TD 04 : Cone contraction and spectral gap

The cross ratio

Let x, y, u, v be pairwise noncollinear elements of a real vector plane V. Let z, w be noncollinear elements of V, so that $\ell = \{\xi(t) = tw + z : t \in \mathbb{R}\}$ is an affine line in V not containing the origin. Assume that ℓ cuts the vector lines $\mathbb{R}x, \mathbb{R}y, \mathbb{R}u, \mathbb{R}v$ in the four points $\xi(t_1)$, $\xi(t_2)$, $\xi(t_3)$ and $\xi(t_4)$ respectively. Compute the cross-ratio $[x, y\,; u, v]$ (as defined in the lectures) in terms of t_1, t_2, t_3, t_4.

Positive matrices

1. Let $\mathbb{R}_+ = [0, +\infty[$ and $n \in \mathbb{N} \smallsetminus \{0\}$. For every $a \in [0, 1]$, let

$$K_a = \{x = (x_1, \dots, x_n) \in (\mathbb{R}_+)^n : \forall\, i, j \in [\![1, n]\!],\ x_i \geq a\, x_j\}\,.$$

Let $M = (m_{ij})_{1 \leq i,j \leq n}$ be a strictly positive matrix.[1] We order the complex eigenvalues $\lambda_1, \dots, \lambda_n$ of M so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

   (a) Assume that $a > 0$. Recall that d_{K_0} denotes the Hilbert projective metric on the pointed cone $K_0^* = K_0 \smallsetminus \{0\}$. Give a (finite) upper bound for $\mathrm{diam}_{K_0} K_a = \sup\limits_{x,y \in K_a^*} d_{K_0}(x, y)$.

   (b) Assume that $a < 1$. Prove that the cone K_a is outer and inner regular.

   (c) Prove that we have $\dfrac{|\lambda_2|}{\lambda_1} \leq \max\limits_{1 \leq i,j,k \leq n} \dfrac{m_{jk} - m_{ik}}{m_{jk} + m_{ik}}$. Prove that this inequality is an equality if $M = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$.

2. **The Collatz-Wielandt formula.** Let $n \in \mathbb{N} \smallsetminus \{0\}$ and let $M \in \mathcal{M}_n(\mathbb{R})$ be a strictly positive matrix. For every $x \in \mathbb{R}^n$, we denote by $x_1, \dots, x_n$ the real numbers such that $x = (x_1, \dots, x_n)$.

   We define a partial order $\preceq$ on $\mathbb{R}^n$ such that for every $x, y \in \mathbb{R}^n$, we have $x \preceq y$ if and only if $y - x \in (\mathbb{R}_+)^n$.

   (a) Prove that M preserves the partial order on $(\mathbb{R}_+)^n$.

   (b) Prove that if $x \in (\mathbb{R}_+)^n \smallsetminus \{0\}$ and $r > 0$ are such that $Mx \preceq rx$ (respectively $Mx \succeq rx$), then we have $\lambda_1 \leq r$ (respectively $\lambda_1 \geq r$).

   (c) Prove that $\lambda_1 = \inf\limits_{x \in (\mathbb{R}_+^*)^n} \max\limits_{1 \leq i \leq n} \dfrac{(Mx)_i}{x_i} = \sup\limits_{x \in K_0 \smallsetminus \{0\}} \min\limits_{1 \leq i \leq n\,:\, x_i \neq 0} \dfrac{(Mx)_i}{x_i}$.

   (d) Give an upper and lower bound on the eigenvalue with maximal absolute value of the matrix $A = \left(\begin{smallmatrix} 4 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$ using the above question (2c) and the vector $x = \left(\begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix}\right)$.

[1] Recall that a strictly positive matrix is a real square matrix all of whose entries are strictly positive.

A family of operators associated with the Gauss map

Let $I = [0,1]$, equipped with the distance $d : (x,x') \mapsto \left|\ln \frac{1+x}{1+x'}\right|$. For every $k \in \mathbb{N} \smallsetminus \{0\}$, let $I_k = \left[\frac{1}{k+1}, \frac{1}{k}\right]$ and let $\psi_k : I_k \to I$ be the C^∞-diffeomorphism defined by $y \mapsto \frac{1}{y+k}$. The sequence $(\psi_k)_{k\in\mathbb{N}\smallsetminus\{0\}}$ is the sequence of inverse branches of the *Gauss map* $\psi :]0,1] \to I$ defined by $x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$. Let X be the Banach space $\mathrm{Lip}(I,d)$ of Lipschitz selfmaps of (I,d). Let $s \in \ell^\infty_+(\mathbb{N} \smallsetminus \{0\}) = \left\{ s : \mathbb{N} \smallsetminus \{0\} \to [0,+\infty[\,:\, \|s\|_\infty = \sup_{k\in\mathbb{N}\smallsetminus\{0\}} |s_k| < +\infty \right\}$ be a family of nonnegative *weights*. We define a *transfer operator* $L_s : X \to X$ by

$$\forall\, \phi \in X,\ \forall\, y \in I, \quad L_s\phi(y) = \sum_{k=1}^{+\infty} s_k \, \frac{1}{(k+y)^2} \, \phi\Big(\frac{1}{k+y}\Big)\,.$$

For every $a > 0$, consider the cone $K_a = \left\{ \phi \in X : \forall\, x,x' \in I,\ 0 \le \phi(x) \le \phi(x')\, e^{a\, d(x,x')} \right\}$ and let $K_a^* = K_a \smallsetminus \{0\}$.

1. Prove that L_s is a bounded linear operator on X.

2. Suppose that $s \neq 0$. Prove that there exist $a \in]0,+\infty[$ and $\sigma \in]0,1[$ such that $L_s(K_a^*) \subset K_{\sigma a}^*$.

3. If $s \neq 0$, estimate the contraction rate of the operator L_s in the Hilbert metric of K_a and prove that L_s admits a spectral gap.

4. In the case when $s = 1$ (the constant function $k \mapsto 1$), prove that $h : x \mapsto \frac{1}{\log(2)} \frac{1}{1+x}$ is an eigenvector of L_1 in X.

5. Prove that the Gauss map ψ admits a unique invariant probability measure on I which is absolutely continous with respect to the Lebesque measure on I and that this measure is strongly mixing (whence ergodic) under ψ.