**Graded Assignment on Dynamical Systems - Due December 2**

*The quality of the writing will be an integral part of the grading.*

**Exercice 1** Let $X$ be a compact space, and $\phi : X \to X$ a continuous map. A *topological factor* of the topological dynamical system $(X, \phi)$ is a topological dynamical system $(Y, \psi)$ equipped with a semi-conjugation (surjective and continuous) $h : (X, \phi) \to (Y, \psi)$. It is said to be *trivial* if $h$ is a homeomorphism or if $Y$ is reduced to a single point.

(1) Give an example of a minimal dynamical system that has a non-trivial topological factor.

(2) A system $(X, \phi)$ is said to be *2-minimal* if for every $x \neq y \in X$, the orbit of $(x, y)$ in $X^2$ under $\phi \times \phi$ is dense in $X^2$.

Prove that if $(X, \phi)$ is 2-minimal, then every Hausdorff topological factor of $(X, \phi)$ is trivial.

**Exercice 2** Let $G$ be a group acting by orientation-preserving homeomorphisms on the circle $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Denote by $\pi : \mathbb{R} \to \mathbb{T}^1$ the canonical projection, and we may denote $\dot{x} = \pi(x)$ for $x \in \mathbb{R}$.

The goal of this exercise is to understand under what conditions there exists a $G$-invariant probability measure $\mu$ on $\mathbb{T}^1$.

(1) In this question, let $G = \mathrm{PSL}_2(\mathbb{R})$, acting on the Alexandrov compactification $\mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ by the usual formula

$$\rho_0(g)(x) = \frac{ax + b}{cx + d}$$

if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Fix a homeomorphism $h : \mathbb{R} \cup \{\infty\} \to \mathbb{T}^1$ and let $G$ act on $\mathbb{T}^1$ via the action $\rho = h\rho_0 h^{-1}$.

(a) For $g \in G$, what are the $\alpha$-limit and $\omega$-limit sets of a point $x \in \mathbb{T}^1$ under $\rho(g)$ ? When does $\rho(g)$ exhibit north-south dynamics on $\mathbb{T}^1$ ? [1]

(b) What is, as a function of $\mathrm{Tr}(g) = a + d$, the rotation number $\tau(\rho(g))$ ?

(c) Assume a homeomorphism $g$ of $\mathbb{T}^1$ has north-south dynamics on $\mathbb{T}^1$. What are the invariant measures under $g$ ?

(d) Prove that there is no measure invariant under all elements of $\rho(G)$ on $\mathbb{T}^1$.

(2) Now consider $G$ arbitrary, and fix an action $\rho : G \to \mathrm{Homeo}^+(\mathbb{T}^1)$ assuming $\rho(G)$ preserves $\mu \in \mathrm{Prob}(\mathbb{T}^1)$. If $\mu$ has at least one atom, show that $G$ has a finite orbit on $\mathbb{T}^1$.

(3) Assume $G$ preserves a measure $\mu$ that has no atoms and whose support is $\mathbb{T}^1$.

(a) Equip $\mathbb{T}^1$ with the usual distance $d_0$, defined by $d_0(\dot{x}, \dot{y}) = \min_{k \in \mathbb{Z}} |x - y + k|$ for $x, y \in \mathbb{R}$ (where $\dot{x}$ (resp. $\dot{y}$) is the projection of $x$ (resp. $y$) in $\mathbb{T}^1$). Show that if $\phi \in \mathrm{Homeo}^+(\mathbb{T}^1)$ is an isometry, then $\phi$ is a rotation.

(b) Define $d(x, y) = \mu([x, y])$ for $x, y \in \mathbb{T}^1$, where $[x, y]$ is a connected component of $\mathbb{T}^1 \setminus \{x, y\}$ minimizing $\mu([x, y])$. Show that $d$ is a distance, and that $(\mathbb{T}^1, d)$ is isometric to $(\mathbb{T}^1, d_0)$.

1. Recall that a matrix $g \in \mathrm{SL}_2(\mathbb{R})$ falls into one of three types :
— *elliptic* if it is conjugate to a rotation matrix,
— *parabolic* if it is conjugate to a matrix of the form $\begin{pmatrix} 1 & x & 0 & 1 \end{pmatrix}$ for some $x \in \mathbb{R}^*$,
— *hyperbolic* if it is conjugate to a matrix of the form $\begin{pmatrix} \lambda & 0 & 0 & \lambda^{-1} \end{pmatrix}$ for some $\lambda \in \mathbb{R}^*$.

(c) Prove that the action of $\rho(G)$ on $\mathbb{T}^1$ is conjugate to an action by rotations.

(4) Assume $G$ preserves a measure $\mu \in \mathrm{Prob}(\mathbb{T}^1)$ that has no atoms. For $t \in [0,1[$, let $f(\dot{t}) = \mu(\pi([0,t]))$. Show that $f$ is a continuous surjective map $f : \mathbb{T}^1 \to \mathbb{T}^1$, and that we can define an action $\rho' : G \to \mathrm{Homeo}^+(\mathbb{T}^1)$ such that $f \circ \rho(g) = \rho'(g) \circ f$ for all $g \in G$.

(5) Show that if every group homomorphism from $G$ to an abelian group has finite image, and if $G$ preserves a measure $\mu \in \mathrm{Prob}(\mathbb{T}^1)$, then $G$ has a finite orbit on $\mathbb{T}^1$.