2\delta \dim(X)$. Thus there is a finite cover $X' \to X$ such that $r(H') > 2\delta \dim(X)$ for any hyperplane H' of X' mapping to $H(a)$ or $H(b)$. In particular each such hyperplane is fully clean.

Let x' denote the image in X' of $\tilde{x}$. Let $\overrightarrow{a'}, \overrightarrow{b'}$ denote the lifts of $\overrightarrow{a}, \overrightarrow{b}$ at x'. We first prove that $A = \langle K_{x';\overrightarrow{a'}}, K_{x';\overrightarrow{b'}} \rangle$ is quasiconvex.

This amounts to showing that the orbit $A.\tilde{x}$ is combinatorially quasiconvex.

Let N_a, N_b denote the regular neighborhoods in $\widetilde{X}$ of the hyperplanes $H(\tilde{a}), H(\tilde{b})$. We observe that $A.\tilde{x}$ is contained in and quasi-isometric to the union T of all translates $\alpha(N_a \cup N_b)$ (with $\alpha \in A$). This is because $K_{x';\overrightarrow{a'}}$ is cocompact on N_a and $K_{x';\overrightarrow{b'}}$ is cocompact on N_b. Note that T is connected.

The subcomplex T is a union of convex subcomplexes, such that two of these subcomplexes either are far away or intersect orthogonally. The quasi-convexity of T is a kind of combinatorial analogue of the following well-known fact : a piecewise geodesic of $\mathbb{H}^2$ whose geodesic subsegments are long enough and whose angles are $\geq \frac{\pi}{2}$ is a quasi-geodesic.

For $\alpha \in A$, we will compare the combinatorial distance $d_T(\tilde{x}, \alpha.\tilde{x})$ in T with the combinatorial distance $d(\tilde{x}, \alpha\tilde{x})$ in $\widetilde{X}$. By the triangle inequality we have $d(\tilde{x}, \alpha\tilde{x}) \leq d_T(\tilde{x}, \alpha\tilde{x})$.

Any nontrivial element $\alpha \in A$ may be expressed as $g_1 \dots g_n$ with $g_k \in K_{x';\overrightarrow{a'}} \cup K_{x';\overrightarrow{b'}}$ and n minimal. In particular, $g_k \notin K_{x';\overrightarrow{a'}} \cap K_{x';\overrightarrow{b'}}$ (for $1 < k \leq n$). We assume $g_1 \in K_{x';\overrightarrow{a'}}$ (the case $g_1 \in K_{x';\overrightarrow{b'}}$ is similar).

We define a sequence of $CAT(0)$ convex subcomplexes $(C_k)_{0 \leq k \leq n}$ by $C_{2i} = (g_1 \dots g_{2i})N_a$ and $C_{2i+1} = (g_1 \dots g_{2i+1})N_b$. Now we introduce a sequence of vertices $q_1 = \Pi_{C_1}(\tilde{x}), q_2 = \Pi_{C_2}(q_1), \cdots, q_n = \Pi_{C_n}(q_{n-1})$.

Using Lemma 13.11 and Corollary 13.16 we note the following:

(1) $\tilde{x} \in C_0$, $\alpha\tilde{x} \in C_n$
(2) For $1 \leq k \leq n$ we have $q_k \in C_{k-1} \cap C_k$
(3) For $1 \leq k \leq n-1$ the δ neighborhoods of $V(C_{k-1})$ and $V(C_{k+1})$ are separated by some hyperplane H_k

Using Lemma 13.17 we see by induction that for all $1 \leq k \leq n$ we have $d(q_k, \Pi_{C_k}(\tilde{x})) \leq \delta$, and also $d(\tilde{x}, q_k) \geq d(\tilde{x}, q_1) + (d(q_1, q_2) - 2\delta) + \cdots + (d(q_{k-1}, q_k) - 2\delta)$. Note that $d(q_{i-1}, q_i) \geq d(C_{i-2}, C_i) \geq 1 + 2\delta$. So each term $d(q_{i-1}, q_i) - 2\delta$ is at least $\frac{1}{1+2\delta} d(q_{i-1}, q_i)$. Now $d(\tilde{x}, \alpha\tilde{x}) = d(\tilde{x}, \Pi_{C_n}(\tilde{x})) + d(\Pi_{C_n}(\tilde{x}), \alpha\tilde{x}) \geq d(\tilde{x}, q_n) + d(q_n, \alpha\tilde{x}) - 2\delta \geq d(\tilde{x}, q_1) + \frac{1}{1+2\delta}(d(q_1, q_2) + \cdots + d(q_{n-1}, q_n)) + d(q_n, \alpha\tilde{x}) - 2\delta \geq \frac{1}{1+2\delta}(d(\tilde{x}, q_1) + d(q_1, q_2) + \cdots + d(q_{n-1}, q_n) + d(q_n, \alpha\tilde{x})) - 2\delta$.

The expression $d(\tilde{x}, q_1) + d(q_1, q_2) + \cdots + d(q_{n-1}, q_n) + d(q_n, \alpha\tilde{x})$ is the length of a path in T joining $\tilde{x}$ to $\alpha\tilde{x}$ (this path is the product of paths in $C_0, \cdots, C_n$). So we get quasiconvexity because:

$$d(\tilde{x}, \alpha\tilde{x}) \geq \frac{1}{1+2\delta} d_T(x, \alpha\tilde{x}) - 2\delta.$$

We now turn to the second assertion of the Lemma. We shall prove that in the based covering $(\bar{X}, \bar{x}) \to (X', x')$ corresponding to the subgroup $A \subset \pi_1(X', x')$, the intersection $N(H(\bar{a})) \cap N(H(\bar{b}))$ is connected. Thus $H(\bar{a})$ and $H(\bar{b})$ do not osculate by Lemma 8.10.

Let $\bar{y}$ denote some vertex in $N(H(\bar{a})) \cap N(H(\bar{b}))$. Let $\tilde{y}_a, \tilde{y}_b$ denote lifts of $\bar{y}$ in N_a, N_b. There is an element $\alpha \in A$ such that $\alpha\tilde{y}_a = \tilde{y}_b$. Using Lemma 8.12 below we see that $\alpha = g_1 g_2$ with $g_1 \in K_{x',\overrightarrow{b'}}$ and $g_2 \in K_{x',\overrightarrow{a'}}$. The vertex $\tilde{y} = g_2\tilde{y}_a = g_1{}^{-1}\tilde{y}_b$ is in $N_a \cap N_b$. Now any path in the $CAT(0)$ convex subcomplex $N_a \cap N_b$ joining $\tilde{x}$ to $\tilde{y}$ projects in $\bar{X}$ to a path of $N(H(\bar{a})) \cap N(H(\bar{b}))$ joining $\bar{x}$ to $\bar{y}$.

The proof is now completed by the following Lemma: □

Lemma 8.12. *Let α be an element in $A - \left(K_{x',\overrightarrow{b'}} K_{x',\overrightarrow{a'}}\right)$. Then $\alpha N_a \cap N_b = \emptyset$.*

Proof. Let $\alpha \in A$. We write $\alpha = g_1 \dots g_n$ with $g_k \in K_{x';\overrightarrow{a'}} \cup K_{x';\overrightarrow{b'}}$ and n minimal.

We first assume $g_1 \in K_{x';\overrightarrow{a'}}$ and $\alpha \notin K_{x';\overrightarrow{a'}}$, i.e. $n > 1$. Under this assumption we prove that $\alpha N_a \cap N_b = \emptyset$.

Let $\eta = n$ if n is even and $\eta = n + 1$ otherwise.

We define a sequence of $CAT(0)$ convex subcomplexes $(C_k)_{-1 \leq k \leq \eta}$ by $C_{-1} = N_b$, $C_k = (g_1 \dots g_k)N_a$ if k is even, and $C_k = (g_1 \dots g_k)N_b$ if $0 < k \leq n$ is odd. Finally when n is odd we set $C_\eta = \alpha N_a$. Now we introduce a sequence of vertices $q_0 = \Pi_{C_0}(\tilde{y}), q_1 = \Pi_{C_1}(q_0), \cdots, q_\eta = \Pi_{C_\eta}(q_{\eta-1})$.

Using Lemma 13.11 and Corollary 13.16 we note the following:

(1) $\tilde{y} \in C_{-1}$, $q_\eta \in \alpha N_a$
(2) For $0 \leq k \leq \eta$ we have $q_k \in C_{k-1} \cap C_k$
(3) For $0 \leq k \leq \eta - 1$ the δ neighborhoods of $V(C_{k-1})$ and $V(C_{k+1})$ are separated by some hyperplane H_k

Now let p_k denote the combinatorial projection of $\tilde{y}$ onto C_k $(k = 0, \dots, \eta)$. We have $p_0 = q_0$. By Lemma 13.17 we have $d(p_1, q_1) \leq \delta$. Applying Lemma 13.17 another $\eta - 1$ times we find that:

$$d(\tilde{y}, p_\eta) > d(\tilde{y}, p_{\eta-1}) > \cdots > d(\tilde{y}, p_1)$$

In particular $d(\tilde{y}, \alpha N_a) > 0$, which concludes the proof.

Assume now that $g_1 \in K_{x',\overrightarrow{b'}}$ and $\alpha \notin K_{x',\overrightarrow{b'}} K_{x',\overrightarrow{a'}}$. Then $\alpha = g_1\alpha'$ with $\alpha' \notin K_{x',\overrightarrow{a'}}$. The decomposition $\alpha' = g_2. \dots .g_n$ is of minimal length, for otherwise there would exist a shorter decomposition for α. And $g_2 \in K_{x',\overrightarrow{a'}}$ by minimality. We may thus apply the first part of the argument: $\alpha' N_a \cap N_b = \emptyset$. If we multiply this relation on the left by g_1 we are done.

□

Theorem 8.13. *Let X be a compact connected nonpositively curved cube complex whose fundamental group is Gromov-hyperbolic. If each quasiconvex subgroup of $\pi_1 X$ is separable then X is virtually special.*

Proof. Applying Corollary 8.9 we may assume that X is (fully) clean.

Let $\overrightarrow{a_1}, \overrightarrow{a_2}$ denote a pair of oriented edges with common origin v such that $\overrightarrow{a_1}, \overrightarrow{a_2}$ are adjacent in link(v, X). Let Y_1, Y_2 denote the hyperplanes dual to a_1, a_2. We prove that there is a finite based cover $(X'', x'') \to (X, x)$ such that the hyperplanes Y''_1, Y''_2 dual to a''_1, a''_2 do not osculate in X''.

By Lemma 8.11 there is a finite cover $X' \to X$ such that the subgroup $A = \langle K_{x';\overrightarrow{a'}}, K_{x';\overrightarrow{b'}} \rangle$ is quasiconvex, and the hyperplanes dual to $\bar{a}$ and $\bar{b}$, do not inter-osculate in the based covering $(\bar{X}, \bar{x}) \to (X', x')$ corresponding to A.

Consider the quasiconvex subcomplex $T = \cup_{\alpha \in A} \big[\alpha.\big(N(H(\tilde{a})) \cup N(H(\tilde{b}))\big] \subset \widetilde{X}$. Then A is cocompact on T. Let $\bar{T}$ denote the image of T inside $\bar{X} = \widetilde{X}/A$. Then $\bar{T}$ is a compact subcomplex, and in fact $\bar{T} = N(H(\bar{a})) \cup N(H(\bar{b}))$.

By the separability of A and Lemma 8.1 there is a finite cover $X'' \to X'$ in which $\bar{T} \to X'$ lifts to an embedding.

We know that $H(\bar{a}), H(\bar{b})$ do not osculate in $\bar{X}$. So the injectivity on $\bar{T}$ of the covering $\bar{X} \to X''$ shows that $H(a''), H(b'')$ do not osculate in X''.

By replacing $X'' \to X$ by a finite regular cover of X we may even assume that no two hyperplanes Y_1'', Y_2'' of X'' mapping onto Y_1, Y_2 inter-osculate.

Let $\hat{X} \to X$ denote a finite cover factoring through the various covers $X'' \to X$ (as $v, \overrightarrow{a_1}, \overrightarrow{a_2}$ vary). By Lemma 3.7, $\hat{X}$ is clean since X is.

We now show that no two hyperplanes of $\hat{X}$ inter-osculate. Indeed, let $\hat{Y}_1, \hat{Y}_2$ denote two intersecting hyperplanes. Let $\overrightarrow{\hat{a}_1}, \overrightarrow{\hat{a}_2}$ denote oriented edges with common origin $\hat{v}$ such that $\hat{H}_i$ is dual to $\hat{a}_i$ and $\overrightarrow{\hat{a}_1}, \overrightarrow{\hat{a}_2}$ are adjacent in $\mathrm{link}(\hat{v}, \hat{X})$. Indeed, when we project this situation to X, we obtain intersecting hyperplanes Y_1, Y_2 of X. Consider the corresponding covering X'' and since $\widehat{X}$ factors through X'', we may project $\hat{Y}_1, \hat{Y}_2$ to Y_1'', Y_2'' in X''. By construction of X'' above, Y_1'' and Y_2'' do not inter-osculate. As in the proof of Lemma 3.7.5 this implies that $\hat{Y}_1, \hat{Y}_2$ do not interosculate.

Finally $\hat{X}$ is (fully) special.

□

Combining Theorem 8.13 with Theorem 4.4 and Proposition 3.10 we have:

Corollary 8.14. *Let X be a nonpositively curved compact connected cube complex whose fundamental group is Gromov-hyperbolic. If each quasiconvex subgroups is separable, then $\pi_1(X)$ is linear.*

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9. A characterization using double cosets.

In this section we give another characterization of being virtually special using the separability of certain single and double cosets.

9.1. **Separability of hyperplane subgroups and hyperplane double cosets.**

Definition 9.1 (separable subsets)**.** The *profinite topology* on a group G is the topology generated by the basis consisting of cosets of finite index subgroups of G. It is easily verified that G is Hausdorff if and only if G is residually finite, which holds precisely if singletons are closed in the profinite topology. A *separable* subgroup H of G is a subgroup that is closed in the profinite topology. More generally, a subset of G is *separable* if it is closed in the profinite topology on G. We will be particularly interested in separable double cosets $H_1 g H_2$.

Lemma 9.2. *Let G be a residually finite group, and let $\rho : G \to G$ denote some retraction morphism, that is ρ is an endomorphism satisfying $\rho^2 = \rho$. Then $\rho(G)$ is a separable subgroup.*

Proof. We denote by N the kernel of ρ. Now we consider the map $f : G \to N$ sending g to $\rho(g)^{-1}g$. This map is continuous in the profinite topology. Furthermore N is residually finite as a subgroup of G. Hence N is Hausdorff.

Now we see that $f(g) = 1$ if and only if $g \in \rho(G)$. So $\rho(G) = f^{-1}(\{1\})$ is closed because it is the preimage of a closed subset by a continous map.

□

Lemma 9.3. *Let G be a residually finite group, and let K be a closed subgroup, and let $\rho : G \to G$ be a retraction such that $\rho(K) \subset K$. Let $H = \rho(G)$. Then HK is a closed subset of G. In particular $H = \rho(G)$ is separable.*

The proof closely follows a criterion for recognizing closed double cosets given by Niblo and we refer to [**?**] for further details.

Proof. Since K is closed, the group $D = G *_{K=\bar{K}} \bar{G}$ obtained by amalgamating two copies of G along the copies of the subgroup K is residually finite and hence Hausdorff.

Since $\rho(K) \subset K$, the retractions $\rho : G \to G$ and $\bar{\rho} : \bar{G} \to \bar{G}$, induce a retraction $D \to D$. By Lemma 9.2, the subgroup generated by H and $\bar{H}$ is a closed subgroup of D.

Consider the map $f : G \to D$ given by $f(g) = g^{-1}\bar{g}$. It is a continuous map since it is the product of two continuous maps. Therefore the preimage of $\langle H, \bar{H}\rangle$ is a closed subset of G.

But as shown in [**?**], this preimage is precisely the double coset KH of G. Thus both it, and its inverse HK is closed. □

Corollary 9.4. *Let G be a right-angled Artin or Coxeter group, and let H and K be subgroups of G generated by subsets of the standard generators of G. Then the double coset HK is closed in G.*

Proof. For each subgroup H generated by a subset of the standard generators of G, we let $\rho : G \to G$ be the retraction induced by fixing the generators of H, and sending the other generators to 1_G. Thus each such subgroup is closed by Lemma 9.2. Moreover, if K is another such subgroup, then clearly $\rho(K) \subset K$. Consequently HK is separable by Lemma 9.3. □

Before stating the double coset characterization, we describe precisely the double cosets we are interested in.

Definition 9.5 (hyperplane double cosets)**.** Let X be a simple cube complex. Let $\overrightarrow{a}$ be an oriented edge of X with origin v. The hyperplane subgroup at $(v; \overrightarrow{a})$, denoted by $K_{v;\overrightarrow{a}}$, has been introduced in Definition 8.7.

Let $\overrightarrow{b}$ be another oriented edge with the same origin v, which is adjacent to $\overrightarrow{a}$ in $\text{link}(v, X)$ by the corner of a square q. We call $K_{v;\overrightarrow{a}}K_{v;\overrightarrow{b}}$ the *hyperplane double coset at* $(v; \overrightarrow{a}, \overrightarrow{b})$.

In this section we will rather use the notation H_a instead of $H(a)$.

Lemma 9.6. *Let X denote a nonpositively curved virtually special cube complex. Let $\overrightarrow{a}$ be an oriented edge of X with origin v.*

Then there exists a fully clean based finite cover $(X', v') \to (X, v)$ on which is defined a cellular retraction map $r : X' \to N(H_{a'})$.

If $\overrightarrow{b}$ is a second edge with origin v, such that $\overrightarrow{a}$, $\overrightarrow{b}$ are adjacent in $\text{link}(v, X)$ *then $r(N(H_{b'})) \subset N(H_{b'})$, and the neighborhoods $N(H_{a'})$ and $N(H_{b'})$ have connected intersection. (Here $\overrightarrow{a}'$, $\overrightarrow{b}'$ denote the lifts of $\overrightarrow{a}$, $\overrightarrow{b}$ at v'.)*

Proof. First choose a finite based cover $(\bar{X}, \bar{v}) \to (X, v)$ such that $\bar{X}$ is fully special, and has a simplicial 1-skeleton (see Remark 6.8). Denote by $\overrightarrow{\bar{a}}$, $\overrightarrow{\bar{b}}$ the lifts of $\overrightarrow{a}$, $\overrightarrow{b}$ at $\bar{v}$.

The neighborhood $N(H_{\bar{a}})$ embeds in $\bar{X}$ by a local-isometry. Applying Corollary 6.7, there is a finite based cover $(X', v') \to (\bar{X}, \bar{v})$, an isomorphism $j : N(H_{\bar{a}}) \to N(H_{a'})$ and a cellular retraction map $r : X' \to N(H_{a'})$.

To conclude let us prove that $r(N(H_{b'})) = N(H_{b'}) \cap N(H_{a'})$. The inclusion $r(N(H_{b'})) \supset N(H_{b'}) \cap N(H_{a'})$ follows because r is a retraction on $N(H_{a'})$.

To prove the reverse inclusion note that $b' \in N(H_{a'})$, so that by the construction in Proposition 6.5, each edge of X parallel to b' is sent by r to an edge of $N(H_{a'})$ parallel to b'.

Finally, let q denote a k-cube of $N(H_{b'})$. There is a k'-cube q' of $N(H_{b'})$ containing q and an edge parallel to b'. The image of q' under r is an ℓ-cube (for $\ell = 1, 2, \ldots$ or k') of $N(H_{a'})$ containing an edge parallel to b'. So $r(q) \subset r(q')$ is inside $N(H_{a'}) \cap N(H_{b'})$.

The connectedness of $N(H_{b'}) \cap N(H_{a'})$ follows because $N(H_{b'}) \cap N(H_{a'}) = r(N(H_{b'}))$ and $N(H_{b'})$ is connected.

□

Proposition 9.7. *In the fundamental group of a compact connected nonpositively curved virtually special cube complex, each hyperplane subgroup and each hyperplane double coset is separable.*

Proof. Fix a vertex v and an oriented edge $\overrightarrow{a}$ containing v.

By Lemma 9.6, there is a finite fully clean based cover $p : (X', v') \to (X, v)$ and a cellular retraction $r : X' \to N(H_{a'})$.

Let $\rho : \pi_1(X', v') \to \pi_1(X', v')$ denote the retraction induced by r. Clearly $\rho(\pi_1(X', v')) = K_{v', \overrightarrow{a'}}$, so by Lemma 9.2 the retract $K_{v', \overrightarrow{a'}}$ is closed in the residually finite $\pi_1(X', v')$. Note that $\pi_1(X', v')$ is closed in $\pi_1(X, v)$, hence $K_{v', \overrightarrow{a'}}$ is closed in $\pi_1(X, v)$.

Observe that $K_{v', \overrightarrow{a'}} = K_{v, \overrightarrow{a}} \cap \pi_1(X', v')$ so $K_{v', \overrightarrow{a'}}$ has finite index in $K_{v, \overrightarrow{a}}$. Choose a finite set $\{g_1, \cdots, g_n\}$ of representatives of $K_{v, \overrightarrow{a}} / K_{v', \overrightarrow{a'}}$. Thus $K_{v, \overrightarrow{a}} = \cup_{i=1}^{i=n} g_i K_{v', \overrightarrow{a'}}$. This implies that $K_{v, \overrightarrow{a}}$ is closed as a finite union of closed sets.

So at this stage we have proved that any hyperplane subgroup of the fundamental group of any special cube complex is closed.

If $\overrightarrow{b}$ is another oriented edge with origin v and $\overrightarrow{a}, \overrightarrow{b}$ are adjacent in $\text{link}(v, X)$, let $\overrightarrow{b}'$ denote the lift at v'. By the first step we have $K_{v', \overrightarrow{b'}}$ is closed in $\pi_1(X', v')$. By Lemma 9.6, we have $\rho(K_{v', \overrightarrow{b'}}) \subset K_{v', \overrightarrow{b'}}$. Lemma 9.3 then shows that the double coset $K_{v', \overrightarrow{a'}} K_{v', \overrightarrow{b'}}$ is closed in $\pi_1(X', v')$.

Now $K_{v', \overrightarrow{b'}}$ has finite index in $K_{v, \overrightarrow{b}}$, so there are elements $\{h_1, \cdots, h_m\}$ such that $K_{v, \overrightarrow{b}} = \cup_{j=1}^{j=n} K_{v', \overrightarrow{b'}} h_j$. We therefore have:

$$K_{v, \overrightarrow{a}} K_{v, \overrightarrow{b}} = \bigcup_{i,j} g_i K_{v', \overrightarrow{a'}} K_{v', \overrightarrow{b'}} h_j$$

Each map $g \mapsto g_i g h_j$ is a homeomorphism of $\pi_1(X, v)$. Thus $K_{v, \overrightarrow{a}} K_{v, \overrightarrow{b}}$ is closed as a finite union of closed subsets.

□

In order to establish the converse of the previous Proposition, we work in the universal cover.

9.2. **Clean and special actions on a $CAT(0)$ cube complex.**

Definition 9.8. G acts *cleanly* on the $CAT(0)$ cube complex $\widetilde{X}$ if for each hyperplane Y with regular neighborhood $N = Y \times I \subset \widetilde{X}$ and boundary components $Y^+ = Y \times \{+1\}$ and $Y^- = Y \times \{-1\}$ we have for every $g \in G$:

(1) if $gY \cap Y \neq \emptyset$ then $gY = Y$
(2) if $gY^+ \cap Y^+ \neq \emptyset$ or $gY^- \cap Y^- \neq \emptyset$ then $gY = Y$

Lemma 9.9. *Let X denote a non positively curved cube complex, and let G denote its fundamental group acting by deck transformations on the universal covering space $\widetilde{X}$.*
Then X is clean if and only if G acts cleanly.

Proof. Assume G acts not cleanly. Let $\widetilde{Y}$ denote some hyperplane with neighbourhood N and let $g \in G$ be some element such that $g\widetilde{Y} \neq \widetilde{Y}$ and either $g\widetilde{Y} \cap \widetilde{Y} \neq \emptyset$, or $g\widetilde{Y}^{\pm} \cap \widetilde{Y}^{\pm} \neq \emptyset$. The wall corresponding to $\widetilde{Y}$ projects to a wall of X, and we denote by Y the corresponding hyperplane of X.

If $g\widetilde{Y} \cap \widetilde{Y} \neq \emptyset$ then Y has a self intersection. And if (for example) $g\widetilde{Y}^+ \cap \widetilde{Y}^+ \neq \emptyset$ then Y has a direct self-osculation.

Conversely assume that a hyperplane Y of X is not clean, in the sense that it self-intersects or directly self-osculates. Choose a hyperplane $\widetilde{Y}$ of $\widetilde{X}$ projecting onto Y. There is a vertex v and two distinct oriented edges $\overrightarrow{a}, \overrightarrow{b}$ dual to Y such that v is the initial point of $\overrightarrow{a}, \overrightarrow{b}$ and either $\overleftarrow{a}, \overrightarrow{b}$ are consecutive in a square (self-intersection), or $\overrightarrow{a} \parallel \overrightarrow{b}$ (direct self-osculation). Consider a sequence of edges $\overrightarrow{a}_1 \parallel \cdots \parallel \overrightarrow{a}_n$ such that two consecutive are elementary parallel, $\overrightarrow{a}_1 = \overrightarrow{a}$ and $a_n = b$ (and even $\overrightarrow{a}_n = \overrightarrow{b}$ in the case of direct self-osculation). Lift this sequence to $\widetilde{X}$ such that the first edge $\overrightarrow{\widetilde{a}}$ is dual to $\widetilde{Y}$. Then the last edge $\widetilde{b}$ is still dual to $\widetilde{Y}$. The edges $\widetilde{a}, \widetilde{b}$ project to a, b. Denote by $\overrightarrow{\widetilde{b}'}$ the lift of $\overrightarrow{b}$ at the initial point $\widetilde{v}$ of $\overrightarrow{\widetilde{a}}$. There is an element $g \in G$ sending $\widetilde{b}$ onto $\widetilde{b}'$. Note that $\widetilde{a} \neq \widetilde{b}'$, thus $\widetilde{a} \nparallel \widetilde{b}'$ because these distinct edges share a vertex and $\widetilde{X}$ is $CAT(0)$ hence clean. We deduce that $\widetilde{b} \nparallel \widetilde{b}'$, so that $g\widetilde{Y} \neq \widetilde{Y}$.

If $\overleftarrow{a}, \overrightarrow{b}$ are consecutive in a square then by lifting we sse that $\overleftarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}'}$ are also consecutive in a square. The center of this square is in $\widetilde{Y} \cap g\widetilde{Y}$.

If $\overrightarrow{a} \parallel \overrightarrow{b}$ then $\overrightarrow{\widetilde{a}} \parallel \overrightarrow{\widetilde{b}}$. Hence v is in $\widetilde{Y}^+ \cap g\widetilde{Y}^+$, where Y^+ denotes the component of $\partial N_{\widetilde{Y}}$ through v.

□

Remark 9.10. Define a group action of G on $\widetilde{X}$ to be fully clean whener for each $g \in G$ and each hyperplane $\widetilde{Y}$ of $\widetilde{X}$ with regular neighbourhood N, we have $gN \cap N \neq \emptyset \Rightarrow g\widetilde{Y} = \widetilde{Y}$. Then X is fully clean if and only if the action of its fundamental group on the universal cover is fully clean. The proof is similar as the proof of Lemma 9.9.

Definition 9.11. G acts *nicely* on the $CAT(0)$ cube complex $\widetilde{X}$ if the following holds for each two intersecting hyperplanes Y and W with regular neighborhoods N_Y and N_W: For each $g \in G$, if $gN_Y \cap N_W \neq \emptyset$ then gY intersects W.

G acts *specially* on the $CAT(0)$ cube complex $\widetilde{X}$ if it acts cleanly and nicely.

Lemma 9.12. *Let X denote a non positively curved cube complex, and let G denote its fundamental group acting by deck transformations on the universal covering space $\widetilde{X}$.*

Then X is special if and only if G acts specially.

Proof. Assume that G acts cleanly and nicely. By Lemma 9.9 we know that X is clean. Let us prove that it is in fact special. So consider two intersecting hyperplanes Y, W. Assume that there are distinct oriented edges $\overrightarrow{a}, \overrightarrow{b}$ dual to Y, W with the same initial point v. Let $\overrightarrow{a}_0, \overrightarrow{b}_0$ denote two oriented edges with the same initial point v_0, such that $\overleftarrow{a}_0, \overrightarrow{b}_0$ are consecutive in a square, and $a_0 \parallel a, b_0 \parallel b$. Lift the vertex v_0 to a vertex $\widetilde{v}_0$ of $\widetilde{X}$, then lift $\overrightarrow{a}_0, \overrightarrow{b}_0$ at $\widetilde{v}_0$. Lift a parallelism from a_0 to a, let $\widetilde{a}$ denote the last edge (projecting onto a). Similarly lift a parallelism from b_0 to b, let $\widetilde{b}$ denote the last edge (projecting onto b). Since $\overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}}$ project to oriented edges with the same origin v there is an element $g \in G$ such that $g\overrightarrow{\widetilde{b}}$ has the same origin as $\overrightarrow{\widetilde{a}}$. Thus $gN_{\widetilde{W}} \cap N_{\widetilde{Y}} \neq \emptyset$ (here $\widetilde{W}$ denotes the hyperplane dual to $\widetilde{b}$ and $\widetilde{Y}$ denotes the hyperplane dual to $\widetilde{a}$). Since G acts nicely we know that $g\widetilde{W}$ intersects $\widetilde{Y}$. Thus there is a square containing $\widetilde{a} \cup g\widetilde{b}$ (X is $CAT(0)$ hence special). Projecting this square in X shows that there is no interosculation of Y, W at $(v; \overrightarrow{a}, \overrightarrow{b})$.

Conversely assume that X is special. By Lemma 9.9 we know that G acts cleanly. Consider two intersecting hyperplanes $\widetilde{Y}$ and $\widetilde{W}$ of $\widetilde{X}$ and an element $g \in G$ such that $gN_{\widetilde{Y}} \cap N_{\widetilde{W}} \neq \emptyset$. Pick a vertex $\widetilde{v}$ in $gN_{\widetilde{Y}} \cap N_{\widetilde{W}}$. Then $\widetilde{v}$ is the origin of oriented edges $\overrightarrow{\widetilde{a}}, g\overrightarrow{\widetilde{b}}$ with $\overrightarrow{\widetilde{a}}, \overrightarrow{\widetilde{b}}$ dual to $\widetilde{Y}, \widetilde{W}$. Note that $\overrightarrow{\widetilde{a}} \neq g\overrightarrow{\widetilde{b}}$ since G acts cleanly. If we project all the situation in X we get intersecting hyperplanes Y, W dual to distinct oriented edges $\overrightarrow{a}, \overrightarrow{b}$ with the same origin. Since X is special $\overleftarrow{a}, \overrightarrow{b}$ are consecutive in some square of X. If we lift this square at $\widetilde{v}$, we see that $g\widetilde{W}$ intersects $\widetilde{Y}$.

□

Remark 9.13. Clearly if G acts cleanly, nicely or specially on $\widetilde{X}$, then so does every subgroup $G' \subset G$. This generalizes the "special" statement in Corollary 3.8.

Lemma 9.14. *Let G denote some cocompact isometry group of a locally compact $CAT(0)$ cube complex X. Assume that for each hyperplane Y of X the hyperplane stabilizer G_Y is separable in G. Then G has a finite index subgroup whose action is (fully) clean.*

Proof. Let N_Y denote the regular neighbourhood of some hyperplane Y in X. Consider the set $I(G,Y) = \{g \in G : gN_Y \cap N_Y \neq \emptyset\}$. We have $G_Y \subset I(G,Y)$ and in fact $I(G,Y)$ is invariant under left- and right-multiplication by G_Y. Set $\text{Bad}(G,Y) = I(G,Y) - G_Y$. The action of the group is fully clean if and only if $\text{Bad}(G,Y) = \emptyset$ for every hyperplane Y.

Since G is cocompact on X the group G_Y is cocompact on the set of edges dual to Y. So mod. G_Y there are finitely many edges in N_Y. Since X is locally finite there are finitely many edges meeting a given edge. Thus there are finitely many elements $b_1, \cdots, b_n \in G$ such that $\mathrm{Bad}(G, Y) = \cup_i G_Y b_i G_Y$.

Since $\{b_1, \cdots, b_n\} \cap G_Y = \emptyset$ and G_Y is separable there is a finite index subgroup $G' \subset G$ containing G_Y and disjoint from $\{b_1, \cdots, b_n\}$. We then have $I(G', Y) = I(G, Y) \cap G'$, so $\mathrm{Bad}(G', Y) = I(G', Y) - G'_Y = I(G, Y) \cap G' - G_Y = \mathrm{Bad}(G, Y) \cap G'$.

But $\mathrm{Bad}(G, Y) \cap G' = \emptyset$. For if $g' \in G'$ belongs to $G_Y b_i G_Y$ for some i, then since $G_Y \subset G'$, we must have $b_i \in G'$, contradiction.

Since G is cocompact on X there are finitely many edges mod. G. So there are hyperplanes $Y_1, \cdots, Y_m$ such that each hyperplane of X is in the G-orbit of one of the Y_i's. Consider finite index subgroups $G'_i \subset G$ such that $\mathrm{Bad}(G'_i, Y_i) = \emptyset$. Then let $\bar{G} \subset G$ denote a finite index normal subgroup contained in $\cap_i G'_i$. We have $\mathrm{Bad}(\bar{G}, Y_i) = \emptyset$ for each i and then by invariance $\mathrm{Bad}(\bar{G}, Y) = \emptyset$ for each hyperplane Y.

□

Lemma 9.15. *Let G denote some cocompact isometry group of a locally compact $CAT(0)$ cube complex X. Assume that for each pair of intersecting hyperplanes Y, W the set $J(G, Y, W) = \{g \in G : gW \cap Y \neq \emptyset\}$ is separable in G. Then G has a finite index subgroup whose action is nice.*

The same if true if we assume that some (G_Y, G_W)-invariant subset $J'(G, Y, W) \subset J(G, Y, W)$ containing $G_Y G_W$ is separable in G.

Proof. Let N_Y, N_W denote the regular neighbourhoods of intersecting hyperplanes Y, W in X. We fix a square Q whose edges are dual to Y or W.

Consider the set $I(G, Y, W) = \{g \in G : gN_W \cap N_Y \neq \emptyset\}$. We clearly have $G_Y G_W \subset J(G, Y, W) \subset I(G, Y, W)$ and in fact $I(G, Y, W)$ is invariant under left-multiplication by G_Y, and right-multiplication by G_W. Set $\mathrm{Bad}(G, Y, W) = I(G, Y, W) - J(G, Y, W)$. The action of the group is nice if and only if $\mathrm{Bad}(G, Y, W) = \emptyset$ for every pair of intersecting hyperplanes Y, W.

Since G is cocompact on X the group G_Y is cocompact on the set of edges dual to Y. So mod. G_Y there are finitely many edges in N_Y. Similarly there are finitely many edges in N_W mod. G_W. So there is a number $R \geq 0$ such that for each $g \in I(G, Y, W)$ there exists $g' \in G_Y g G_W$ mapping an edge b' dual to W and at combinatorial distance $\leq R$ of Q to an edge meeting N_Y, with gb' at combinatorial distance $\leq R$ of Q. Since X is locally finite there are finitely many edges meeting a given edge. Thus there are finitely many elements $b_1, \cdots, b_n \in G$ such that $\mathrm{Bad}(G, Y, W) = \cup_i G_Y b_i G_W$.

Let us prove the most general form of the Lemma. So let $J'(G, Y, W) \subset J(G, Y, W)$ denote some (G_Y, G_W)-double coset containing $G_Y G_W$, which we assume to be closed in the profinite topology on G.

Since $\{b_1, \cdots, b_n\} \cap J'(G, Y, W) = \emptyset$ and $J'(G, Y, W)$ is separable there is a finite index subgroup $G' \subset G$ such that $(b_1 G' \cup \cdots \cup b_n G') \cap J'(G, Y, W) = \emptyset$. We may and will assume that G' is normal in G.

We then have $I(G', Y, W) = I(G, Y, W) \cap G'$ and $J(G', Y, W) = J(G, Y, W) \cap G'$, so $\mathrm{Bad}(G', Y, W) = I(G', Y, W) - J(G', Y, W) = \mathrm{Bad}(G, Y, W) \cap G'$.

Assume $\mathrm{Bad}(G, Y, W) \cap G' \neq \emptyset$. Then let $g' \in G'$ belong to $G_Y b_i G_W$ for some i: we may write $g' = y b_i w$ with $y \in G_Y, w \in G_W$. We rewrite this in the form $b_i(w^{-1} {g'}^{-1} w) = yw$.

Since G' is normal we deduce that $b_iG' \cap G_YG_W \neq \emptyset$, thus $b_iG' \cap J'(G,Y,W) \neq \emptyset$, contradiction.

So at this stage we were able to find a finite index subgroup $G' \subset G$ such that $\mathrm{Bad}(G',Y,W) = \emptyset$.

Since G is cocompact on X there are finitely many squares mod. G. So there are finitely many pairs of intersecting hyperplanes $(Y_1,W_1),\cdots,(Y_m,W_m)$ such that each pair of intersecting hyperplane of X is in the G-orbit of one of the (Y_i,W_i)'s. Consider finite index subgroups ${G'}_i \subset G$ such that $\mathrm{Bad}({G'}_i,Y_i,W_i) = \emptyset$. Then let $\bar{G} \subset G$ denote a finite index normal subgroup contained in $\cap_i {G'}_i$. We have $\mathrm{Bad}(\bar{G},Y_i,W_i) = \emptyset$ for each i and then by invariance $\mathrm{Bad}(\bar{G},Y,W) = \emptyset$ for each pair of intersecting hyperplanes (Y,W).

□

Lemma 9.16. *Let G act on a $CAT(0)$ cube complex. Assume that there are finitely many hyperplanes mod. G (for example G is cocompact). If G acts cleanly then it has a finite index subgroup acting freely.*

Proof. An automorphism g of the $CAT(0)$ cube complex X has a fixed point if and only if it preserves some k-cube Q.

For each hyperplane Y of X we may consider the orbit $G \cdot Y$. Since G acts cleanly any two distinct hyperplanes of $G \cdot Y$ are disjoint. We may then consider the tree T_Y dual to the covering of X by the closed convex subsets obtained by taking the closure of a connected component of $X - G\dot{Y}$. The group G maps to an automorphism group of T_Y. The automorphism group of a tree always has an index two subgroup consisting in those element that preserve the bipartite structure of the tree. So for each (G-orbit of) hyperplane Y we get a subgroup $G_Y \subset G$ of index ≤ 2 acting without inversion on T_Y.

By assumption there are finitely many G-orbit of hyperplanes: say $G \cdot Y_1, \cdots, G \cdot Y_n$. The intersection $G' = \cap_i G_{Y_i}$ is a finite index subgroup of G. We claim that G' acts freely on X. Indeed assume that $g \in G'$ preserves some k-cube Q of X. Let $W_1, \cdots, W_k$ denots the hyperplanes dual to an edge of Q. Since G' acts without inversion on each tree T_{W_i} it follows that g has to fix each vertex of Q.

Let v denote some vertex in Q. Assume that $g \neq \mathrm{id}_X$. Let w denote a vertex of X such that $gw \neq w$ and the combinatorial distance between v and w is as small as possible. Then consider the last edge e of a combinatorial geodesic from v to w. One of the vertices of e is w, let v' denote the second vertex. By minimality of $d(v,w)$ we have $gv' = v'$. Note that $ge \neq e$ hence the hyperplane dual to ge is distinct of the hyperplane dual to e. This contradicts the cleanliness of the action of G.

□

Lemma 9.17. *Let G denote a group acting on a set X.*

1. *Let Y denote some subset of X. Assume G has a finite index subgroup G' for which the stabilizer ${G'}_Y$ is separable in G'. Then the full stabilizer G_Y is separable in G.*
2. *Let Y,W denote some pair of subsets of X. Assume G has a finite index subgroup G' for which the double coset ${G'}_Y{G'}_W$ is separable in G'. Then the full double coset G_YG_W is separable in G.*

Proof. (1) Since $G' \subset G$ is of finite index, the subgroup ${G'}_Y \subset G_Y$ is of finite index too. Thus there are finitely many elements $g_1, \ldots, g_k$ such that $G_Y = \cup_i g_i {G'}_Y$. The subgroup G' is obviously closed in G, hence ${G'}_Y$ is infact closed in G. Then G_Y is closed as a finite union of closed subsets.

(2) Write as above $G_Y = \cup_{i=1}^{i=n} g_i {G'}_Y$, and write similarly $G_W = \cup_{j=1}^{i=m} {G'}_W h_j$. Then $G_Y G_W = \cup_{i,j} g_i {G'}_Y {G'}_W h_j$. By assumption ${G'}_Y {G'}_W$ is closed in G', hence in G. It follows that $G_Y G_W$ is closed as a finite union of closed subsets (left and right translations are homeomorphisms). □

Remark 9.18. Using Lemma 9.16, Lemma 9.17 and Lemma 8.8 provides an alternative proof of Lemma 9.14.

Theorem 9.19. *Let G act cocompactly on a locally finite $CAT(0)$ cube complex. The following are equivalent:*

1. *G has a finite index subgroup which acts specially.*
2. *For each hyperplane Y the hyperplane stabilizer G_Y is separable, and for any two intersecting hyperplanes Y, W the double coset $G_Y G_W$ is separable.*
3. *Each hyperplane stabilizer is separable, and for any two intersecting hyperplanes Y, W, the set $\{g \in G : gW \cap Y \neq \emptyset\}$ is separable.*

Proof. By Lemma 9.17 it is enough to prove $1 \Rightarrow 2$ for a finite index subgroup of G. By Lemma 9.16 it is enough to prove $1 \Rightarrow 2$ when G acts freely. But in this case $1 \Rightarrow 2$ follows from Proposition 9.7, because of the equivalence of definitions contained in Lemma 9.12. (Note that the hyperplanes subgroups of a compact non positively curved cube complex X are the groups G_Y, and the hyperplane double cosets are the $G_Y G_W$.)

Let us turn to the proof of the implication $2 \Rightarrow 3$. As we have seen in the proof of Lemma 9.15 the set $J = \{g \in G : gW \cap Y \neq \emptyset\}$ is a finite union of (G_Y, G_W)-double classes. Thus we may write $J = \cup_{i=1}^{i=n} G_Y g_i G_W$, and $g_i W \cap Y \neq \emptyset$. If we set $W_i = g_i W$ we have $g_i G_W = G_{W_i} g_i$ and W_i intersects Y. Thus $J = \cup_{i=1}^{i=n} G_Y G_{W_i} g_i$ is closed as a finite union of right translates of closed double cosets.

The implication $3 \Rightarrow 1$ follows by Lemma 9.14 and Lemma 9.15.

□

Using Lemma 9.12 we deduce the following:

Corollary 9.20. *Let X denote a compact, connected non positively curved cube complex. Then X is virtually special if and only if the following hold:*

1. *Each hyperplane subgroup is separable.*
2. *Each hyperplane double coset is separable.*

10. A linear version of Rips's short exact sequence

In [**?**], Rips gave a simple but very useful construction which given a finitely presented group Q, produces a finitely presented $C'(1/6)$ group with a finitely generated normal subgroup N, such that $Q \cong G/N$. Most pathological properties for a group Q, will lift to a suitably reinterpreted pathology of G, and this construction has thus proven very useful for producing interesting examples of $C'(1/6)$ groups. Several variations on Rips's

FIGURE 2. Squared Polygons: The five polygons above which are subdivided into squares correspond to the standard 2-cells of X. From top to bottom they correspond to relators of the form $R_t = W_t$, $x_j^{a_s} = X_{sj+}$, $x_j^{a_s^{-1}} = X_{sj-}$, $y_j^{a_s} = Y_{sj+}$, $y_j^{a_s^{-1}} = Y_{sj-}$. The simple arrows correspond to a_s letters, the white triangular arrows correspond to x_j letters, and the black triangular arrows correspond to the y_j letters. The left part of the relations sits always at the bottom of the picture.

construction have appeared. In [**?**], a version was given where G is the fundamental group of a nonpositively curved 2-complex. More recently a version was given in [**?**] where G is a residually finite $C'(1/6)$ group. In this section we describe a variation of Rips's construction where G is now the fundamental group of a finite nonpositively curved $\mathcal{VH}$-complex. This brings Rips's construction to the realm of linear groups through Theorem 5.8.

The following construction is similar to that given in [**?**].

Theorem 10.1. *Let Q be a finitely presented group. Then there exists a short exact sequence $1 \to N \to G \to Q \to 1$, where G is the fundamental group of a compact nonpositively curved thin $\mathcal{VH}$-complex, and $N \subset G$ is a finitely generated normal subgroup.*

Proof. Let Q be the finitely presented group

$$Q = \langle a_1, \ldots, a_S \mid R_1, \ldots, R_T \rangle.$$

We will form a nonpositively curved $\mathcal{VH}$-complex X, such that letting $G = \pi_1 X$, there is a finitely generated normal subgroup $N \subset G$ such that $1 \to N \to G \to Q \to 1$.

The complex X will be a subdivision of the standard 2-complex of a presentation of the following form:

$$\left\langle \begin{array}{c} a_1, \cdots, a_S \\ x_1, \cdots, x_J \\ y_1, \cdots, y_J \end{array} \middle| \; R_t = W_t \quad \begin{array}{c} x_j^{a_s} = X_{sj+}, \\ x_j^{a_s^{-1}} = X_{sj-}, \\ y_j^{a_s} = Y_{sj+}, \\ y_j^{a_s^{-1}} = Y_{sj-}, \end{array} \right\rangle \qquad \begin{pmatrix} s \in \{1, \ldots, S\} \\ j \in \{1, \ldots, J\} \\ t \in \{1, \ldots, T\} \end{pmatrix}$$

where the number J is to be determined later, and $X_{sj\pm}$, $Y_{sj\pm}$, and W_t are words in $\{x_1, \ldots, x_J, y_1, \ldots, y_J\}$ that will be specified later.

Note that the four last families of relations imply that the subgroup N generated by the x_j's and the y_j's is normal. And using the first family of relations we see that $\pi_1(X)/N$ has the same presentation as the initial group Q.

We will choose the words $\{W_t, X_{sj+}, X_{sj-}, X_{sj-}, Y_{sj-}\}$ so that the patterns of x_j and y_j letters conform to the patterns indicated in Figure 2. This will ensure that X is a $\mathcal{VH}$-complex where the a_s-edges and x_j-edges are horizontal and where the y_j-edges are vertical. To see this, observe that boundary vertices of the polygons in Figure 2 where two edges of the same or different types meet have respectively an even or an odd number of squares meeting.

Since X will be a $\mathcal{VH}$-complex, in order that X be nonpositively curved it is necessary that there be no cycles of length 2 in the links of vertices of X. We need only consider a cycle corresponding to a pair of corners of relators inside one of the $\{W_t, X_{sj+}, X_{sj-}, X_{sj-}, Y_{sj-}\}$ words. Therefore it is enough to choose these words so that they have no 2-letter repetitions.

We show that fixing Q, if J is sufficiently large then there exists a set of words $\{W_t, X_{sj+}, X_{sj-}, X_{sj-}, Y_{sj-}\}$ with no 2-letter repetitions, and which have the x-y form indicated in the pictures.

Consider $L(Q,J)$ which we define to be the sum of the lengths of the required words, and which obviously depends on Q and J.

$$L(Q,J) = \sum_t |W_t| + \sum_{s,j} |X_{sj+}| + \sum_{s,j} |X_{sj-}| + \sum_{s,j} |Y_{sj+}| + \sum_{s,j} |Y_{sj-}|.$$

$$L(Q,J) = \sum_t (|R_t| + 16) + \sum_{s,j} 19 + \sum_{s,j} 19 + \sum_{s,j} 25 + \sum_{s,j} 25$$

$$L(Q,J) = \sum_t (|R_t| + 16) + 19SJ + 19SJ + 25SJ + 25SJ$$

$$L(Q,J) = 88SJ + 16T + \sum_t |R_t| < KJ$$

where K is some constant that depends on the given finite presentation of Q (for instance $K = 100(S+T) + \sum_t |R_t|$).

We will now show that if $J \geq K+2$ then a set of appropriate words $\{W_t, X_{sj+}, X_{sj-}, X_{sj-}, Y_{sj-}\}$ can be chosen. Consider the sequence:

$$\Sigma_2 = (z_2)$$
$$\Sigma_3 = (z_2)(z_3 z_1 z_3)$$
$$\Sigma_4 = (z_2)(z_3 z_1 z_3)(z_4 z_1 z_4 z_2 z_4)$$

In general, define Σ_J to be:

$$\Sigma_J = (z_2)(z_3 z_1 z_3)(z_4 z_1 z_4 z_2 z_4)\cdots(z_J z_1 z_J z_2 \cdots z_J z_{J-3} z_J z_{J-2} z_J) \tag{3}$$

Observe that

$$|\Sigma_J| = \sum_{i=2}^{J} (2i-3) = (J-1)^2.$$

Furthermore, Σ_J has no repetitions of a 2-letter subword. Moreover, any subscript preserving way of changing the z_i letters to $x_i^{\pm 1}$ and $y_i^{\pm 1}$ letters yields a word which still has no 2-letter repetitions.

Clearly for $J \geq K+2$, the length of Σ_J is greater than JK and so we can use consecutive disjoint subwords of Σ_J for the $\{W_t, X_{sj+}, X_{sj-}, X_{sj-}, Y_{sj-}\}$ and substitute x_i and y_i letters for the z_i letters as needed. This completes the construction of the nonpositively curved $\mathcal{VH}$-complex X.

We will prove in Lemma 10.2 below that X is thin. Theorem 5.5 implies that X is virtually clean since X is thin, and hence $\pi_1 X$ is linear by Theorem 5.8. □

Lemma 10.2. *The complex X above is thin.*

Proof. According to Definition 5.4, it suffices to show that for some $n \geq 1$ there is no $\mathcal{VH}$-immersion $f : I_n \times I_2 \to X$ where I_k denotes the graph homeomorphic to a segment with k edges, and where edges parallel to I_n are horizontal, and edges parallel to I_2 are vertical.

Let J denote the horizontal path along the center of $I_n \times I_2$. We argue by considering the restriction $f : J \to X$.

First observe that if J passes through the interior of some polygon then provided n is greater than the diameter of any of the polygons, J emerges at either a yy-corner or a ya-corner.

The ya-corner case is impossible since all ya-corners have angle $\frac{3\pi}{2}$, we see that the interior of $I_n \times I_2$ cannot pass through such a corner.

In the yy-corner case, since there are no 2-letter repetitions, J cannot enter the interior of another polygon on the other side, and hence J travels along a horizontal path on the boundaries of polygons.

We now bound the length of a path J which is extendible to $I_n \times I_2$, where J does not enter the interior of any polygon. Assuming that $n \geq$ twice the diameter of any polygon, we see that J must contain an xxx-subpath which occurs along ∂P where P is a polygon whose interior $I_n \times I_2$ intersects. Since all x-pieces have length 1, and since each square in each polygon has at most two boundary edges, we see that it is impossible to extend J to $I_n \times I_2$ at each internal vertex of the xxx-subpath, and we are done. $\square$

Example 10.3. Let Q be a finitely presented group that is not residually finite. Applying Theorem 10.1 we obtain a short exact sequence $1 \to N \to G \to Q \to 1$ where $G = \pi_1 X$ and X is special.

Since Q is not residually finite, we see that the finitely generated subgroup N is not separable in G. This shows that Corollary 7.4 is sharp in the sense that its conclusion may fail to hold for nonquasiconvex subgroups.

On the other hand, if we choose Q to be infinite but residually finite, then N is separable, but not quasiconvex. So not every separable subgroup is quasiconvex.

11. Problems

The work in this paper motivates some problems:

Problem 11.1. Is every compact nonpositively curved clean complex virtually special?

Problem 11.2. Does every regular locally compact $CAT(0)$ cube complex admit an isometry into the universal cover of $ART(\Gamma)$ for some finite graph Γ? Here "regular" means for example: the automorphism group is cocompact.

Problem 11.3. Are all $CAT(0)$-quasiconvex subgroups of right-angled Artin groups separable?

Problem 11.4. Give conditions for building new special complexes out of old ones. It seems a graph product of special complexes is special.

In particular:

Problem 11.5. When is a graph of (virtually) special complexes (virtually) special? (It appears there is a theorem hiding here which places various malnormality and local-isometry conditions on the attaching maps of edge spaces.)

Problem 11.6. Does every finitely generated Coxeter group have a finite index subgroup that is π_1 of a special cube complex?

Problem 11.7. Let X be a compact nonpositively curved cube complex, and suppose that $\pi_1 X$ is word-hyperbolic. Is X virtually special?

This will be hard, but worth aiming for:

Problem 11.8. Is every Artin group virtually the fundamental group of a special cube complex?

Conjecture 11.9. *Let M be a hyperbolic 3-manifold with $\pi_1 M$ finitely generated. Then $\pi_1 M$ contains a finite index subgroup G such that G is the fundamental group of a special cube complex.*

In particular, $\pi_1 M \subset SL_n(\mathbb{Z})$ for some n.

While at face value, Conjecture 11.9 seems like a stretch, it is remarkably equivalent to the combination of two plausible but well-known conjectures: Hyperbolic 3-manifolds contain many immersed incompressible surfaces, and $\pi_1 M$ has separable quasiconvex subgroups.

12. Appendix A: Locally special cube complexes.

In this Appendix we give the proof of two lemmas used in the article. First we introduce some notations useful in the sequel.

For an integer $k \geq 2$, we let Q_k denote the 2-skeleton of the k-cube I^k and let C_k denote the subcomplex of I^k union of all squares containing $v_k = (1, \cdots, 1)$. We also introduce the union T_k of edges of I^k containing v_k.

For any edge e containing v_k let σ_e denote the euclidean reflection of the cube leaving e invariant. Let G_k be the group of automorphisms of I^k generated by the reflections σ_e. For any vertex v of I^k we will denote by σ_v the unique element of G_k sending v_k to v.

Definition 12.1. Let X be any cube complex. A *k-corner of X* is a combinatorial map $C_k \to X$. Two k-corners $c_1 : C_k \to X, c_2 : C_k \to X$ are said to be *adjacent along an edge* e containing v_k if $c_2 \circ \sigma_e = c_1$ on the union of squares of C_k containing e.

Note that a k-corner is an immersion. Using corners we may reinterpret familiar notions.

Lemma 12.2. *Let X denote any cube complex. Then:*

1. *X is* simple *if and only if the 2-skeleton X^2 is simple and for all integer $k \geq 3$, any k-corner $C_k \to X$ extends to* at most *one k-cube $I^k \to X$.*
2. *X is* nonpositively curved *if and only if the 2-skeleton X^2 is simple and for all integer $k \geq 3$, any k-corner $C_k \to X$ extends to* exactly *one k-cube $I^k \to X$.*

Proof.

(1) By definition X is simple if and only if for any vertex x of X and any integer $k \geq 2$, any set of k vertices of link(x) is the set of vertices of at most one $(k-1)$ simplex of link(x). But this is equivalent to ask that for any integer $k \geq 2$, any combinatorial map $T_k \to X$ extends to at most one k-cube.

So if X is simple then a fortiori any combinatorial map $C_k \to X$ extends to at most one k-cube, and X^2 is simple.

Conversely assume X^2 is simple and there is at most one cubical extension for each corner. Let $t : T_k \to X$ denote some combinatorial map extending to $f : I^k \to X$ and $g : I^k \to X$. As X^2 is simple we have $f = g$ on C_k, hence in fact $f = g$.

(2) By the first part of the Lemma we need only note that in a cube complex every complete graph of the link of a vertex is the 1-skeleton of some simplex if and only if for each integer $k \geq 3$, any k-corner extends to some k-cube.

□

Remark 12.3. Assume X is a cube complex such that X^2 is simple. Then two k-corners adjacent to the same k-corner $c : C_k \to X$ along the same edge e are equal. Because they are equal on the union of squares containing e, hence on T_k, hence everywhere.

In the sequel we will denote by $\sigma_e(c)$ the unique k-corner adjacent to c along e (whenever it exists).

For any vertex v and any combinatorial map $f : Q_k \to X$ or $f : I^k \to X$ we define a k-corner by $f_v = f \circ \sigma_{v|C_k}$. Clearly if v, w are adjacent and e is the unique edge containing v_k such that $\sigma_w = \sigma_v \sigma_e$ then f_v, f_w are adjacent along e.

Proof of Lemma 2.5.

We first check the uniqueness of an extension. For any extension $\bar{f} : X \to Y$ of $f : X^2 \to Y$, and for any k-cube $q : I^k \to X$ the composition $\bar{q} = \bar{f} \circ q$ is a k-cube of Y whose restriction $\bar{q}_{v_k}$ is entirely determined by f. As Y is nonpositively curved it is simple. By Lemma 12.2 we see that two extensions are equal on each cube.

To prove the existence of the extension it suffices to extend cube by cube. So we prove that any combinatorial map $q : Q_k \to Y$ extends to a (unique) k-cube $\bar{q} : I^k \to Y$.

Consider the collection of corners $\{q_v\}_v$ defined by q. By Lemma 12.2 each corner q_v extends to a k-cube $\bar{q}_v : I^k \to Y$. When v, w are adjacent the adjacency of q_v, q_w shows that $\bar{q}_w = \bar{q}_v \sigma_e$ on T_k. So by simplicity of Y we have $\bar{q}_w = \bar{q}_v \sigma_e$. Finally for each vertex v we have $\bar{q}_v = \bar{q}_{v_k} \sigma_v$.

So $\bar{q}_{v_k}$ is a k-cube equal to q on each $\sigma_v(C_k)$, hence it extends q.

□

The following is a first step in the construction of cube completions.

Lemma 12.4. *Let X denote any simple cube complex. Then there is a combinatorial embedding $j : X \to \bar{X}$ where $\bar{X}$ is simple, j induces an isomorphism between 2-skeleta and any combinatorial map $Q_k \to \bar{X}$ extends to a unique k-cube of $\bar{X}$.*

Proof. We define by induction a sequence $X_2 \to X_3 \to \cdots \to X_n \to \cdots$, where $X_2 = X$, each $X_{k-1} \to X_k$ is a combinatorial embedding of simple cube complexes inducing an isomorphism of the 2-skeleta, for each $\ell \leq k$, and any combinatorial map $Q_\ell \to X_k$ extends to a unique ℓ-cube $I^\ell \to X_k$.

Assume the sequence has been defined until X_n (for some $n \geq 2$).

For any combinatorial map $q : Q_{n+1} \to X_n$ the restriction of q to the intersection of each codimension 1 face of I^{n+1} with Q_{n+1} corresponds to a combinatorial map $Q_n \to X_n$, thus q extends to a map $\bar{q} : \partial I^{n+1} \to X_n$. This extension is unique because X_n is simple.

Declare two combinatorial maps $f : Q_k \to X_n, g : Q_k \to X_n$ to be *equivalent* if there is an automorphism φ of I^{n+1} such that $g = f\varphi$. Cleafly if f, g are equivalent and f extends to I^k, then so does g (and the extension are conjugate by an autormophism of the cube).

Now, in each equivalence class of combinatorial map $Q_{n+1} \to X_n$ that does not extend to a $(n+1)$-cube of X_n pick a representative $q_\alpha : Q_{n+1} \to X_n$, and consider its extension $\bar{q}_\alpha : \partial I^{n+1} \to X_n$. Attach a $(n+1)$-cube to X_n for each such $\bar{q}_\alpha$. Let X_{n+1} denote the resulting cube complex.

Clearly $X_n \subset X_{n+1}$ and both complexes have the same 2-skeleton.

For $\ell \leq n$, any combinatorial map $Q_\ell \to X_{n+1}$ extends to a some ℓ-cube $I^\ell \to X_n$. Any such extension has range in X_n, hence is unique.

Consider a combinatorial map $f : Q_{n+1} \to X_{n+1}$. Its range is contained in X_n. If f extends to a $(n+1)$-cube of X_n, then any extension to X_{n+1} has range in X_n, so there is exactly one extension.

Assume f does not extend to a n-cube of X_n. Then by definition f is equivalent to a unique q_α. By construction of X_{n+1} this map extends to a unique combinatorial map $I^{n+1} \to X_{n+1}$, hence so does f.

It remains to prove that X_{n+1} is simple. As X_n is already simple, we just need to check that any $(n+1)$-corner of X_{n+1} admits at most one extension to a $(n+1)$-cube of X_{n+1}.

Suppose $f, g : I^{n+1} \to X_{n+1}$ extend some corner $c : C_{n+1} \to X_{n+1}$. By Remark 12.3 all the $(n+1)$-corners defined by f, g are equal. Thus f, g extend the same $q : Q_{n+1} \to X_{n+1}$, and they are equal.

We let $\bar{X}$ denote the limit of the system of inclusion $X_2 \to \cdots \to X_n \to \cdots$. The inclusion $X \to \bar{X}$ has the required properties.

□

Now we link special cube complexes to completable cube complexes. We need a new notion. In what follows B_n denotes the strip $I_n \times I_1$ consisting in n squares, and we call the linear subgraph $I_n \times \{0\}$ (respectively $I_n \times \{1\}$) the *bottom horizontal path* (resp: the *top horizontal path*).

Definition 12.5 (locally special). Let X be any cube complex. We say that X is *locally special* whenever X is simple and the following two conditions hold:

(1) any combinatorial map $B_2 \to X$ sending the bottom horizontal path to two consecutive edges of a square also sends the top horizontal path to two consecutive edges of a square;

(2) any combinatorial map $B_4 \to X$ sending the bottom horizontal path to the boundary of a square of X identifies the two extreme vertical edges.

The following is clear:

Lemma 12.6. *Let X denote a simple cube complex such that*

(1) *no two hyperplane inter-osculate;*
(2) *no hyperplane directly self-osculates.*

Then X is locally special. In particular if X special it is locally special.

Remark 12.7. If $X' \to X$ is a covering of cube complexes, then X is locally special if and only if X' is locally special. So in view of the previous Lemma any virtually special cube complex is locally special.

Lemma 12.8. *Let X be a simple cube complex. Then X is locally special if and only if the following holds for all integers $k \geq 3$:*

(1) *for any edge e containing v_k, any k-corner has a k-corner adjacent along e (denoted $\sigma_e(c)$, see Remark 12.3);*
(2) *for any two edges a, b containing v_k and any k-corner $c : C_k \to X$ we have $\sigma_a(\sigma_b(c)) = \sigma_b(\sigma_a(c))$.*

Proof. For $k = 3$ the conditions above are a reformulation of the conditions defining a locally special complex. And the conditions hold for $k = 3$ if and only if they hold for any $k \geq 3$.

□

The group G_k is generated by the σ_e, subject to the relations $\sigma_e^2 = 1$ and $\sigma_a\sigma_b = \sigma_b\sigma_a$. Hence in a locally special cube complex there is an action of G_k on the set of k-corners (such that $\sigma_e(c) = c \circ \sigma_e$ on the union of squares containing e).

Lemma 12.9. *Let X denotes a locally special cube complex. Then any k-corner extends to a unique combinatorial map $Q_k \to X$.*

Proof. Consider a k-corner $c : C_k \to X$. For each vertex v of I^k use the action of G_k to define a new k-corner $c_v = \sigma_v(c)$. Now define a combinatorial map on $\sigma_v(C_k)$ by $q^v = c_v \circ \sigma_v^{-1}$.

If v, w are adjacent let e denote the edge of T_k such that $\sigma_w = \sigma_v\sigma_e$. Then we have $q^w = c_w \circ \sigma_e^{-1} \circ \sigma_v^{-1}$. But c_v, c_w are adjacent along e, so $q^w = q^v$ on the union of squares containing v and w. Thus the maps q^v fit together to produce a combinatorial map $q : Q_k \to X$ extending c.

If a corner has two extensions on Q_k, then by simplicity of X the corners defined by these extensions have to be equal (see Remark 12.3), hence the extensions are equal. □

Proof of Lemma 3.13.

By Lemma 12.6 we are done if we prove that if X is a locally special square complex then it is completable.

As X is a simple square complex we may apply Lemma 12.4, thus getting a combinatorial embedding $j : X \to \bar{X}$ that induces an isomorphism onto the 2-skeleton of $\bar{X}$, $\bar{X}$ is simple, and such that any $Q_k \to \bar{X}$ extends to a unique k-cube.

For any k-corner $c : C_k \to \bar{X}$ there is by Lemma 12.9 a unique extension $q : Q_k \to X$, which in turns extends to a unique k-cube $\bar{q} : I^k \to \bar{X}$. So $\bar{X}$ is nonpositively curved by Lemma 12.2.

□

13. Appendix B: Combinatorial geometry of $CAT(0)$ cube complexes.

In this appendix we establish some general results about the combinatorial distance on the set of vertices of a $CAT(0)$ cube complex.

In this Appendix, we always denote by $d(p, q)$ the combinatorial distance between two vertices p, q of some cube complex. So $d(p, q)$ is the smallest length n of a combinatorial (vertex) path $(p_0 = p, p_1, \ldots, p_n = q)$ (with $p_i = p_{i+1}$ or p_i, p_{i+1} are the endpoints of an edge of the complex). A combinatorial path between p and q of length $d(p, q)$ is a *geodesic*.

We will usually denote by X the $CAT(0)$ cube complex and by V its set of vertices. A subset W of V is *combinatorially convex* if each geodesic whose endpoints lie in W is itself entirely contained in W.

We recall first two basic facts about the combinatorial distance.

Lemma 13.1. *[see* [?], [?]*]*

Let X be a $CAT(0)$ cube complex. Then the distance $d(p, q)$ is the number of hyperplanes separating p from q. A path $(p_0 = p, p_1, \ldots, p_n = q)$ is a geodesic if and only if $p_{i+1} \neq p_i$ and there is no repetition in the sequence of walls through $e_1, \ldots, e_n$ (with $e_i = \{p_{i-1}, p_i\}$).

Definition 13.2. A *(combinatorial) half-space* defined by a hyperplane H of X is the the set of vertices contained in a given connected component of $X - H$.

A hyperplane H *separates* a set of vertices W if the two combinatorial half-spaces defined by H intersect W non-trivially. We first note a non-separation result:

Lemma 13.3. *Let Y denote a $CAT(0)$ convex subcomplex of a $CAT(0)$ cube complex X. Let $\overrightarrow{e}$ denote an oriented edge whose origin is in Y and whose endpoint is outside Y. Then the hyperplane H dual to e is disjoint from Y. In other words the half-space defined by H containing the origin of $\overrightarrow{e}$ in fact contains all vertices of Y.*

Proof. Let y denote the origin of $\overrightarrow{e}$ and let x denote its extremity. We see that the angle at y between the edge e and Y is $\geq \frac{\pi}{2}$. So e is orthogonal both to the hyperplane H and to Y, which implies in the $CAT(0)$ space X that $Y \cap H_E = \emptyset$.

□

Lemma 13.4. *Let X be a $CAT(0)$ cube complex, and let H be a hyperplane in X. Then the set of vertices of the neighborhood $N(H)$ is convex. Moreover, each half-space defined by H is convex.*

Proof. Assume a combinatorial geodesic g of X has its endpoints in $N(H)$, but has a vertex outside $N(H)$. Consider the first such vertex x. Then there is an oriented edge $\overrightarrow{e}$ with origin y in Y and endpoint x, so that (y, x) is a subpath of g.

By Lemma 13.3 the hyperplane $H(e)$ separates x from $N(H)$, hence from the endpoint of g.

Thus the path g has to cross again the hyperplane $H(e)$ after x, contradicting Lemma 13.1.

Let V^+ denote one of the combinatorial half-spaces defined by H. Any geodesic g with origin in V^+ and crossing once the hyperplane H cannot cross it twice by Lemma 13.1. Thus the endpoint of g is outside V^+, and V^+ is convex.

□

Remark 13.5. Suppose H is a hyperplane of a $CAT(0)$ cube complex, and that p, q are vertices of $N(H)$.

By Lemma 13.1, if p, q are not separated by H, then no geodesic joining p to q crosses H.

Otherwise, let e denote the unique edge dual to H containing p, and let p' denote its second vertex. Then (p, p') followed by any geodesic from p' to q yields a geodesic from p to q. This again follows from Lemma 13.1.

Lemma 13.6. *The (combinatorial) convex hull of a set of vertices W is the intersection of (combinatorial) half-spaces of V containing W.*

Proof. The convex hull of W is by definition the intersection of all convex subsets containing W. Half-spaces are convex by Lemma 13.4. Hence it suffices to check that any convex subset is the intersection of half-spaces containing it.

If C is convex and x, y are vertices with $x \in C$, $y \notin C$ and x, y are the endpoints of an edge e, we claim that the hyperplane H_e separates y from C. Otherwise there would be a vertex y' in $N(H(e)) \cap C$ on the same side than y. But then by Remark 13.5 there is a geodesic from x to y' whose second point is y. Hence by convexity $y \in C$, which is a contradiction. □

Proposition 13.7. *Let X be a $CAT(0)$ cube complex and let Y be a $CAT(0)$-convex subcomplex. Then the set of vertices of Y is convex. Conversely any combinatorially convex set of vertices W is the set of vertices of a $CAT(0)$-convex subcomplex.*

Proof. Let V denote the set of vertices of X and let $V(Y)$ denote the set of vertices of the $CAT(0)$ convex subcomplex Y.

If e is an edge with endpoints x, y, and $x \notin Y$ but $y \in Y$, then $Y \cap H_E = \emptyset$ by Lemma 13.3. By Lemma 13.6 we see that x is not in the convex hull of $V(Y)$. Thus no vertex outside Y lies in the convex hull.

To prove the converse statement observe that by Lemma 13.6 we know that W is the intersection of all half-spaces containing it. Now for each half-space V^+ containing W and associated to some hyperplane H the union X^+ of cubes of X whose vertices are in V^+ is a $CAT(0)$-convex subcomplex. This is because boundary vertices v of X^+ are in the neighborhood $N(H)$, they are contained in exactly one edge e dual to H, and the link $\text{link}(v, X^+)$ is the complement in $\text{link}(v, X)$ of $\text{St}(e, \text{link}(v, X))$, which is a flag complex by nonpositive curvature.

□

Lemma 13.8. *Let X be a $CAT(0)$ cube complex. Let $x \in X^0$ and let $C \subset X$ be a nonempty $CAT(0)$ convex subcomplex with vertex set $V(C)$.*

Then there is a unique vertex $p = \Pi_C(x)$ *in* C *such that for any vertex* $y \in C$ *we have* $d(x,y) = d(x,p) + d(p,y)$*. It is characterized by the property* $p \in C$ *and* $d(x,p) \leq d(x,y)$ *for any vertex* $y \in C$*.*

Any wall separating x *from* p *also separates* x *from* C*.*

The vertex $\Pi_C(x)$ *is* the (combinatorial) projection of x onto C.

Proof. Observe that $V(C)$ is combinatorially convex by Lemma 13.7.

The relation $d(x,y) = d(x,p) + d(p,y)$ immediately implies that $d(x,V(C)) = d(x,p)$. So any p satisfying the relation has to be one of the vertices of C at minimal distance from x. Any other such vertex p' also satisfies $d(x,p') = d(x,V(C))$ and finally $d(p,p') = d(x,p') - d(x,p) = 0$. Uniqueness follows.

Now choose a vertex p such that $d(x,p) = d(x,V(C))$. Choose any vertex $y \in C$. Apply the property of median graphs to vertices x,p,y (see [?]). There are three combinatorial geodesics with endpoints $\{x,y\}$, $\{x,p\}$, $\{p,y\}$ passing through a common point m. This point m is on a geodesic from $p \in C$ to $y \in C$, thus by combinatorial convexity of $V(C)$ we see that $m \in C$. And m is on a geodesic from x to p: by the relation $d(x,p) = d(x,V(C))$ we see that $m = p$. Using the third geodesic we get $d(x,y) = d(x,p) + d(p,y)$.

Let us fix a combinatorial geodesic $\gamma = (x_0, x_1 \dots, x_d)$ from $x_0 = x$ to $x_d = p$. By Lemma 13.1 a wall separates x from p $\iff$ it passes through some edge e_i of γ. Thus the set of walls separating x from p is $\{W_0, W_1, \dots, W_{d-1}\}$.

Let $H_0, H_1, \dots, H_{d-1}$ denote the hyperplanes corresponding to $W_0, W_1, \dots, W_{d-1}$. Assume H_i disconnects C, so it passes through some edge of C. Observe that for each vertex $v \in C$ we have $d(x_{i+1}, v) \geq d(x_{i+1}, p)$, for otherwise we would not have $d(x,p) = d(x,V(C))$. Thus $p = \Pi_C(x_{i+1})$. Let y denote any vertex in $C \cap N(H_i)$. We know that for any geodesic g from p to y, the product $(x_{i+1}, \cdots, x_d).g$ is a geodesic from x_{i+1} to y. But $N(H_i)$ is combinatorially convex. Hence $p \in C \cap N(H_i)$.

The vertices x_{i+1}, p are not separated by W_i else $(x_{i+1}, \cdots, p)$ would not be a combinatorial geodesic. Let p' denote the unique vertex of $N(H_i)$ such that $\{p,p'\}$ is an edge e'_i with $e'_i \parallel e_i$. Then by reflection in $N(W_i)$ we have $d(p', x_i) = d(p, x_{i+1})$.

We claim that $p' \in C$. Indeed, let e denote some edge of H_i and containing p. Then $N(H(e_i)) \cap C$ is combinatorially convex and contains $p \cup e$. Consider a geodesic g from p to $\Pi_e(p)$: it has no edge in W_i. Then by reflecting g in $N(H_i)$ we get a geodesic g' starting at p', ending inside e, such that $(p,p').g'$ is still a geodesic. Thus by convexity $p' \in C$.

Thus we have $d(x,p') \leq d(x,x_i) + d(x_i,p') = d-1$ and $p' \in C$, which is a contradiction.

To conclude we have just seen that $H_i \cap C = \emptyset$. Furthermore H_i separates x from p, $p \in C$ and C is connected. Thus H_i separates x from C.

□

Remark 13.9 (projection and canonical retraction)**.** By Corollary 6.7 there is a cellular retraction $r : X \to C$. We briefly verify that for any vertex x of X we have $r(x) = \Pi_C(x)$.

We argue by induction on $d(x,V(C))$. Let $\gamma = (x_0, x_1 \dots, x_{d+1})$ denote a combinatorial geodesic from $x = x_0$ to $x_{d+1} = \Pi_C(x)$. As observed above we have $\Pi_C(x_1) = \Pi_C(x_0)$. It is thus enough to prove that $r(x_0) = r(x_1)$, that is, r "shrinks" the edge $e_0 = \{x_0, x_1\}$ to a vertex of C.

In the terminology of Proposition 6.5, r shrinks an edge to a point if and only if this edge is horizontal. But in our situation an edge of X is horizontal if and only if it is not parallel to an edge of C. And this is true for e_0, because by Lemma 13.8 we know that $W(e_0)$ does not intersect C.

Corollary 13.10. *Let X be a $CAT(0)$ cube complex. Let C, C' be nonempty $CAT(0)$ convex subcomplexes and let $\gamma = (x_0, x_1 \dots, x_m)$ be a minimal length combinatorial geodesic between the set of vertices of C and C'.*

Then each hyperplane H_i dual to the edge $e_i = \{x_{i-1}, x_i\}$ separates C from C'.

Conversely any hyperplane separating C from C' is dual to some e_i.

Proof. Let us prove the first statement. By the first part of Lemma 13.8 we see that $x_m = \Pi_{C'}(x_0)$ and $x_0 = \Pi_C(x_m)$. By the last part of Lemma 13.8 each wall $H(e_i)$ separates x_0 from C' and x_m from C. But $x_0 \in C$ and C is connected and disjoint from H_i, so in fact H_i separates C from C'.

The converse is true because any wall separating C from C' also separates $x_0 \in C$ from $x_m \in C'$.

□

Lemma 13.11. *Let X denote a $CAT(0)$ cube complex. Let C, C' denote $CAT(0)$ convex subcomplexes. Assume $C \cap C' \neq \emptyset$. Then for any vertex $x \in C$ we have $\Pi_{C\cap C'}(x) = \Pi_{C'}(x)$. In particular $\Pi_{C'}(x) \in C \cap C'$.*

Proof. By Lemma 13.8 there is a geodesic from x to $\Pi_{C\cap C'}(x)$ through $\Pi_{C'}(x)$. By combinatorial convexity of C we get $\Pi_{C'}(x) \in C$. Thus $\Pi_{C'}(x) \in C \cap C'$ and using the characterization by distances, we see that $d(x, \Pi_{C'}(x)) \leq d(x, \Pi_{C\cap C'}(x))$ (because $\Pi_{C\cap C'}(x) \in C'$), so $\Pi_{C'}(x) = \Pi_{C\cap C'}(x)$.

□

Definition 13.12. Let X denote any cube complex. Set $\dim(X) = \sup \dim q$, where the sup is taken over all possible cubes q of X. We call $\dim(X)$ the *dimension of* X. We say that X is *finite dimensional* whenever $\dim(X) < \infty$.

Lemma 13.13. *Let X denote a $CAT(0)$ cube complex. Assume that $C_1, \cdots, C_k$ is a family of $CAT(0)$ convex subcomplexes that pairwise intersect. Then in fact $C_1 \cap \dots \cap C_k$ is not empty.*

In particular if $H_1, \cdots, H_k$ is a family of pairwise distinct and intersecting hyperplanes then there is a cube containing edges dual to $H_1, \cdots, H_k$, hence $k \leq \dim(X)$.

Proof. Let us prove the first part of the lemma. The assertion is obvious for $k \leq 2$.

We first concentrate on $k = 3$. So assume that C_1, C_2, C_3 are $CAT(0)$ convex subcomplexes with $C_{ij} = C_i \cap C_j$ nonempty, but $C_1 \cap C_2 \cap C_3 = \emptyset$.

Consider a combinatorial geodesic $\gamma = (x_0, x_1 \dots, x_m)$ between the set of vertices of C_{12} and C_3 of minimal length ($m > 0$ by assumption). By Corollary 13.10 the hyperplane H_0 dual to $\{x_0, x_1\}$ separates C_{12} from C_3. Suppose H_0 disconnects C_1. Then by Lemma 13.3 the edge e_0 is contained in C_1, thus $x_1 \in C_1$.

Necessarily either H_0 is disjoint of C_1 or H_0 is disjoint from C_2, because $x_1 \notin C_{12}$ by definition of the geodesic γ. This implies that H_0 separates C_3 either from C_1 or from C_2, contradicting $C_{13} \neq \emptyset, C_{23} \neq \emptyset$.

Let us conclude by induction on k.

So assume $C_1, \cdots, C_{k+1}$ is a family of $CAT(0)$ convex subcomplexes that pairwise intersect. By the case $k = 3$ we know that the $CAT(0)$ convex subcomplexes $C_i' = C_i \cap C_{k+1}$ (for $1 \leq i \leq k$) pairwise intersect. We conclude by induction.

We now apply this result to the family of neighborhoods $N(H_1), \cdots, N(H_k)$ of k distinct pairwise intersecting hyperplanes. So we get that $N(H_1) \cap \cdots \cap N(H_k) \neq \emptyset$. Choose any vertex x in this intersection. Then there are edges $e_1, \cdots, e_k$ containing x and dual to $H_1, \cdots, H_k$. But a $CAT(0)$ cube complex is special and $H(e_i), H(e_j)$ intersect: so there is a square in X containing $e_i \cup e_j$. Using nonpositive curvature in X again we see that there is a cube containing all the e_i's.

□

Definition 13.14. Let X denote any $CAT(0)$ cube complex, and Y any subcomplex. The *cubical neighborhood of* Y is the subcomplex $U(Y)$ union of those cubes that meet Y. The *combinatorial neighborhood* of a subset V of the set of vertices of X is the set $\beta(V)$ of those vertices of X at combinatorial distance ≤ 1 of V.

We will consider iterated neighborhoods $U^k(Y), \beta^k(V)$ (defined inductively by $U^{k+1}(Y) = U(U^k(Y))$ and $\beta^{k+1}(V) = \beta(\beta^k(V))$).

Lemma 13.15. *Let X denote any $CAT(0)$ cube complex and let Y denote any $CAT(0)$ convex subcomplex with vertex set V.*

Then $U(Y)$ is still a $CAT(0)$ convex subcomplex. If furthermore X is finite dimensional then the set of vertices of $U(Y)$ is contained in $\beta^{\dim(X)}(V)$.

Proof. Assume $U(Y)$ is not $CAT(0)$-convex. Thus its set of vertices V' is not combinatorially convex by Proposition 13.7.

So there are vertices x, x' in $U(Y)$ and a combinatorial geodesic g from x to x' with $g \not\subset V'$. We choose x, x' such that $d(x, x')$ is minimal. Let z denote the second vertex of g and let e denote the edge $e = \{x, z\}$.

Observe that by minimality we have $z \notin V'$.

There are vertices y, y' in Y such that $a = \{y, x\}$ and $a' = \{y', x'\}$ are edges of X. Each cube containing a is in $U(Y)$.

By Lemma 13.3 we know that the hyperplane $H(a)$ does not separate V, so y' and y are on the same side of $H(a)$.

We claim that $H(a)$ contains no edge of g. Else g would have an initial path g_0 joining x to another vertex of $N(H(a))$. By convexity we would have $g_0 \subset N(H(a))$. In particular $e \subset N(H(a))$, so there would exist a square in X containing $a \cup e$, contradicting the fact that $z \notin V'$.

The previous observation shows that x and x' are on the same side of $H(a)$. Hence in fact $H(a) = H(a')$. By convexity of $N(H(a))$ we see that $g \subset N(H(a))$ and this again contradicts the fact that $z \notin V'$.

The inclusion $V' \subset \beta^{\dim(X)}(V)$ follows by definition.

□

Corollary 13.16. *Let X be a finite dimensional $CAT(0)$ cube complex. Let C_1, C_2 denote two $CAT(0)$-convex subcomplexes with vertex sets V_1, V_2, such that $d(V_1, V_2) > 2\delta \dim(X)$. Then there exists a wall W separating $\beta^\delta(V_1)$ from $\beta^\delta(V_2)$.*

Proof. Consider the subcomplexes $C_1' = U^\delta(C_1)$ and $C_2' = U^\delta(C_2)$. By Lemma 13.15 the subcomplex C_i' is $CAT(0)$ convex and its vertices are in $\beta^{\delta \dim(X)}(V_i)$.

By assumption $\beta^{\delta \dim(X)}(V_1) \cap \beta^{\delta \dim(X)}(V_2) = \emptyset$. Thus the convex subcomplexes C_1' and C_2' are disjoint and so by Corollary 13.10 they are separated by some hyperplane. This concludes because clearly $\beta^\delta(V_i) \subset C_i'$.

□

Lemma 13.17. *Let X be a δ-hyperbolic $CAT(0)$ cube complex where δ is an integer thinness constant for combinatorial geodesic triangles.*

Let C_1, C_2, D_1, D_2 denote nonempty $CAT(0)$ convex subcomplexes. We assume $D_1 \subset C_1$ and $D_2 = C_1 \cap C_2$.

For any vertex x in X and any vertex $q_1 \in D_1$ define vertices p_1, p_2, q_2 in $C_1 \cup C_2$ by the relations:

$$p_1 = \Pi_{C_1}(x), \quad p_2 = \Pi_{C_2}(x), \quad q_2 = \Pi_{C_2}(q_1)$$

Suppose that q_1 is a quasi-projection in the sense that $d(p_1, q_1) \leq \delta$, and that there exists a hyperplane H separating the δ-neighborhoods of V_{D_1} and V_{C_2}. Then $q_2 \in D_2$, $d(p_2, q_2) \leq \delta$, $d(x, p_2) > d(x, p_1)$ and $d(x, q_2) \geq d(x, q_1) + d(q_1, q_2) - 2\delta$.

Proof. Observe first that by Lemma 13.11 we have $p_2' = \Pi_{D_2}(p_1') \in D_2$ and also $q_2 = \Pi_{D_2}(q_1) \in D_2$.

Consider geodesics g_0, g_1, g_2, g_3, g_4 joining x to p_1, p_1 to q_1, q_1 to q_2, x to p_2 and p_2 to q_2.

We note that $g_0 g_1$ and $g_3 g_4$ are geodesics. The geodesic triangle $g_0 g_1 \cup g_2 \cup g_3 g_4$ is δ-thin. Thus p_2 is at distance $\leq \delta$ of either g_2 or $g_0 g_1$.

In the first case we note that $g_2 (g_4)^{-1}$ is also a geodesic, hence we obtain $d(p_2, q_2) = d(p_2, g_2) \leq \delta$.

Let us prove now that the second case does not occur. First $d(p_2, g_1) > \delta$, for otherwise there would exist a path of length $\leq 2\delta$ from p_2 to q_1, contradicting the fact that there is a wall separating the δ-neighborhoods of D_1 and C_2.

Assume now that $d(p_2, g_0) \leq \delta$. Let V^+ denote the halfspace defined as the set of vertices of X in the connected component of $X - H$ containing D_1. By Lemma 13.11 we see that $x \in V^+$ because its projection p_1 is in V^+. By convexity of V^+ we get $g_0 \subset V^+$. Then the relation $d(p_2, g_0) \leq \delta$ implies that H is dual to an edge of the δ-neighborhood of C_2, contradiction.

Let us introduce a new projection $p_2' = \Pi_{C_2}(p_1')$, and two new geodesics g_1', g_4' from p_1 to p_2', and from p_2 to p_2'. Note that $g_3 g_4'$ and $g_1'(g_4')^{-1}$ are geodesics. The geodesic triangle $g_0 \cup g_1' \cup g_3 g_4'$ is δ-thin and $d(p_2, g_0) > \delta$, hence $d(p_2, p_2') = d(p_2, g_1') \leq \delta$.

We have

$$d(x, p_2) = d(x, p_2') - d(p_2, p_2') \geq d(x, p_2') - \delta = d(x, p_1) + d(p_1, p_2') - \delta$$

$$\geq d(x, p_1) + d(q_1, p_2') - 2\delta.$$

But the separation hypothesis implies that $d(q_1, p_2') > 2\delta$, hence $d(x, p_2) > d(x, p_1)$.

To conclude we also have:

$$d(x, q_2) = d(x, p_1) + d(p_1, q_2) \geq (d(x, q_1) - \delta) + d(q_1, q_2) - \delta)$$

and we get the last inequality.

□

Remark 13.18. Using the projections defined in Lemma 13.11 and Proposition A.1 in [**?**] one can show that combinatorial quasiconvex subsets of a $CAT(0)$ cube complex are within a finite distance of convex subcomplexes.

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