Dynamical systems arising from classification of geometric structures

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Classification of geometric structures: A source of interesting dynamical systems

- Lie and Klein (1872): A *geometry* in the classical sense consists of the properties of a space X invariant under the transitive action of a Lie group G.
- Ehresmann (1936): Manifolds locally modeled on (G, X).
- Fix a topological manifold Σ.
- Classifying such (G, X)-structures on Σ leads to an action of the *mapping class group* $\mathrm{Mod}(\Sigma) := \pi_0\big(\mathrm{Homeo}(\Sigma)\big)$ on a *deformation space* $\mathrm{Def}_{(G,X)}(\Sigma)$ of (G, X)-structures.
- $\mathrm{Def}_{(G,X)}(\Sigma)$ itself is locally modeled on $\mathrm{Rep}\big(\pi_1(\Sigma), G\big)$
- The $\mathrm{Mod}(\Sigma)$-action on $\mathrm{Def}_{(G,X)}(\Sigma)$ corresponds to the $\mathrm{Out}(\pi)$-action on $\mathrm{Rep}\big(\pi_1(\Sigma), G\big)$.

Coordinate atlases and development

- *Geometry:* Homogeneous space $X = G/H$.
- *Topology:* Topological manifold Σ with universal covering $\widetilde{\Sigma} \longrightarrow \Sigma$ and fundamental group π.
- *Marking:* Homeomorphism $\Sigma \xrightarrow{f} M$; the geometry on M will vary, but the topology of Σ remains fixed.
  - Patches $U \subset M$; Coordinate atlas of charts $U \longrightarrow X$ defining local coordinates on U modeled on X.
  - On overlapping patches the change of coordinates are restrictions of transformations of X lying in G.
  - Charts globalize to immersion $\widetilde{\Sigma} \to X$, equivariant respecting the *holonomy homomorphism* $\pi \longrightarrow G$.
  - Holonomy globalizes coordinate changes.
- M *(G, X)-manifold,* (M, f) *marked* (G, X)*-structure* on Σ.

Ehresmann-Weil-Thurston principle

- Construct a *deformation space* of marked (G, X)-structures on Σ up to appropriate equivalence relation.
- Holonomy defines a mapping

$$\mathsf{Def}_{(G,X)}(\Sigma) \xrightarrow{\mathcal{H}} \mathsf{Hom}\big(\pi_1(\Sigma), G\big)/\mathsf{Inn}(G)$$

- Best cases (e.g. hyperbolic manifolds): stratify into smooth manifolds and $\mathcal{H}$ local diffeomorphism.
- Changing the marking corresponds to an action of the *mapping class group*

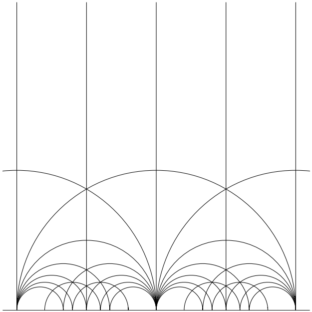
$$\mathsf{Mod}(\Sigma) := \pi_0\big(\mathsf{Homeo}(\Sigma)\big)$$

on $\mathsf{Rep}(\pi, G)$ whose orbit structure defines the *moduli space* of (G, X)-structures on Σ.

Example of trivial dynamics: Hyperbolic surfaces

- Suppose $X = \mathsf{H}^2$ and $G = \mathsf{Isom}(\mathsf{H}^2) \cong \mathsf{PGL}(2, \mathbb{R})$.
- Then $\mathsf{Def}_{(G,X)}(\Sigma)$ is the *Fricke space* $\mathfrak{F}(\Sigma)$, which identifies with the *Teichmüller space* by the uniformization theorem.
- $\mathcal{H}$ embeds $\mathfrak{F}(\Sigma)$ as a connected component of $\mathsf{Rep}(\pi, G)$:
  - *Open:* Weil (1960).
  - *Closed:* Chuckrow(1968), Kazhdan-Margulis (1968)
  - *Connected:* $\mathfrak{F}(\Sigma)$ is a cell:
    - Teichmüller (1943)+ uniformization;
    - direct hyperbolic-geometry proofs: Fenchel-Nielsen ($\sim$ 1940?), Fricke-Klein ($\sim$ 1900?).
- *Trivial dynamics:* Action of Mod on $\mathfrak{F}(\Sigma)$ is proper (Fricke ?). Its quotient is the *Riemann moduli space* of smooth Riemann surfaces of fixed topology.
- For $\Sigma = T^2$, the deformation space of unit-area Euclidean structures is the upper half-plane H^2 with action the modular group $\mathsf{Mod}(\Sigma) \cong \mathsf{GL}(2, \mathbb{Z})$ acting *properly* by linear fractional transformations.

Examples of nontrivial dynamics



- In contrast, the deformation space of complete affine structures on T^2 is homeomorphic to $\mathbb{R}^2$, with the Euclidean structures corresponding to the origin. (O. Baues 2000)
  - $\mathrm{Mod}(T^2)$-action is usual *linear action* of $\mathrm{GL}(2, \mathbb{Z})$ on $\mathbb{R}^2$.
  - This chaotic action admits no reasonable quotient.
- Therefore, the classification of geometric structures is a *dynamical system,* since the moduli space (its quotient) is often intractable.

Symplectic/Poisson structure

- When $G = \mathrm{SL}(2)$, then the character variety $\mathrm{Rep}(\pi, G)$ admits a symplectic structure extending:
  - Weil-Petersson Kähler form on Teichmüller component for $G = \mathrm{SL}(2, \mathbb{R})$;
  - Narasimhan-Atiyah-Bott Kähler form for $G = \mathrm{SU}(2)$.
- When $\partial\Sigma \neq \emptyset$, then $\mathrm{Rep}(\pi, G)$ inherits a *Poisson structure* with restriction mapping

$$\mathrm{Rep}(\pi, G) \longrightarrow \mathrm{Rep}\big(\pi_1(\partial\Sigma), G\big)$$

as universal Casimir. The level sets (relative character varieties) are its *symplectic leaves.*

Ergodicity for compact groups

- Let G be a compact Lie group with Levi factor K and Σ a compact orientable surface. If $\partial\Sigma = \emptyset$, then $\Gamma := \mathsf{Mod}(\Sigma) = \mathsf{Out}(\pi)$ (Nielsen).
- Components of $\mathsf{Rep}(\pi, G)$ parametrized by $\pi_1(K)$.
- Γ acts ergodically on each compoeenent of $\mathsf{Rep}(\pi, G)$ (Pickrell-Xia).
  - Also known for all surfaces of genus > 1.
  - Case of local products of U(1) and SU(2), and all surfaces ealier (Goldman).

Character functions and Hamiltonian twist flows

- Elements $\gamma \in \pi_1(\Sigma)$ define *character functions* on $\mathsf{Rep}(\pi, G)$:

$$\mathsf{Rep}(\pi, G) \xrightarrow{f_\gamma} \mathbb{R}$$
$$[\rho] \mapsto \Re\big(\mathsf{Tr}\rho(\gamma)\big)$$

  with Hamiltonian vector fields $\mathsf{Ham}(f_\gamma)$.
- For the Fricke-Teichmüller component when $G = \mathsf{PSL}(2, \mathbb{R})$, and γ corresponding to a *simple loop*, $\mathsf{Ham}(f_\gamma)$ generates the *Fenchel-Nielsen twist flows*, reparametrized (Wolpert 1982).
  - γ determines an *oriented cycle* on Σ and the Killing vector field generating the holonomy $\rho(\gamma)$ defines a coefficient in the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, giving a *infinitesimal deformation* of ρ in

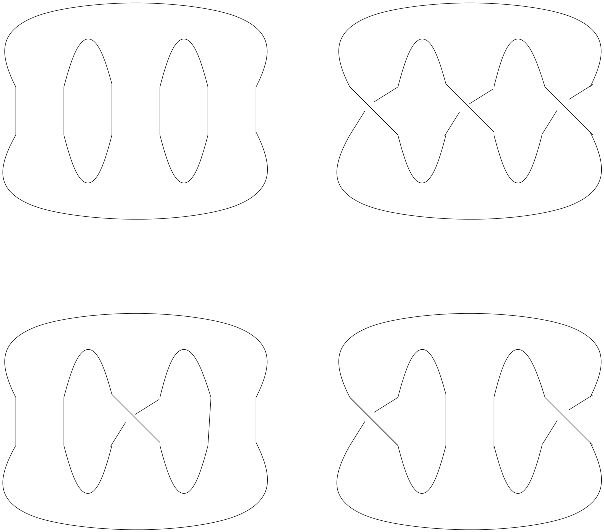
$$T_{[\rho]}\mathsf{Hom}\big(\pi_1(\Sigma), G\big)/G\big) \cong H_1(\Sigma, \mathfrak{sl}(2, \mathbb{R})_{\mathsf{Ad}\rho}\big)$$

  - This deformation is *supported* on the cycle γ.

Hamiltonian flows and Dehn twists

- Dehn twist Tw_γ generates lattice inside $\mathbb{R}$-action corresponding to $\mathrm{Ham}(f_\gamma)$-orbits.
- $\rho(\gamma)$ elliptic element of $G = \mathrm{SL}(2, \mathbb{R}) \Longrightarrow$ Integral curves of $\mathrm{Ham}(f_\gamma)$ are circles S^γ_ρ.
- For almost every value of f_γ, the Dehn twist Tw_γ defines an ergodic translation of S^γ_ρ;
- Ergodic decomposition: Every Tw_γ-invariant function is a a.e. $\mathrm{Ham}(f_\gamma)$-invariant.
  - For SL(2), a family of simple curves exist so that f_γ generate the coordinate ring of $\mathrm{Rep}(\pi, G)$
  - Flows of $\mathrm{Ham}(f_\gamma)$ generate transitive action on each connected component of where the vector fields span.
- $\mathrm{Mod}(\Sigma)$-action ergodic on regions where simple loops have elliptic holonomy.

Surfaces with $\pi \cong F_2$



Vogt-Fricke theorem and F_2

- Let $F_2 = \langle X, Y \rangle$ be free of rank two. Then

$$\mathrm{Hom}(F_2, SL(2)) \cong SL(2) \times SL(2)$$

  and $\mathrm{Rep}(F_2, SL(2))$ is its quotient under $\mathrm{Inn}\big(SL(2)\big)$.
- The $\mathrm{Inn}\big(SL(2)\big)$-invariant mapping

$$\mathrm{Hom}(F_2, SL(2)) \longrightarrow \mathbb{C}^3$$
$$\rho \longmapsto \begin{bmatrix} \xi := & \mathrm{Tr}\big(\rho(X)\big) \\ \eta := & \mathrm{Tr}\big(\rho(Y)\big) \\ \zeta := & \mathrm{Tr}\big(\rho(XY)\big) \end{bmatrix}$$

  defines an isomorphism

$$\mathrm{Rep}(F_2, SL(2)) \xrightarrow{\cong} \mathbb{C}^3.$$

Polynomial automorphisms

- $\mathrm{Out}(\mathsf{F}_2)$-invariant commutator trace function:

$$\begin{aligned} \mathrm{Rep}(\mathsf{F}_2, \mathrm{SL}(2)) \cong \mathbb{C}^3 &\xrightarrow{\kappa} \mathbb{C} \\ (\xi, \eta, \zeta) &\longmapsto \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 \\ &\qquad = \mathrm{Tr}[\rho(X), \rho(Y)] \end{aligned}$$

  - Casimir (∂-trace) for one-holed torus $\Sigma_{1,1}$.
  - (Nielsen): $\mathrm{Out}(\mathsf{F}_2) \cong \mathrm{GL}(2, \mathbb{Z}) = \mathrm{Mod}(\Sigma_{1,1})$.
- Nonlinear automorphisms generated by *Vieta involutions:*

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \longmapsto \begin{bmatrix} \eta\zeta - \xi \\ \eta \\ \zeta \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \longmapsto \begin{bmatrix} \xi \\ \xi\zeta - \eta \\ \zeta \end{bmatrix}, \quad \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \longmapsto \begin{bmatrix} \xi \\ \eta \\ \xi\eta - \zeta \end{bmatrix}$$

  - Coordinate projections are double Galois coverings
  - Vieta involutions are deck transformations.

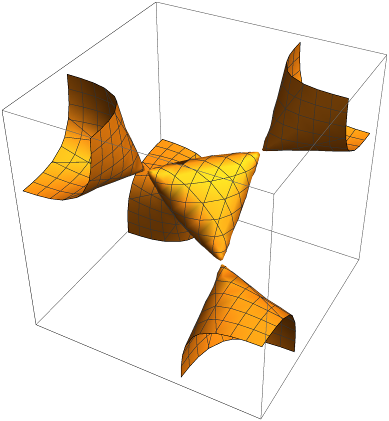
Cayley cubic $\xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta = 4$

- Reducible representations correspond precisely to $\kappa^{-1}(2)$.
  - Quotient of $\mathbb{C}^* \times \mathbb{C}^*$ by the involution

$$(a, b) \longmapsto (a^{-1}, b^{-1}).$$

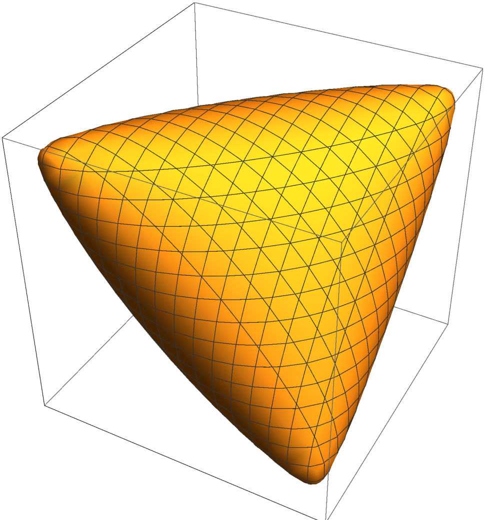
$$\xi = a + a^{-1}, \qquad \eta = b + b^{-1}, \qquad \zeta = ab + (ab)^{-1}$$

  - *Homogeneous dynamics:* $\mathrm{GL}(2, \mathbb{Z})$-action on $(\mathbb{C}^* \times \mathbb{C}^*)/(\mathbb{Z}/2)$.



$\mathbb{R}$-points: Unitary representations

- $\mathbb{R}$-points correspond to representations into $\mathbb{R}$-forms of SL(2): either $\mathrm{SL}(2,\mathbb{R})$ or SU(2).
- Characters in $[-2,2]^3$ with $\kappa \leq 2 \longleftrightarrow$ SU(2)-representations.



$\mathbb{R}$-points: Hyperbolic structures on one-holed tori

- Hyperbolic structures on $\Sigma_{1,1}$ correspond to real characters $(\xi, \eta, \zeta) \in \mathbb{R}^3$ with commutator trace $k := \kappa(\xi, \eta, \zeta) < -2$ corresponding to the boundary length:

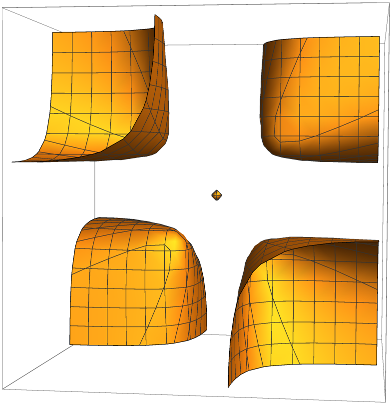
$$k = -2\cosh\big(\ell_{\partial\Sigma}/2)\big)$$

- The level set $\mathbb{R}^3 \cap \kappa^{-1}(-2)$ corresponds to hyperbolic structures on a once-punctured torus, that is, the end of Σ corresponding to $\partial\Sigma$ is a *cusp.*
- Level sets $\mathbb{R}^3 \cap \kappa^{-1}(k)$ where $-2 < k < 2$ correspond to hyperbolic tori with one *cone point of angle* θ:

$$k = -2\cos\big(\theta/2)\big),$$

  - *Generalized Fricke space* $\mathfrak{F}'(\Sigma)$ comprises hyperbolic structures on Σ with funnels, cusps or discs containing cone points.

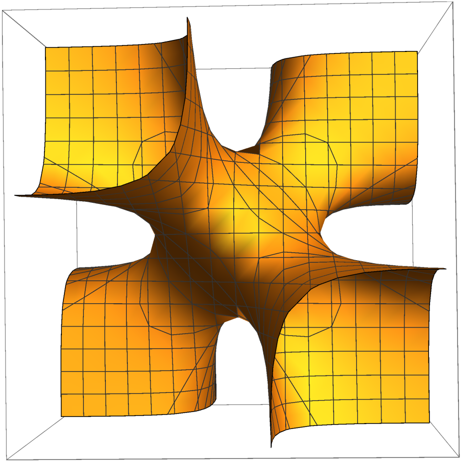
Example: The Markoff surface $x^2 + y^2 + z^2 = xyz$



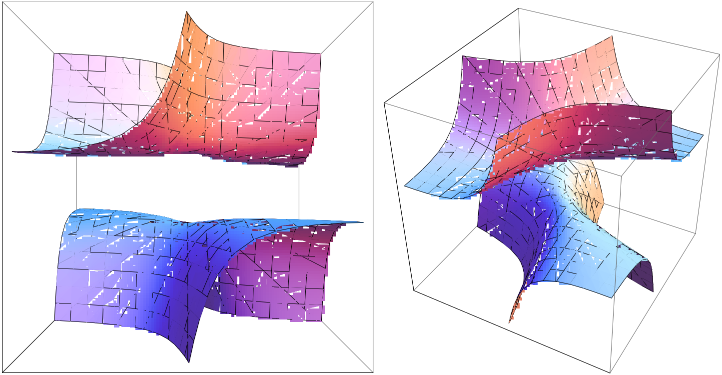
$\mathbb{R}^3 \cap \kappa^{-1}(-2)$ parametrizes hyperbolic structures on the punctured torus. The origin $(0,0,0)$ corresponds to the unique SU(2)-representation with $k = -2$. The famous *Markoff triples* correspond to triply symmetric hyperbolic punctured tori.

Fricke orbits define wandering domains for $k > 2$

- Homotopy equivalences $\Sigma_{1,1} \to \Sigma_{0,3}$ define embeddings of Fricke spaces $\mathfrak{F}(\Sigma_{0,3})$ in $\kappa^{-1}(k)$ for $k > 18$;
- For $k \leq 18$, action is ergodic.
- For $k > 18$, action is ergodic on complement of Fricke orbit



Relative character variety for one-holed Klein bottle $C_{1,1}$



Let $k > 2$ be the commutator trace. The relative character variety is defined by:

$$-x^2 - y^2 + z^2 + xyz = k + 2$$

Each component projects diffeomorphically to the (x, y)-plane.

Structures on $C_{1,1}$

- The Generalized Fricke space $\mathfrak{F}'(C_{1,1})$ of $C_{1,1}$ identifies with the subset defined by $z > 2$ and

$$Q_z(x,y) = x^2 + y^2 - zxy < 0.$$

- Trace function z corresponding to two-sided interior curve Z.
- The boundary trace is:

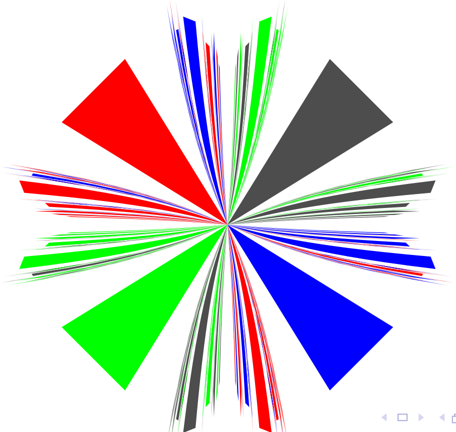
$$\delta := Q_z(x,y) + 2 = z^2 - k =$$

$$\begin{cases} -2\cosh(\ell/2) & \text{for a funnel with closed geodesic of length } \ell; \\ -2 & \text{for a cusp;} \\ -2\cos(\theta/2) & \text{for a point with cone angle } \theta; \end{cases}$$

- Goldman – McShane – Stantchev – Ser Peow Tan *Automorphisms of two-generator free groups and spaces of isometric actions on the hyperbolic plane,* DG.1509.03790

The level set $\kappa^{-1}(k)$ for $k > 2$

- Generalized Fricke space $\mathfrak{F}'(C_{1,1})$ of $C_{1,1}$ projects to a linear sector in $\mathbb{R}^2$ invariant under
$$\mathrm{Mod}(C_{1,1}) \cong \mathbb{Z}/2 \times \left(\mathbb{Z}/2 \star \mathbb{Z}/2\right) \sim \langle \mathrm{Tw}_Z \rangle \cong \mathbb{Z}.$$
- Wandering domain under Γ whose orbit is open and dense. What is the Hausdorff dimension of its complement?



HAPPY BIRTHDAY, GRISHA!