

### TD 02 : Dynamical systems and invariant measures : The doubling map

Let  $I = [0, 1]$ , let  $\mathcal{B}$  be the Borel  $\sigma$ -algebra of  $I$  and let  $\lambda$  be the Lebesgue measure on  $I$ . We consider the (discontinuous) *doubling map*  $T$  on  $I$  defined by  $T : x \mapsto 2x \bmod 1$ . Recall that  $\lambda(T^{-1}A) = \lambda(A)$  for every  $A \in \mathcal{B}$  (i.e. that  $\lambda$  is  $T$ -invariant). Below we consider two ways of proving the strong mixing (whence ergodicity) property of  $T$ .

#### Composing $f$ with $T^n$

1. Let  $a, b \in I$  with  $a < b$ , let  $J = [a, b]$  and let  $f = \mathbf{1}_J$  be the characteristic function of  $J$ . Write  $f \circ T^n$  as a sum of characteristic functions.
2. Prove that for every  $g \in C^0(I)$ , we have

$$\lim_{n \rightarrow +\infty} \int_I f \circ T^n g \, d\lambda = \int_I f \, d\lambda \int_I g \, d\lambda.$$

(Hint: Reformulate the first integral as a suitable Riemann sum):

3. Prove that the above limit also holds, first for all  $f \in L^\infty(I, \lambda)$  and  $g \in C^0(I)$ , then for all  $f \in L^\infty(I, \lambda)$  and  $g \in L^1(I, \lambda)$ .
4. We denote by  $\delta_z$  the unit Dirac mass at any point  $z \in I$ . Fix  $a \in [0, 1]$  and for every  $n \in \mathbb{N}$ , let

$$f_n = \frac{1}{2^n} \sum_{\xi \in I : T^n(\xi) = a} \delta_\xi.$$

Prove that for every  $g \in C^0(I)$  we have  $\lim_{n \rightarrow +\infty} \int_I f_n g \, d\lambda = \int_I g \, d\lambda$ , but that this may fail if  $g \in L^1(I)$ . Why is this situation different from the previous one where we integrated against  $f \circ T^n$ ?

#### Composing by pre-images of $T$

5. Let  $I, \mathcal{B}, \lambda$  be as above, and consider the transfer operator  $L : L^1(I, \lambda) \rightarrow L^1(I, \lambda)$  defined, for all  $\phi \in L^1(I, \lambda)$  and  $y \in I$  by

$$L\phi(y) = \frac{1}{2} \left( \phi\left(\frac{y}{2}\right) + \phi\left(\frac{1+y}{2}\right) \right).$$

6. Prove that for  $\phi \in C^1(I)$ , we have  $\left\| L^n \phi(x) - \int_0^1 \phi(t) \, dt \right\|_\infty \leq \frac{1}{2^n} \|\phi'\|_\infty$ . (Hint: Use the mean value theorem).

7. Use the previous result to prove that for all  $f \in L^\infty(I, \lambda)$  and  $g \in C^1(I)$ , we have

$$\lim_{n \rightarrow +\infty} \int_I f \circ T^n g \, d\lambda = \lim_{n \rightarrow +\infty} \int_I f (L^n g) \, d\lambda = \int_I f \, d\lambda \int_I g \, d\lambda.$$

Prove that for all  $f \in L^\infty(I, \lambda)$  and  $g \in L^1(I, \lambda)$ , we have

$$\lim_{n \rightarrow +\infty} \int_I f \circ T^n g \, d\lambda = \int_I f \, d\lambda \int_I g \, d\lambda.$$