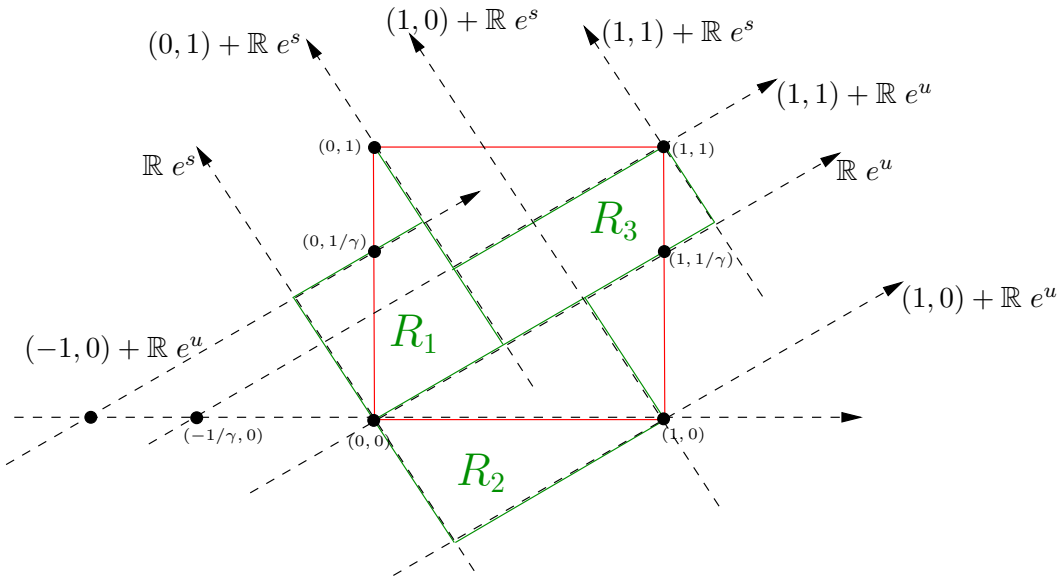
TD 3 : Hyperbolic maps, symbolic dynamics and zeta-functions.

(1) (a)




(1) (b) The linear map $A$ fixes $0$ and acts by a dilation of ratio $\gamma$ on the line $\mathbb{R}\ e^u$, and by the composition of the flip $x \mapsto -x$ with the contraction of ratio $\frac{1}{\gamma}$ on the line $\mathbb{R}\ e^s$. Hence (with nonempty intersection of the interiors), we have $f(R_1) \subset R_2$, $f(R_2) \subset R_1 \cup R_3$ and $f(R_3) \subset R_1 \cup R_2$. Therefore, we have

$$t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} .$$

The characteristic polynomial of $t$ is

$$\chi_t(X) = -X(X^2-1) - (-X-1) = (X+1)(1+X-X^2) .$$

Hence its eigenvalues are $-1$, $\gamma$ and $-\frac{1}{\gamma}$, that are real and pairwise distinct. For all $n \in \mathbb{N}$ and $\xi = (\xi_k)_{k\in\mathbb{Z}} \in \Sigma_t$, let $K_n = \bigcap_{-n\le k\le n} f^{-k}(R_{\xi_k})$. Since for every $i \in [\![1,3]\!]$, the linear automorphism $f$ contracts the sides of $R_i$ parallel to the image of $\mathbb{R}\ e^s$ in $\mathbb{T}^2$ and $f^{-1}$ contracts the sides of $R_i$ parallel to the image of $\mathbb{R}\ e^u$ in $\mathbb{T}^2$, the diameter of the nonincreasing sequence of compact subsets $(K_n)_{n\in\mathbb{N}}$ tends to $0$. Hence there exists one and only one $x \in \mathbb{T}^2$ such that $\bigcap_{k\in\mathbb{Z}} f^{-k}(R_{\xi_k}) = \{x\}$. Let us define $\pi(\xi) = x$.

The map $\pi$ is continuous, and semi-conjugates the subshift $\sigma_t$ to the linear automorphism $f$. The image of $\pi$ is compact, since $\pi$ is continuous, $\mathbb{T}^2$ is Hausdorff, and $\Sigma_t$, being closed in the compact product set $\Sigma_3$ by Tychonoff's theorem, is compact. The image of $\pi$ is dense by the itinery map: for almost every $x \in \mathbb{T}^2$ (not belonging to the full orbit of the union of the boundaries of $R_1$, $R_2$, $R_3$), there exists a unique sequence $(\xi_k)_{k\in\mathbb{Z}} \in \Sigma_3$ such that $f^k(z) \in \overset{\circ}{R}_{\xi_k}$ for every $k \in \mathbb{Z}$, and hence $(\xi_k)_{k\in\mathbb{Z}} \in \Sigma_t$. Therefore the image of $\pi$ is equal to $\mathbb{T}^2$.

(1) (c) The full orbit of the union of the boundaries of $R_1$, $R_2$, $R_3$ is contained in the union $L$ of the images in $\mathbb{T}^2$ of the eigenlines $\mathbb{R}\ e^u$ and $\mathbb{R}\ e^s$. As said above, the map $\pi$ is injective outside

this union. The only periodic point of $f$ contained in this union is $0 \in \mathbb{T}^2$. By TD01, for every $n \in \mathbb{N} \smallsetminus \{0\}$, we have

$$\mathrm{card}(\mathrm{Fix}({f_A}^n)) = |\det(I_2 - A^n)| = (\gamma^n - 1)\Big(1 - \Big(-\frac{1}{\gamma}\Big)^n\Big)\,.$$

Hence $\lim\limits_{n\to\infty} \dfrac{1}{n} \ln\big(\mathrm{card}(\mathrm{Fix}(f^n))\big) = \ln\gamma$.

(1) (d) For every $a \in \mathbb{C} \smallsetminus \{0\}$, since we have $\frac{1}{1-w} = \sum_{n=1}^{+\infty} w^{n-1}$ for every $w \in \mathbb{C}$ such that $|w| < 1$, the power series $\sum_{n=1}^{\infty} \frac{a^n}{n}\, z^n$ has radius of convergence $\frac{1}{|a|}$ and for every $z \in \mathbb{C}$ such that $|z| < \frac{1}{|a|}$, we have $\sum_{n=1}^{\infty} \frac{a^n}{n}\, z^n = -\ln(1 - az)$. As seen in the lectures and by Assertion (1) (b), for every $n \in \mathbb{N} \smallsetminus \{0\}$, we have $\mathrm{card}(\mathrm{Fix}({\sigma_t}^n)) = \mathrm{trace}(t^n) = \gamma^n + \big(-\frac{1}{\gamma}\big)^n + (-1)^n$. Therefore for every $z \in \mathbb{C}$ such that $|z| < \frac{1}{\gamma}$, we have

$$\begin{aligned}
\zeta_{\sigma_t}(z) &= \exp\Big(\sum_{n=1}^{\infty} \frac{\gamma^n}{n}\, z^n + \sum_{n=1}^{\infty} \frac{(-\frac{1}{\gamma})^n}{n}\, z^n + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\, z^n\Big) \\
&= \exp\Big(-\ln(1-\gamma z) - \ln\Big(1 + \frac{1}{\gamma} z\Big) - \ln(1+z)\Big) \\
&= \frac{\gamma}{(1-\gamma z)(\gamma + z)(1+z)}\,.
\end{aligned}$$

The fixed point $0$ of $f$ is a quadruple point on the boundary of the rectangles $R_1$, $R_2$, $R_3$ (it belongs once to $R_1$, once to $R_3$ and twice to $R_3$). Its inverse image by the coding $\pi$ is the union $\{x_0, \sigma_t(x_0), x_0', \sigma_t(x_0')\}$ of the two periodic orbits of period $2$ defined by $x_0 = (\xi_i)_{i\in\mathbb{Z}}$ with $\xi_i = 1$ if $i$ is odd and $\xi_i = 2$ if $i$ is even, and by $x_0' = (\xi_i')_{i\in\mathbb{Z}}$ with $\xi_i' = 3$ if $i$ is odd and $\xi_i' = 2$ if $i$ is even. Hence for every $n \in \mathbb{N}$, instead of contributing to $\mathrm{card}(\mathrm{Fix}(f^n))$ by $1$, the fixed point $0$ contributes to $\mathrm{card}(\mathrm{Fix}({\sigma_t}^n))$ by $4$ if $n$ is even, and by $0$ if $n$ is odd. Therefore for every $n \in \mathbb{N} \smallsetminus \{0\}$, we have

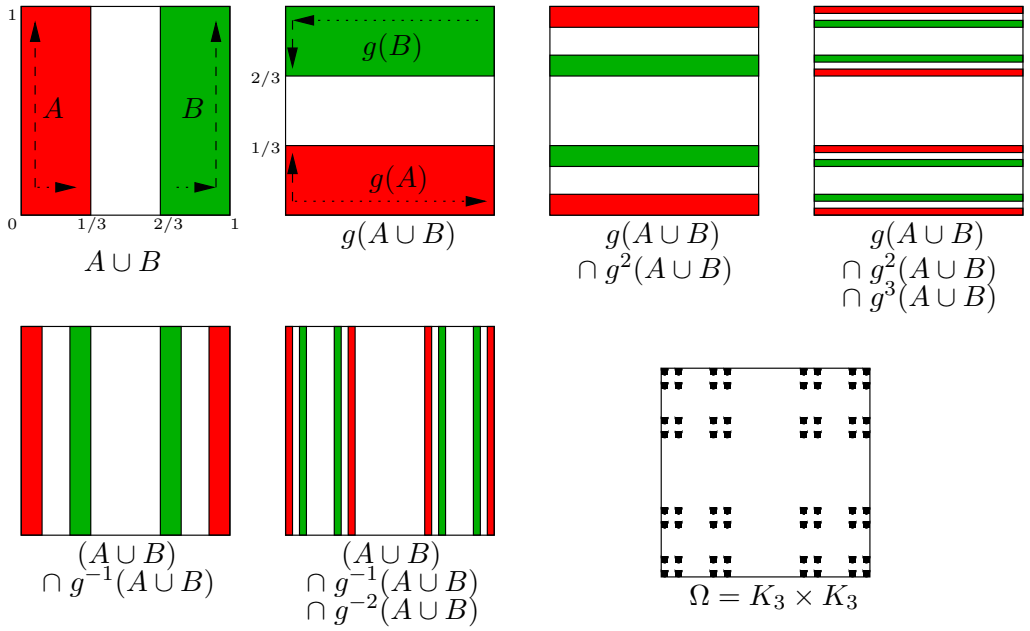
$$\mathrm{card}(\mathrm{Fix}(f^n)) - 1 = \mathrm{card}(\mathrm{Fix}({\sigma_t}^n)) - 2(-1)^n - 2\,.$$

Thus as above, we have

$$\begin{aligned}
\zeta_f(z) &= \exp\Big(\sum_{n=1}^{\infty} \frac{\gamma^n}{n}\, z^n + \sum_{n=1}^{\infty} \frac{(-\frac{1}{\gamma})^n}{n}\, z^n - \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\, z^n - \sum_{n=1}^{\infty} \frac{1}{n}\, z^n\Big) \\
&= \exp\Big(-\ln(1-\gamma z) - \ln\Big(1 + \frac{1}{\gamma} z\Big) + \ln(1+z) + \ln(1-z)\Big) \\
&= \frac{\gamma(1+z)(1-z)}{(1-\gamma z)(\gamma + z)}\,,
\end{aligned}$$

which is a rational function of $z$ with poles $-\gamma$ and $\frac{1}{\gamma}$.

(2)




Let $I_0 = [0, \frac{1}{3}]$, $I_1 = [\frac{2}{3}, 1]$, and $\Omega_1^+ = g(A \cup B) = [0,1] \times (I_0 \cup I_1)$. By induction, for every $n \in \mathbb{N} \smallsetminus \{0\}$, the set $\Omega_n^+ = \bigcap_{1 \leq k \leq n} g^k(A \cup B)$ is equal to $[0,1] \times \big( \bigcup_{(i_1,\dots,i_n) \in \{0,1\}^n} I_{i_1,\dots,i_n} \big)$ where the sets $I_{i_1,\dots,i_n}$ for $(i_1, \dots, i_n) \in \{0,1\}^n$ are pairwise disjoint closed intervals with common length $\frac{1}{3^n}$. The set $\Omega_{n+1}^+$ is obtained by replacing, for every $(i_1, \dots, i_n) \in \{0,1\}^n$, the interval $I_{i_1,\dots,i_n} = [a,b]$ by its first third $I_{i_1,\dots,i_n,0} = [a, a + \frac{b-a}{3}]$ and its last third $I_{i_1,\dots,i_n,1} = [b - \frac{b-a}{3}, b]$. By the standard construction of the triadic Cantor set $K_3 = \big\{ \sum_{i=1} \frac{a_i}{3^i} : (a_{i+1})_{i \in \mathbb{N}} \in \{0,2\}^{\mathbb{N}} \big\}$, we hence have

$$\Omega^+ = \bigcap_{1 \leq k \leq +\infty} g^k(A \cup B) = \bigcap_{n \in \mathbb{N} \smallsetminus \{0\}} \Omega_n^+ = [0,1] \times K_3 \,.$$

Similarly, let $J_0 = A$, $J_1 = B$ and $\Omega_0^- = J_0 \cup J_1$. For every $n \in \mathbb{N}$, the set $\Omega_n^- = \bigcap_{-n \leq k \leq 0} g^k(A \cup B)$ is equal to $\big( \bigcup_{(i_{-n},\dots,i_0) \in \{0,1\}^{n+1}} J_{i_{-n},\dots,i_0} \big) \times [0,1]$ where the sets $J_{i_{-n},\dots,i_0}$ for $(i_{-n}, \dots, i_0) \in \{0,1\}^{n+1}$ are pairwise disjoint closed intervals with common length $\frac{1}{3^{n+1}}$. We have

$$\Omega^- = \bigcap_{-\infty \leq k \leq 0} g^k(A \cup B) = \bigcap_{n \in \mathbb{N}} \Omega_n^- = K_3 \times [0,1] \,.$$

Therefore we have $\Omega = \bigcap_{k \in \mathbb{Z}} g^k(A \cup B) = \Omega^- \cap \Omega^+ = K_3 \times K_3$, which is a Cantor subset of $[0,1]^2$.

(2) (a) We have $R_A = (K_3 \cap [0, \frac{1}{3}]) \times K_3$ and for every $(x,y) \in R_A$, the horizontal segment $[0, \frac{1}{3}] \times \{y\}$, on which $g$ dilates by the factor 3, is a local unstable leaf of $g$ through $(x,y)$, and the vertical segment $\{x\} \times [0,1]$, on which $g$ contracts by the factor $\frac{1}{3}$, is a local stable leaf of $g$ through $(x,y)$. The result is similar for $R_B$.

(2) (b) For all $x, x' \in \Omega$ and $k \in \mathbb{Z}$, we have $g^k(x) \in A \cup B = R_A \cup R_B$ by the invariance of $\Omega$ under $g$, and the union $R_A \cap R_B$ is disjoint. Let $j(x) = (x_k)_{k \in \mathbb{Z}} \in \Sigma_2 = \{A, B\}^{\mathbb{Z}}$ be the *itinary* of $x$ in the Markov partition $R_A \cup R_B$ of $\Omega$, where $x_k = A$ if $g^k(x) \in R_A$ and $x_k = B$ if $g^k(x) \in R_B$. Since $g^{k+1} = g^k \circ g$, if $y = g(x)$, then $y_k = x_{k+1}$. Therefore $j \circ g = \sigma \circ j$. We endow the set $\{A, B\}$ with the discrete topology and $\Sigma_2 = \{A, B\}^{\mathbb{Z}}$ with the

product topology. The map $j$ is continuous, since for every $n \in \mathbb{N} \smallsetminus \{0\}$, the diameter of each connected component of $\Omega_n^- \cap \Omega_n^+$ is at most $\frac{2}{3^n}$, hence tends to $0$ as $n \to +\infty$, and if $x$ and $x'$ belong to the same connected component, then we have $x_k = x'_k$ for all $k \in [\![-n+1, n-1]\!]$. The map $j$ is bijective since the intersection of a nonincreasing sequence of compact subsets of $[0,1]^2$ whose diameters converge to $0$ contains one and only one point. The continuous bijective map from the compact space $\Omega$ (a Cantor set, as said above) to the compact space $\Sigma_2$ (by Tychonoff's theorem) is an homeomorphism. Hence the two topological dynamical systems $(\Omega, g)$ and $(\Sigma_2, \sigma)$ are topologically conjugated.

(2) (c) As already seen in Assertion (1) (d), for every $z \in \mathbb{C}$ such that $|z| < \frac{1}{2}$, we have $\sum_{n=1}^{\infty} \frac{2^n}{n} z^n = -\ln(1-2z)$. For every $n \in \mathbb{N} \smallsetminus \{0\}$, a sequence $(x_k)_{k\in\mathbb{Z}} \in \Sigma_2$ is fixed by $\sigma^n$ if and only if $x_{i+n} = x_i$ for every $i \in \mathbb{Z}$. Hence $\mathrm{card}(\mathrm{Fix}(\sigma^n)) = 2^n$. Since the Artin-Mazur zeta function is invariant by topological conjugacy and by Assertion (2) (b), for every $z \in \mathbb{C}$ such that $|z| < \frac{1}{2}$, we have

$$\zeta_g(z) = \zeta_\sigma(z) = \exp\Big(\sum_{n=1}^{\infty} \frac{2^n}{n}\, z^n\Big) = \exp(-\ln(1-2z)) = \frac{1}{1-2z}\,.$$