

(5) The upper half space Poincaré, Lobatchevski, Bolyai

$SL_2(\mathbb{R})$ acts on $\mathcal{H} = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$ by homographies $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \mapsto \frac{az+b}{cz+d}$ *various verifications left as exc.*
 C^∞ and transitively $z \mapsto z+b, z \mapsto a^2 z$

$$\mathcal{H} \underset{C^\infty\text{-diffeo}}{\simeq} SL_2(\mathbb{R})/SO(2) \simeq PSL_2(\mathbb{R})/PSO(2) \leftarrow \text{isomorphic to } SO(2)$$

(6) The space of volume 1 ellipsoids

Let $n \in \mathbb{N} \setminus \{0\}$, $\mathcal{S}_n^1 = \{f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \mid \text{bilinear, sym, pos definite, } \det(M_f) = 1\}$

$SL_n(\mathbb{R})$ acts transitively on $\mathcal{S}_n^1$ by $(g, f) \mapsto f \circ (g^{-1} \cdot g)$ *(by reduction theory)*

Exer: $\mathcal{S}_n^1 \subset C^\infty$ submanifold of $\text{Sym}_n = \{f: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \mid \text{bilin sym}\}$, $\text{codim } 1$
 $\frac{n(n+1)}{2} - 1 \dim$

$$\text{Stab}_{SL_n(\mathbb{R})}(\text{usual scalar product}) = SO(n)$$

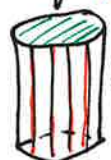
$$\mathcal{S}_n^1 \underset{C^\infty\text{-diffeo}}{\simeq} SL_n(\mathbb{R})/SO(n)$$

II

Vector bundles and connections

1 Fiber bundles fibrations

Def: * A C^∞ fiber bundle is $p: E \rightarrow B$ C^∞ where E, B C^∞ manifolds st
 $\forall b \in B, \exists F$ nonempty C^∞ manifold, $\exists U$ open neigh of b ,
 $\exists \phi: p^{-1}(U) \rightarrow U \times F$ C^∞ -diffeo st $\phi^{-1}(U) \rightarrow U \times F$ commutes



* $B = \text{base}$, $E = \text{total space}$, $p^{-1}(b) = \text{fiber over } b$, $U = \text{distinguished open set}$, $\phi = \text{local trivialisation of } p \text{ over } U$. If F independent of b , say fiber bundle with fiber F .

* $\forall p: E \rightarrow B, p': E' \rightarrow B'$ C^∞ fiber bundles, a morphism (of fiber bundles) from p to p' is pair (f, g) C^∞ maps st $E \xrightarrow{f} E'$ commutes.

If $B=B', g=id$, say morphism over id over B . One defines obviously composition isomorphism after the case context clear.
 Beware global sections don't always exist.

Ex: (1) $\forall M, F$ C^∞ manifolds, $pr_1: M \times F \rightarrow M$ trivial fiber bundle

A fiber bundle isomorphic to trivial one is trivialisable

$\uparrow$ fin covs 6 (2026) $\rightarrow$ global sections exist
 $\uparrow$ fin covs 6?

A fiber bundle is loc trivialisable $\Rightarrow$ admits local sections.

For instance $\mathbb{R}^{n+1} \ni (y, z) \mapsto [0, +\infty[$ fiber bundle with fiber $\mathbb{R}_n$,
 $x \mapsto \|x\|$ trivialisable

(2) C^∞ covering maps $\Leftrightarrow C^\infty$ fiber bundles with discrete fibers

(3) C^∞ fiber bundle $\Rightarrow$ surjective C^∞ submersion

$\mathbb{R} \ni (x, y) \mapsto \mathbb{R}$ counter-example to $\Leftarrow$
 $x \mapsto x^2 - 3x$

Theo (Ehresmann)

admitted
 not true complex analytic

$f: M \rightarrow N$ proper surjective C^∞ submersion $\Rightarrow f$ C^∞ fiber bundle

OK if M compact

A crucial family of examples is the following one.

Theo:
 OK complex analytic

Let M C^∞ manifold, endowed with proper, free, smooth (left) action of G (real) Lie group. Then top. quotient space $G \backslash M$ has a unique C^∞ manifold structure st $\pi: M \xrightarrow{\text{can}} G \backslash M$ C^∞ submersion.

$\uparrow$
 quotient manifold of M by G

Furthermore π C^∞ fiber bundle

Def: A principal fiber bundle with structural group G is $p: M \rightarrow N$ C^∞ fiber bundle st M endowed with proper, free, smooth action of G , preserving fibers of p , st p induces $G \backslash M \xrightarrow{\sim} N$ C^∞ diffeo.

Pf:

Step 1: $G \backslash M$ Hausdorff with countable basis: seen in §I.4.1.

Step 2: $\forall x \in M$, $G \cdot x$ is loc closed $\xRightarrow{\S I.4.2} G \cdot x$ submanifold and $\Theta_x: G \rightarrow G \cdot x$ C^∞ diffeo since action free

Pf: Let U open neigh of x with compact closure. Action proper $\Rightarrow K = \{g \in G \mid gx \in U\}$ compact. Hence $(G \cdot x) \cap U = (K \cdot x) \cap U$ is closed in U .

Step 3: C^∞ manifold structure on $G \backslash M$.

Pf: Let $p = \dim G$, $n = \dim M$. $\Delta: x \mapsto T_x(G \cdot x)$ is C^∞ p -plane field. Since bracket of 2 vector fields tangent to submanifold again tangent, by Step 2, by Frobenius, Δ is integrable, tangent to foliation $\mathcal{F}$. By uniqueness, leaves of $\mathcal{F}$ = connected components of G orbits.

$\forall x \in M$, let $\psi_x: U_x \rightarrow \mathbb{R}^p \times \mathbb{R}^{n-p}$ foliated local chart, U_x = open neigh of x with compact closure. Up to reducing U_x , let us prove that every orbit of G meets U_x in $\emptyset$ or 1 local leaf. Let V open neigh of e in G st $V \cdot x \subset U_x$. Otherwise, $\forall n \in \mathbb{N}$, $\exists x_n \in U_x$, $\exists g_n \in G \setminus V$ st $x_n \rightarrow x$, $g_n x_n \rightarrow x$. Action proper $\Rightarrow$ seq to extraction, $g_n \rightarrow g \in G \setminus V$.

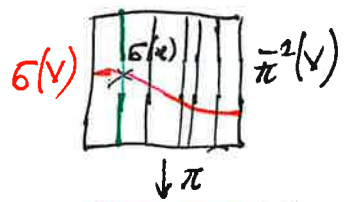
By continuity, $gx = x$. Action free $\Rightarrow g = e$, contradiction.

As in §I.4.2, $(\pi(U_x), \pi_2 \circ \psi_x)_{x \in M}$ C^∞ atlas of charts on $G \backslash M$ st π C^∞ subm.

Step 4: π C^∞ fiber bundle with fiber G .

Pf: As π submersion, $\forall y \in G \backslash M$, $\exists V$ open neigh of y , $\exists \phi: V \rightarrow M$ C^∞ section of π . Let us prove that $\Theta: V \times G \rightarrow \pi^{-1}(V)$ C^∞ diffeo, OK (then Θ^{-1} is a local trivialisation of π) $(x, g) \mapsto g \phi(x)$. Θ is C^∞ bijective since action is free. Only need to check that Θ submersion at every $(x, g) \in V \times G$ (since bijective submersions are C^∞ diffeo).

As G acts by differ, and θ is G -equivariant
 But $G \rightarrow M$ is C^∞ -diffeo onto G . $\sigma(x) = \pi^{-1}(x)$ by step 2
 $g \mapsto g\sigma(x)$
 Line π submersion and σ section of π , $T_{\sigma(x)}M = T_x\sigma(T_xV) \oplus T_x(\pi^{-1}(x))$



Hence linear map $T_x \theta$ onto.
 since its image contains two supplementary subspaces

□

Coro: A principal fiber bundle is trivialisable $\Leftrightarrow$ it admits a C^∞ section.
 de la démonstration ci-dessus. and TD!

Ex: (Hopf fibration, see "Dimension" online movie) Alvarez-Ghys. Leys.
 $\forall n \geq 1$, Lie group $U = (S^1, \times)$ acts on sphere $S_{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} \mid |z_0|^2 + \dots + |z_n|^2 = 1\}$
 diagonally by $z \cdot (z_0, \dots, z_n) = (zz_0, \dots, zz_n)$, C^∞ freely, properly.
 Hence can proj $\pi: S_{2n+1} \rightarrow \mathbb{P}_n(\mathbb{C}) = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^\times$ is C^∞ fiber bundle
 with fiber S^1 . since $\mathbb{P}_1(\mathbb{C}) \cong S^2$, $|S^1$ fibers over $|S^2$ with fiber $|S^1$.

2 Vector bundles

2.1 Definitions

Def: A vector bundle ξ over K with rank n is the data that depends C^∞ on b .
 $* p: E \rightarrow B$ C^∞ fiber bundle
 $* \forall b \in B$, a structure ξ_b K -vector space on $p^{-1}(b)$ (by abuse $\xi_b = p^{-1}(b)$)
 st $\forall b \in B$, $\exists U$ open neigh of b , $\exists \phi: p^{-1}(U) \rightarrow U \times F$
 local trivialisation of ξ over U st $\forall y \in U$, $p \circ \phi|_{p^{-1}(y)}: \xi_y \rightarrow F$ K -linear isom.
 $U =$ distinguished open set, $\phi =$ local trivialisation of ξ over U

Def: Let ξ, ξ' vector bundles over K , with projections $p: E \rightarrow B$ and $p': E' \rightarrow B'$.
morphism (of vector bundles) from ξ to $\xi' =$ morphism (of fiber bundles) (f, g)
 from p to p' st $\forall x \in B$, $f|_{p^{-1}(x)}: \xi_x \rightarrow \xi'_x$ K -linear. $\mapsto$ category
 We define obviously composition, identity morphism, isomorphism, automorphism,
 morphism over id.

NB: $\Gamma(\xi) = \{ C^\infty \text{ sections of } p \}$ is a $C^\infty(B, K)$ -module for
 $\sigma + \sigma': x \mapsto \sigma(x) + \sigma'(x)$, $f\sigma: x \mapsto f(x)\sigma(x)$. pointwise operations
 $* \text{rank } n \text{ vector bundle} \Rightarrow \text{fiber bundle with fibers } K^n$

Ex: (1) $\forall M$ C^∞ manifold, $\forall V$ K -vect space dim n data $p_{\perp}: M \times V \rightarrow M$ and $\forall x \in M$,
 obvious K -vect space structure on $\{x\} \times V$ is a rank n vector bundle,
 called trivial vector bundle. A vector bundle is trivialisable if isomorphic
 to trivial vector bundle.

NB: ξ trivialisable $\Leftrightarrow \exists \sigma_1, \dots, \sigma_n: B \rightarrow E$ C^∞ sections global over
 st $\forall b \in B$, $(\sigma_1(b), \dots, \sigma_n(b))$ K -basis of ξ_b
 for line bundle, $\Rightarrow \exists$ nonvanishing section $\Rightarrow$ lin indep $\Rightarrow$ generates
 ($\neq$ nonzero)

Ex 2: Let $M \subset \mathbb{R}^n$ manifold with $\dim n$. Its tangent bundle $\pi: TM \rightarrow M + \mathbb{R}$ -vector space structure on its fibres, is a (real) vector bundle of rank n . If $f: M \rightarrow N \subset \mathbb{R}^n$ then (Tf, f) is a vector bundle morphism from π_M to π_N and $\Gamma(TM) = C^\infty(M, \mathbb{R})$ -module of vector fields on M . *that has already been denoted*

NB: If (U, φ) local chart of M , then $(TU, T\varphi)$ local chart of TM and $TU \rightarrow U \times \mathbb{R}^n$ local trivialisation of vector bundle π_M .
 $v \mapsto (\pi_M^{-1}(u), \varphi_*^{-1}(v))$

Def: M is parallelisable if vector bundle TM is trivialisable.
every manifold is locally parallelisable, not necessarily globally.

vector (can get the metric hermitian)

Ex 3: If ξ complex vector bundle, its conjugated vector bundle $\bar{\xi}$ has same projection, fibres $\bar{\xi}_b =$ conjugated vector space of ξ_b .

Ex 4: V $\mathbb{K}$ vector space of $\dim n$, $\forall k \in [0, n]$,
 $\pi: \mathcal{G}_k(V) = \{(v, u) \in \mathcal{G}_k(V) \times V \mid v \in \mathcal{G}_k(V)\} \rightarrow \mathcal{G}_k(V)$ with obvious $\mathbb{K}$ -linear structure on fibres.
 $(v, u) \mapsto u$
 is a rank k vector bundle over $\mathcal{G}_k(V)$ called tangentological vector bundle above $\mathcal{G}_k(V)$.

Exer: Prove that $T\mathbb{S}^1$ is trivialisable *tangent bundle of the circle*
 Prove that $T_1(\mathbb{R}^2)$ is not trivialisable *open cylinder*
open Möbius band

Ex 5: Every Lie group G is parallelisable (including $\mathbb{S}^1$) ($\mathbb{S}^n$ parallelisable $\Leftrightarrow n=1, 3, 7$)
Pf: $(X_i)_{1 \leq i \leq n}$ basis of $\text{Lie}(G) \Rightarrow (g_i: g \mapsto T_g X_i)_{1 \leq i \leq n}$ are C^∞ sections of TG lin independent at each point. $\square$
 In what follows, all vector bundles are over $\mathbb{K}, C^\infty$, over same base B , and morphisms are over id .

2.2 Subbundles and quotient bundles

Let ξ vector bundle, projection $p: E \rightarrow B$.

Def: A vector subbundle of ξ is $E' \subset E$ st
 $\forall b \in B, \exists \varphi: \varphi^{-1}(U) \rightarrow U \times F$ local trivialisation of ξ above U ,
 open neigh of $b, \exists F'$ linear subspace of F st $\varphi(E' \cap \varphi^{-1}(U)) = U \times F'$.
NB: $\forall b \in B, p^{-1}(b) \cap E'$ linear subspace of ξ_b , E' closed C^∞ submanifold of E ,
 and $p' = p|_{E'}: E' \rightarrow B$ linear subspace structure on fibres of p' is a vector bundle,
 and inclusion $E' \hookrightarrow E$ is vector bundle morphism.

Ex: $\mathcal{G}_k(V)$ subbundle of trivial bundle $\mathcal{G}_k(V) \times V$
 $E' = \bigcup_{b \in B} \xi_b / \xi'_b \xrightarrow{p'} B$ quotient

Exercice: (1) Let ξ' vector subbundle of ξ . Prove $E' = \bigcup_{b \in B} \xi'_b$ has unique vector bundle structure st can proj $E \rightarrow E'$ is vector bundle morphism with base B for ξ' .
 (2) Let ϕ vector bundle morphism from ξ to ξ' and $k \in \mathbb{N}$ st $\forall b \in B, \phi|_{\xi_b}: \xi_b \rightarrow \xi'_b$ has rank k . Prove that $\ker \phi = \bigcup_{b \in B} \ker \phi|_{\xi_b}$ and $\text{im } \phi = \bigcup_{b \in B} \text{im } \phi|_{\xi_b}$ are vector subbundles and that $\text{im } \phi$ and $\xi / \ker \phi$ are isomorphic.

2.3 Reminder of linear algebra

Let K field. All linear spaces and multilinear maps are over K .
 Let $E, E_1, \dots, E_m, F, F_1, \dots, F_m$ finite dim lin spaces.

- * $\mathcal{L}(E_1, \dots, E_m; E) = \{ \mu: E_1 \times \dots \times E_m \rightarrow E \text{ m-linear} \}$ linear space
- * $E^* = \mathcal{L}(E; K)$ dual $E \subseteq E^{**}$ bidual $\forall \alpha \in \mathcal{L}(E, F) \quad \alpha: F^* \rightarrow E^*$ dual map
 $x \mapsto \{ \ell \mapsto \ell(x) \}$ linear isomorphism by which we identify them
 $\ell \mapsto \ell \circ \alpha$ of α
- * $E_1 \otimes \dots \otimes E_m = \mathcal{L}(E_1^*, \dots, E_m^*; K)$ tensor product of $E_1, \dots, E_m$. pure tensor
 - $\forall (x_i)_{1 \leq i \leq m} \in \prod_{i=1}^m E_i, \quad x_1 \otimes \dots \otimes x_m: E_1^* \times \dots \times E_m^* \rightarrow K$
 $(\ell_1, \dots, \ell_m) \mapsto \prod_{i=1}^m \ell_i(x_i)$
 - If $(e_i)_i$ basis of $E, (f_j)_j$ basis of F , then $(e_i \otimes f_j)_{i,j}$ basis of $E \otimes F$.
 - $\forall (\mu_i)_{1 \leq i \leq m} \in \prod_{i=1}^m \mathcal{L}(E_i; F_i), \quad \mu_1 \otimes \dots \otimes \mu_m: E_1 \otimes \dots \otimes E_m \rightarrow F_1 \otimes \dots \otimes F_m$
 $\text{linear} + x_1 \otimes \dots \otimes x_m \mapsto \mu_1(x_1) \otimes \dots \otimes \mu_m(x_m)$
 - $(E^*)^{\otimes s} \otimes E^{\otimes t} =$ (s, t) power tensor of $E \quad E^* \otimes E \subseteq \text{End}(E)$ linear
 $E \otimes \dots \otimes E$ s times $+ \ell \otimes x \mapsto \{ y \mapsto \ell(y)x \}$
 - K neutral element for $\otimes, \otimes$ commutative, associative by canonical isomorphisms.

* $\omega: E^m \rightarrow K$ m -linear is symmetric if $\forall (x_1, \dots, x_m) \in E^m, \forall \sigma \in \mathcal{I}_m$
 $\omega(x_{\sigma(1)}, \dots, x_{\sigma(m)}) = \omega(x_1, \dots, x_m)$
 alternate if $\forall (x_1, \dots, x_m) \in E^m$ if $\exists i \neq j \quad x_i = x_j$, then $\omega(x_1, \dots, x_m) = 0$.

m th exterior power of E $\Lambda^m E^* = \{ \text{alternate } m\text{-linear forms on } E \}$
 m th symmetric power of E $S^m E^* = \{ \text{symmetric } m\text{-linear forms on } E \}$
 $\Lambda^m E^* \subseteq (E^*)^{\otimes m}$ linear subspaces
 $\forall f \in \mathcal{L}(E, F), \quad f^*: \Lambda^m F^* \rightarrow \Lambda^m E^*$ linear
 $\omega \mapsto f^* \omega: (x_1, \dots, x_m) \mapsto \omega(f(x_1), \dots, f(x_m))$

The natural operations on linear spaces may be extended fiberwise to vector bundles.

2.4 Fiberwise operations on vector bundles

Let ξ, ξ', ξ_i for $1 \leq i \leq m$ vector bundles, with projections $p: E \rightarrow B, p': E' \rightarrow B, p_i: E_i \rightarrow B$

Def: Let $\pi: E \oplus E' \rightarrow B$ $\pi|_B = \text{id}_B$. If $\varphi: \pi^{-1}(U) \rightarrow U \times F$ and $\psi: (\pi')^{-1}(U) \rightarrow U \times F'$ are local trivialisations of ξ, ξ' above U , let $\Psi: \pi^{-1}(U) \rightarrow U \times (F \oplus F')$
 $x \oplus y \in \xi_b \oplus \xi'_b \mapsto (b, p_2 \circ \varphi|_{\xi_b}(x) \oplus p_2 \circ \psi|_{\xi'_b}(y))$ admitted
 direct sum linear space structure on its fibres, has unique vector bundle structure st these Ψ are local trivialisations, denoted $\xi \oplus \xi'$ direct sum vector bundle.

NB: * If $\sigma: U \rightarrow E$ and $\sigma': U \rightarrow E'$ are C^∞ sections of ξ, ξ' then $\sigma \oplus \sigma'$:
 $x \mapsto \sigma(x) \oplus \sigma'(x)$ is C^∞ section of $\xi \oplus \xi'$. characterizing property
 * $\text{rank}(\xi \oplus \xi') = \text{rank } \xi + \text{rank } \xi'$.

Def: Let $\pi: \mathcal{L}(E_1, \dots, E_m; E) = \bigsqcup_{b \in B} \mathcal{L}((\xi_1)_b, \dots, (\xi_m)_b; \xi_b) \rightarrow B$
 with linear space structure on its fibers is endowed as above
 with natural vector bundle structure, denoted $\mathcal{L}(\xi_1, \dots, \xi_m; \xi)$,
 of rank $\text{rank } \xi \times \prod_{i=1}^m \text{rank } \xi_i$.

Ex: Let $\xi_K = \text{trivial line bundle } B \times K \rightarrow B$. same rank, fibers = dual of fibers

(1) dual vector bundle of ξ : $\xi^* = \mathcal{L}(\xi; \xi_K)$ cotangent bundle of M : $(TM)^*$ denoted T^*M

(2) tensor product vector bundle of $\xi_1, \dots, \xi_m$: $\xi_1 \otimes \dots \otimes \xi_m = \mathcal{L}(\xi_1^*, \dots, \xi_m^*; \xi_K)$
 rank = product of rank, fibers = tensor product of fibers

NB: If $\sigma_i: U \rightarrow E_i$ C^∞ section of ξ_i , then $\sigma_1 \otimes \dots \otimes \sigma_m: x \mapsto \sigma_1(x) \otimes \dots \otimes \sigma_m(x)$
 C^∞ section of $\xi_1 \otimes \dots \otimes \xi_m$, and $\Gamma(\xi_1 \otimes \dots \otimes \xi_m) = \{ \text{locally finite } C^\infty(M) \text{-linear combinations of such } \sigma_1 \otimes \dots \otimes \sigma_m \}$ If M compact, forget if locally C^∞ property

(3) (s,t) power tensor vector bundle of ξ : $(\xi^*)^{\otimes s} \otimes \xi^{\otimes t}$

NB: If M C^∞ manifold, $T_s^t M = (T^*M)^{\otimes s} \otimes TM^{\otimes t}$ eg $T_0^1 M = TM$,
 $T_1^0 M = T^*M$, $T_2^0 M = \otimes^2 T^*M = \mathcal{L}(TM, TM; M \times K)$ vector bundle of bilinear forms on tangent spaces of M .
 $\Gamma(T_s^t M) = \{ s \text{ times covariant, } t \text{ times contravariant tensors on } M \}$

Def: m-th exterior power vector bundle $\Lambda^m \xi^*$ of $\xi =$ vector subbundle of $\otimes^m \xi^*$ whose fibers over $b \in B$ are $\Lambda^m (\xi_b^*)$ alternating m-linear forms on ξ_b

NB: $\Lambda^* \xi^* = \bigoplus_{m \in \mathbb{N}} \Lambda^m \xi^*$, $\Lambda^m T^*M =$ vector bundle of differential m-forms on M .
 $\Gamma(\Lambda^m T^*M \otimes \xi) = \{ \text{differential m-forms with values in } \xi \}$
 $\uparrow$ fibers over $x \in M =$ alternate m-linear map $(T_x M)^m \rightarrow \xi_x$
 a real vector bundle $C \otimes \xi^* = \mathcal{L}(\xi, \xi; \xi_K)$

Def: A Euclidean metric on ξ is $g \in \Gamma(S^2 \xi^*)$ st $\forall b \in B$
 $\langle \cdot, \cdot \rangle_b = g_b = g(b): \xi_b \times \xi_b \rightarrow \mathbb{R}$ Euclidean scalar product: it's the data of a Euclidean scalar product on each fiber ξ_b of ξ depending C^∞ in $b \in B$.

Prop: Every real vector bundle has a Euclidean metric (at least one! actually ∞ many)
Dem: OK: if ξ trivial, hence trivialisable, hence locally OK,
 one patches using partition of unity $\leftarrow$ main tool to pass from local to global $\square$

Ex: If ξ real vector bundle and $\langle \cdot, \cdot \rangle$ Euclidean metric, prove that $\xi \rightarrow \xi^*$ vector bundle isomorphism.

2.5 Pull-back vector bundle (or inverse image?) $f^* \xi$ $\leftarrow$ be a C^∞ manifold

Def: The pull-back of ξ by f is the vector bundle $f^* \xi$ over M with projection $\tilde{p}: f^* \xi \rightarrow M$ $f^* \xi = \{ (x, y) \in M \times E \mid f(x) = p(y) \} = \bigsqcup_{x \in M} \xi_{f(x)} \rightarrow M$,
 where fibers $\xi_{f(x)}$ over x is endowed with its linear space structure.

If M submanifold of B and $f: M \hookrightarrow B$ inclusion, then $f^*\xi$ is called restriction of ξ to M

Pf: **Step 1** $f^*\xi$ is C^∞ submanifold of product manifold $M \times F$

Transversality theorem: Let M', N C^∞ manifolds, $S \subset N$ C^∞ submanifold ^{not necessarily direct}

$\phi: M' \rightarrow N$ C^∞ transverse to $S: \forall u \in M', \phi(u) \in S \Rightarrow T_{\phi(u)} N = T_{\phi(u)} S + T_u \phi(T_u M')$

Then $\phi^{-1}(S)$ C^∞ submanifold of M' . $\square$

Its dim is $\dim M' + \dim S - \dim N$. Write S locally as preimage of a point by a submersion, transversality to S implies precomposition by ϕ stays a submersion.

Let $M' = M \times E, N = B \times B, S = \text{diagonal of } N, \phi: (x, y) \mapsto (f(x), p(y))$ so that $\phi^{-1}(S) = f^*E$. ϕ is transverse to S as if $\phi(x, y) \in S (\Leftrightarrow f(x) = p(y))$

then $\forall (x, y) \in T_{\phi(x, y)}(B \times B) = T_{f(x)} B \times T_{p(y)} B$, and recall that $f(x) = p(y)$

$$(X, Y) = (X, X) + (0, Y - X) \quad \text{since } p \text{ submersion.}$$

$$T_{\phi(x, y)} \phi(T_{(x, y)} M') = T_x f(T_x M) \times T_y p(T_y E)$$

Step 2 If $\psi: p^{-1}(U) \rightarrow U \times F$ local trivialisation of ξ above U , then $\tilde{\psi}: p^{-1}(f^{-1}(U)) \rightarrow f^{-1}(U) \times F$ is " of $f^*\xi$ above $f^{-1}(U)$.

$$(x, y) \mapsto (x, p_2 \circ \psi(y))$$

NB: ξ trivialisable $\Rightarrow f^*\xi$ trivialisable.

Classification theorem of vector bundles

(admitted) Let ξ vector bundle over K with compact base B and rank k

(1) $\exists n \geq k, \exists f: B \rightarrow \text{Gr}_k(K^n) \subset \mathbb{R}^n$ st ξ isomorphic to $f^* \tau_k(K^n)$ where $\tau_k(K^n) \rightarrow \text{Gr}_k(K^n)$ tautological vector bundle.

(2) $\forall f, g: B \rightarrow \text{Gr}_k(K^n) \subset \mathbb{R}^n, f^* \tau_k(K^n)$ and $g^* \tau_k(K^n)$ are isomorphic $\Leftrightarrow \exists n' \geq n$ st f, g homotopic in $\text{Gr}_k(K^{n'})$. $\square$

NB: $\{ \text{vector bundles over } K \text{ with base } B, \text{ rank } k \} / \text{isom} \xrightarrow{\text{bij}} [B, \text{Gr}_k(K^\infty)] \xleftarrow{\text{inductive limit}} \text{set of homotopy classes}$

Coro: Every vector bundle with compact base is isomorphic to vector subbundle of trivial vector bundle. $\hookrightarrow$ starting point of K -theory $\square$

Pf: $\tau_k(K^n)$ is, OK by full-back.

Lemma for theorem $\forall \xi$ vector bundle over $(0, 1) \times B$, the vector bundles pull-backs of ξ by $b \mapsto (0, b)$ and $b \mapsto (1, b)$ are isomorphic.

Pf: $\forall t \in (0, 1)$, let ξ_t pull-back of ξ by $b \mapsto (t, b)$. By connectivity and compactness of $(0, 1)$, only need to prove that $\forall t_0 \in (0, 1)$, if t close enough to t_0 , then ξ_t is isomorphic to ξ_{t_0} . Use local trivialisation of ξ to locally pass from ξ_t to ξ_{t_0} , glue by partition of unity. $\square$

Coro: Every vector bundle ξ with contractible basis B is trivialisable

Pf: Let $b_0 \in B$ and $h: [0, 1] \times B \rightarrow B$ C^∞ st $h(0, b) = b, h(1, b) = b_0$ (by smoothing homotopy) Then pull-backs of $\xi = h^* \xi$ by $b \mapsto (0, b)$ and $b \mapsto (1, b)$ are isomorphic to ξ and to trivial vector bundle with fiber ξ_{b_0} . $\square$