DM 1 - due on October 14th

Let $(X, \mathscr{B}, \mu, \phi)$ be a conservative and ergodic measure-preserving dynamical system in discrete time. We assume that the measure μ is σ-finite. We fix $A \in \mathscr{B}$ with a non-zero and finite measure. We denote by $\tau_A : X \to \mathbb{N}^* \cup \{+\infty\}$ the first return time, and by τ_k the k-th return time to A, defined recursively by :

$$\tau_1(x) = \tau_A(x) = \min\{n \geq 1 \mid \phi^n(x) \in A\},$$

and for $k \geq 1$,

$$\tau_{k+1}(x) = \min\{n > \tau_k(x) \mid \phi^n(x) \in A\}.$$

Finally, we denote by $\phi_A : A \to A$ the induced map, defined as $\phi_A(x) = \phi^{\tau_A(x)}(x)$ if $x \in A$ such that $\tau_A(x) < +\infty$ (and $\phi_A(x) = x$ otherwise). We denote by μ_A the measure μ restricted to A (i.e., if $B \subset A$ is measurable, then $\mu_A(B) = \mu(B)$), and we assume that ϕ_A preserves the measure μ_A and that the induced dynamical system $(A, \mathscr{B}_A, \mu_A, \phi_A)$ is ergodic (see TD1).

Part I — The goal of this part is to verify that mixing does not behave well with respect to the induced map. To do so, we consider an invertible and mixing probabilistic dynamical system $(Y, \mathscr{B}_Y, \mu_Y, \psi)$, and for fixed $n \geq 2$, we define $X = (\mathbb{Z}/n\mathbb{Z}) \times Y$ equipped with the product measure of the uniform measure on $\mathbb{Z}/n\mathbb{Z}$ and μ_Y. We define $\phi : X \to X$ by $(k, y) \mapsto (k+1, y)$. Let $A = \{(0, y) \mid y \in Y\}$.

(1) Verify that ϕ preserves the measure, is ergodic, and that the induced map ϕ_A is well-defined and mixing.

(2) Prove that ϕ is not mixing.

Part II — The objective of this part is to show that for all functions $f, g \in \mathbb{L}^1(X, \mu)$, with $\int_X g \, d\mu \neq 0$, we have for μ-almost every x :

$$\text{(H)} \qquad \lim_{n \to \infty} \frac{\Sigma_n f(x)}{\Sigma_n g(x)} = \frac{\int_X f \, d\mu}{\int_X g \, d\mu},$$

(where we have denoted $\Sigma_n f(x) = \sum_{k=0}^{n-1} f \circ \phi^k(x)$).

(1) Prove equation (H) when μ is finite.

(2) We return to the general case. Explain why there exists a set $X' \subset X$ with $\mu(X \setminus X') = 0$, such that for all $x \in X'$ and for all $k \in \mathbb{N}^*$, we have $\tau_k(x) < +\infty$.

(3) (a) Let $x \in A$ and $k \in \mathbb{N}^*$. Show that for any measurable subset $B \subset A$ and for all $n \in]\tau_k(x), \tau_{k+1}(x)]$, we have

$$\Sigma_n(\mathbb{1}_B)(x) = \sum_{j=0}^{k} \mathbb{1}_B(\phi_A^j(x)).$$

Specify the value of this sum when $A = B$.

(b) Deduce that equation (H) holds when $f = \mathbb{1}_B$ and $g = \mathbb{1}_A$.

(4) Now fix $f \in \mathbb{L}^1(X, \mu)$ and assume $f \geq 0$. Define for $x \in X$,

$$f^A(x) = \sum_{k=0}^{\tau_A(x)-1} f(\phi^k(x)).$$

(a) Let $k \in \mathbb{N}^*$. Show that for all $x \in A$ and for all $n \in]\tau_k(x), \tau_{k+1}(x)]$, we have

$$\frac{\sum_{j=0}^{k-1} f^A(\phi_A^j(x))}{k+1} \leq \frac{\Sigma_n f(x)}{\Sigma_n(\mathbb{1}_A)(x)} \leq \frac{\sum_{j=0}^{k} f^A(\phi_A^j(x))}{k+1}.$$

(b) We want to prove that

$$\int_A f^A(x)d\mu_A(x) = \int_X f(x)d\mu(x) \tag{*}$$

(i) Show that it suffices to prove equation (*) for $f = \mathbb{1}_B$ with $B \in \mathscr{B}$ of finite measure, and then that we can assume B disjoint from A.

(ii) Assume B is disjoint from A. Consider the dynamical system $(A \cup B, \phi_{A\cup B})$ with the induced measure. Using Kac's theorem on the return time to A in this system, prove equation (*).

(c) Deduce from the above that for μ_A-almost every x, we have

$$\lim_{n\to+\infty} \frac{\Sigma_n f(x)}{\Sigma_n(\mathbb{1}_A)(x)} = \frac{\int_X f\, d\mu}{\mu(A)}.$$

(d) Conclude.