

TD 01 : Introduction to dynamical systems

Toral endo- and automorphisms

(1) Recall that the determinant of a square matrix with entries in a commutative ring is a polynomial in its entries. If $A \in \text{GL}_n(\mathbb{Z})$, then $\det A \in \mathbb{Z}$ is an invertible element in $\mathbb{Z}$ (with inverse $\det(A^{-1})$), hence is equal to ± 1 . Conversely, if $A \in L_n$ satisfies that $|\det A| = 1$, then $\det A = \pm 1$ and A is invertible over the field $\mathbb{Q}$. Since the comatrix A' of A also has entries in $\mathbb{Z}$ and since $A^{-1} = \frac{1}{\det A} A'$, we have $A^{-1} \in L_n$.

(2) Since $A\mathbb{Z}^n \subset \mathbb{Z}^n$, the map $\pi \circ A$ induces by taking quotient a continuous map $f_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$ such that $\pi \circ A = f_A \circ \pi$. Since $f_A([x]) = \pi(Ax)$, such a map is unique. By uniqueness, we have $f_{I_n} = \text{id}_{\mathbb{T}^n}$ and $f_{AB} = f_A \circ f_B$ for all $A, B \in L_n$, hence if $A \in \text{GL}_n(\mathbb{Z})$, then f_A is bijective with inverse $f_{A^{-1}}$.

(3) If $[x] \in \text{Per}_p(f_A)$, since $\pi(A^p x) = f_A^p(\pi(x)) = \pi(x)$, we have $A^p x - x \in \mathbb{Z}^n$, as wanted.

(4) If 1 is an eigenvalue of A , let E_1 be its associated eigenspace (which has dimension at least 1, then $\pi(E_1)$ is an uncountable set such that for every $x \in E_1$, we have $f_A^p([x]) = \pi(A^p x) = \pi(x)$, hence $[x] \in \text{Per}_p(f_A)$, and $\text{Per}_p(f_A)$ is uncountable. If 1 is not an eigenvalue of A , then $I_n - A^p$ is invertible over $\mathbb{Q}$. Let $q = \det(I_n - A^p)$ so that $(I_n - A^p)^{-1}\mathbb{Z}^n \subset \frac{1}{q}\mathbb{Z}^n$. Since $\pi(\frac{1}{q}\mathbb{Z}^n) = \frac{1}{q}\mathbb{Z}^n \bmod \mathbb{Z}^n$ is finite, the set $\text{Per}_p(f_A)$ is finite.

(5) This has been seen in (4).

(6) The set $\frac{1}{q}\mathbb{Z}^n$ is invariant under A by linearity and since A has integral entries. Hence the set $\pi(\frac{1}{q}\mathbb{Z}^n)$ is invariant under f_A and, being finite, all its points have finite orbit.

(7) Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, its image $\pi(\mathbb{Q}^n)$ is dense in $\mathbb{T}^n$. Since every point of $\mathbb{Q}^n$ belongs to $\frac{1}{q}\mathbb{Z}^n$ for some $q \in \mathbb{Z} \setminus \{0\}$ by common reduction of denominators, the result follows from (6).

(8) Since A is diagonalisable over $\mathbb{R}$ with distinct eigenvalues $\lambda = \frac{3+\sqrt{5}}{2} > 1$ and $\mu = \frac{3-\sqrt{5}}{2} \in]0, 1[$, there are only two A -invariant lines, the corresponding eigenspaces, that we denote respectively by E^u and E^s . For every $x \in E^s$, we have $A^k x = \mu^k x$, which tends to 0 as $k \rightarrow +\infty$. For every $x \in E^u$, we have $A^k x = \lambda^k x$, which tends to 0 as $k \rightarrow -\infty$.

(9) We only prove the result in the stable case, the proof for the unstable one is similar. We have $f_A(W^s(0)) = f_A(\pi(E^s)) = \pi(A E^s) = \pi(E^s) = W^s(0)$, hence $W^s(0)$ is invariant by f_A .

If $x \in E^s$, since $f_A^k([x]) = \pi(A^k(x))$ by the commutative diagram in (2), since π is continuous, and by (8), we have $\lim_{k \rightarrow +\infty} f_A^k([x]) = \pi(0) = 0$. Conversely, if $\lim_{k \rightarrow +\infty} f_A^k([x]) = 0$, then since $f_A^k([x]) = \pi(A^k(x))$ and since π is a covering hence a local homeomorphism, there exists a neighborhood U of 0 in $\mathbb{R}^2$ such that for k large enough, $A^k(x)$ belongs to U and $\lim_{k \rightarrow +\infty} A^k(x) = 0$. Since $x = x^- + x^+$ with $x^- \in E^-$ and $x^+ \in E^+$, then since $\lim_{k \rightarrow +\infty} A^k x^+ = 0$, we have $\lim_{k \rightarrow +\infty} A^k x^- = 0$.

Since $A^k x^- = \lambda^k x^-$ and $\lim_{k \rightarrow +\infty} \lambda^k = +\infty$, the fact that $A^k x$ belongs to U implies that $x^- = 0$.

(10) We only prove that $W^s(0)$ is dense, the proof that $W^u(0)$ is dense is similar. Since π is a local homeomorphism, we only have to prove that $E^s + \mathbb{Z}^2$ is dense in $\mathbb{R}^2$. For every vertical line $D = \{a\} \times \mathbb{R}$, the intersection of D with $E^s + \mathbb{Z}^2$ is $\{a\} \times (\mathbb{Z} + \mu\mathbb{Z})$. Since μ is irrational, the subgroup $\mathbb{Z} + \mu\mathbb{Z}$ is dense in $\mathbb{R}$. Hence $E^s + \mathbb{Z}^2$ is dense in $\mathbb{T}^2$.

(11) If $\xi = W^s(0) \cap W^u(0)$ is a homoclinic point, then by (9), we have $\lim_{k \rightarrow +\infty} f_A^k(\xi) = 0$ and $\lim_{k \rightarrow -\infty} f_A^k(\xi) = 0$.

Topological dynamical systems

(1) • If $x \in X$ is periodic of period p , then $h(x)$ is periodic of period dividing p , since we have $\phi^p(h(x)) = h(\phi^p(x)) = h(x)$. Hence $h(\text{Per}(\phi)) \subset \text{Per}(\phi')$. Since h is surjective, continuous and conjugates ϕ to ϕ' , we therefore have $X' = h(X) = h(\overline{\text{Per}(\phi)}) \subset \overline{h(\text{Per}(\phi))} \subset \overline{\text{Per}(\phi')}$. Thus $\text{Per}(\phi')$ is dense in X' .

• Since h is continuous, for every $x \in X$, we have

$$\begin{aligned} h(\omega(x)) &= \bigcap_{n \in \mathbb{N}} h(\overline{\{\phi^k(x) : k \geq n\}}) \\ &\subset \bigcap_{n \in \mathbb{N}} \overline{\{h(\phi^k(x)) : k \geq n\}} = \bigcap_{n \in \mathbb{N}} \overline{\{\phi'^k(h(x)) : k \geq n\}} = \omega(h(x)). \end{aligned}$$

Hence if $x \in \omega(x)$, then $h(x) \in h(\omega(x)) \subset \omega(h(x))$, and $h(x)$ is positively recurrent.

• Assume that (X, ϕ) is topologically transitive. For all nonempty open subsets U and V in X' , since $h^{-1}(U)$ and $h^{-1}(V)$ are nonempty open subsets of X , there exists $n \in \mathbb{Z}$ such that the intersection $h^{-1}(U) \cap \phi^{-n}(h^{-1}(V))$ is non empty. Since the image of an intersection is contained in the intersection of the images, since $\phi^{-n}(h^{-1}(V)) = (h \circ \phi^n)^{-1}(V) = (\phi'^n \circ h)^{-1}(V) = h^{-1}(\phi'^{-n}(V))$, the intersection $U \cap \phi'^{-n}(V)$, which contains $h(h^{-1}(U) \cap \phi^{-n}(h^{-1}(V)))$, is hence nonempty. Therefore (X', ϕ') is topologically transitive.

• Assume that (X, ϕ) is topologically mixing. For all nonempty open subsets U and V in X' , since $h^{-1}(U)$ and $h^{-1}(V)$ are nonempty open subsets of X , there exists $n \in \mathbb{N}$ such that for every $k \geq n$, the intersection $h^{-1}(U) \cap \phi^{-k}(h^{-1}(V))$ is non empty. Since the image of an intersection is contained in the intersection of the images, since $\phi^{-k}(h^{-1}(V)) = (h \circ \phi^k)^{-1}(V) = (\phi'^k \circ h)^{-1}(V) = h^{-1}(\phi'^{-k}(V))$, the intersection $U \cap \phi'^{-k}(V)$, which contains $h(h^{-1}(U) \cap \phi^{-k}(h^{-1}(V)))$, is hence nonempty. Therefore (X', ϕ') is topologically mixing.

• Assume that (X, ϕ) is minimal and let $x' \in X'$. Since h is surjective, let $x \in X$ be such that $h(x) = x'$. Since $\mathcal{O}(x)$ is dense in X and since h is continuous and conjugates ϕ to ϕ' , we have $X' = h(X) = h(\overline{\mathcal{O}(x)}) \subset \overline{h(\mathcal{O}(x))} = \overline{\mathcal{O}(h(x))}$. Hence the orbit of $h(x)$ is dense in X' .

(2) Since the closure of any orbit is nonempty, closed and invariant, if the only nonempty closed invariant subset of X is X , then every orbit is dense. Conversely, assume that every orbit is dense. Let F be a nonempty closed invariant subset of X and let $x \in F$. Then since F is invariant and closed, we have $X = \overline{\mathcal{O}(x)} \subset \overline{F} = F$. Hence the only nonempty closed invariant subset of X is X .

(3) Assume that there exists a point x in X whose orbit is dense in X . Let U and V be two nonempty open subsets in X . Since $\mathcal{O}(x)$ is dense in X and since U and V are nonempty, there exists $n, m \in \mathbb{Z}$ such that $\phi^n(x) \in U$ and $\phi^m(x) \in V$. Hence the intersection $U \cap \phi^{-(m-n)}(V)$, which contains $\phi^n(x)$, is nonempty. Conversely, assume that (X, ϕ) is topologically transitive. Since X is a separable metric space, there exists a countable basis $(U_i)_{i \in \mathbb{N}}$ of open sets of X . Let $Y = \bigcup_{i \in \mathbb{N}} \bigcup_{n \in \mathbb{Z}} \phi^{-n}(U_i)$. Since (X, ϕ) is topologically transitive, the open set $\bigcup_{n \in \mathbb{Z}} \phi^{-n}(U_i)$ is dense. By the Baire property of the complete separable metric space X , the G_δ -set Y is dense in X , and in particular nonempty.
