

TD 02 : Dynamical systems and invariant measures

The doubling map

(1) Assume that $a > 0$ or that $b < 1$. By induction, we have $T^{-n}(J) = \bigsqcup_{k=0}^{2^n-1} [\frac{a}{2^n}, \frac{b}{2^n}] + \frac{k}{2^n}$. Hence we have

$$\mathbf{1}_J \circ T^n = \mathbf{1}_{T^{-n}(J)} = \sum_{k=0}^{2^n-1} \mathbf{1}_{[\frac{a}{2^n}, \frac{b}{2^n}] + \frac{k}{2^n}}.$$

If $a = 0$ and $b = 1$, this equality holds almost everywhere (actually, we have $\mathbf{1}_J \circ T^n = \mathbf{1}_J$).

(2) Since g is continuous on the compact interval I , it is uniformly continuous. Hence for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and $k \in [0, 2^n - 1] \cap \mathbb{N}$, for every $t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]$, we have $|g(t) - g(\frac{k}{2^n})| \leq \epsilon$. Therefore

$$\begin{aligned} \left| \int_I f \circ T^n g \, d\lambda - \int_I f \, d\lambda \int_I g \, d\lambda \right| &= \left| \sum_{k=0}^{2^n-1} \int_{\frac{a}{2^n} + \frac{k}{2^n}}^{\frac{b}{2^n} + \frac{k}{2^n}} g(t) \, dt - (b-a) \int_I g \, d\lambda \right| \\ &= \left| \sum_{k=0}^{2^n-1} \int_{\frac{a}{2^n} + \frac{k}{2^n}}^{\frac{b}{2^n} + \frac{k}{2^n}} \left(g(t) - g\left(\frac{k}{2^n}\right) \right) dt + \sum_{k=0}^{2^n-1} \frac{b-a}{2^n} g\left(\frac{k}{2^n}\right) - (b-a) \int_I g \, d\lambda \right| \\ &\leq (b-a) \left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} g\left(\frac{k}{2^n}\right) - \int_I g \, d\lambda \right| + \epsilon \end{aligned}$$

By letting $n \rightarrow +\infty$ and since Riemann sums of continuous functions approximate their integrals, then by letting $\epsilon \rightarrow 0$, the result follows.

(3) For every $n \in \mathbb{N}$, the n -th correlation coefficient

$$(f, g) \mapsto c_n(f, g) = \int_I f \circ T^n g \, d\lambda - \int_I f \, d\lambda \int_I g \, d\lambda$$

from $L^\infty(I, \lambda) \times L^1(I, \lambda)$ to $\mathbb{C}$ is bilinear and continuous uniformly in n : for all $f \in L^\infty(I, \lambda)$ and $g \in L^1(I, \lambda)$, we have $|c_n(f, g)| \leq 2 \|f\|_\infty \|g\|_1$. Let $g \in C^0(I)$. By Assertion (2), if f is a characteristic function, we have $\lim_{n \rightarrow +\infty} c_n(f, g) = 0$. Hence by linearity on the left of the correlation coefficients, if f is a simple function (a finite linear combination of characteristic functions), we also have $\lim_{n \rightarrow +\infty} c_n(f, g) = 0$. Since the set of simple functions is dense in $L^\infty(I, \lambda)$ and by the continuity property on the left of the correlation coefficients, for every $f \in L^\infty(I, \lambda)$, we have $\lim_{n \rightarrow +\infty} c_n(f, g) = 0$. Since $C^0(I)$ is dense in $L^1(I, \lambda)$, by the continuity property on the right of the correlation coefficients, for all $f \in L^\infty(I, \lambda)$ and $g \in L^1(I, \lambda)$, we have $\lim_{n \rightarrow +\infty} c_n(f, g) = 0$.

(4) The convergence of $\int_I f_n g \, d\lambda$ towards $\int_I f g \, d\lambda$ proceeds as in Assertion (2) by the uniform continuity of g and an approximation of the integral of g by Riemann sums. It fails if $g \in L^1(I, \lambda)$ since

g could be taken to be 0 on the countable set $\bigcup_{n \in \mathbb{N}} T^{-1}(\{a\})$ while having nonzero integral. The situation is different, since f_n is almost everywhere 0.

(5) The map h from I to $\mathbb{C}$ defined by $h : y \mapsto \frac{1}{2}(\phi(\frac{y}{2}) + \phi(\frac{1+y}{2}))$ is measurable and by two changes of variable, we have

$$\begin{aligned} \|h\|_1 &= \int_I |h| \, d\lambda \leq \frac{1}{2} \left(\int_I \left| \phi\left(\frac{y}{2}\right) \right| \, d\lambda(y) + \int_I \left| \phi\left(\frac{1+y}{2}\right) \right| \, d\lambda(y) \right) \\ &= \int_0^{\frac{1}{2}} |\phi(y')| \, d\lambda(y') + \int_{\frac{1}{2}}^1 |\phi(y'')| \, d\lambda(y'') = \|\phi\|_1. \end{aligned}$$

Hence $L\phi$ is well defined and belongs to $L^1(I, \lambda)$. Note that L is linear.

(6) By induction, for all $n \in \mathbb{N}$ and $\phi \in L^1(I, \lambda)$, we have $L^n \phi : y \mapsto \frac{1}{2^n} \sum_{k=0}^{2^n-1} \phi(\frac{y+k}{2^n})$. For all $n \in \mathbb{N}$, $k \in [0, 2^n - 1] \cap \mathbb{N}$, $y \in I$ and $t \in [\frac{k}{2^n}, \frac{k+1}{2^n}]$, by the mean value theorem, we have $|\phi(t) - \phi(\frac{y+k}{2^n})| \leq |t - \frac{y+k}{2^n}| \|\phi'\|_\infty \leq \frac{1}{2^n} \|\phi'\|_\infty$. Hence for every $y \in I$, we have

$$\begin{aligned} \left| L^n \phi(y) - \int_0^1 \phi(t) \, dt \right| &= \left| \frac{1}{2^n} \sum_{k=0}^{2^n-1} \phi\left(\frac{y+k}{2^n}\right) - \sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \phi(t) \, dt \right| \\ &= \left| \sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \left(\phi\left(\frac{y+k}{2^n}\right) - \phi(t) \right) \, dt \right| \leq \sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \left| \phi\left(\frac{y+k}{2^n}\right) - \phi(t) \right| \, dt \\ &\leq \sum_{k=0}^{2^n-1} \int_{\frac{k}{2^n}}^{\frac{k+1}{2^n}} \frac{1}{2^n} \|\phi'\|_\infty \, dt = \frac{1}{2^n} \|\phi'\|_\infty. \end{aligned}$$

(7) By two changes of variables, for all $f \in L^\infty(I, \lambda)$ and $g \in L^1(I, \lambda)$, we have

$$\begin{aligned} \int_I f(Lg) \, d\lambda &= \frac{1}{2} \left(\int_0^1 f(t) g\left(\frac{t}{2}\right) \, dt + \int_0^1 f(t) g\left(\frac{1+t}{2}\right) \, dt \right) \\ &= \int_0^{\frac{1}{2}} f(2t) g(t) \, dt + \int_{\frac{1}{2}}^1 f(2t-1) g(t) \, dt = \int_I (f \circ T) g \, d\lambda. \end{aligned}$$

Hence by induction, for every $n \in \mathbb{N}$, we have $\int_I f(L^n g) \, d\lambda = \int_I (f \circ T^n) g \, d\lambda$. Furthermore, by Assertion (6), we have

$$\begin{aligned} \left| \int_I f(L^n g) \, d\lambda - \int_I f \, d\lambda \int_I g \, d\lambda \right| &\leq \int_I |f| \left| L^n g - \int_I g \, d\lambda \right| \, d\lambda \\ &\leq \|f\|_\infty \left\| L^n g - \int_I g \, d\lambda \right\|_\infty, \end{aligned}$$

which tends to 0 as $n \rightarrow +\infty$. The last claim of Assertion (7) follows again by the density of $C^1(I)$ in $L^1(I, \lambda)$ and by the continuity property on the right of the correlation coefficients, as in Assertion (3).