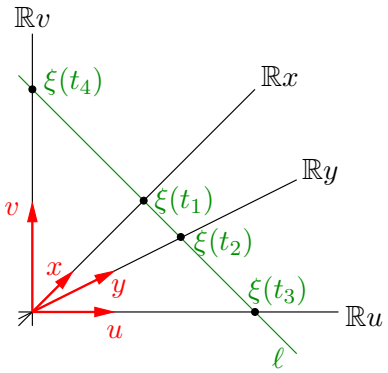
TD 4 : Cone contraction and spectral gap

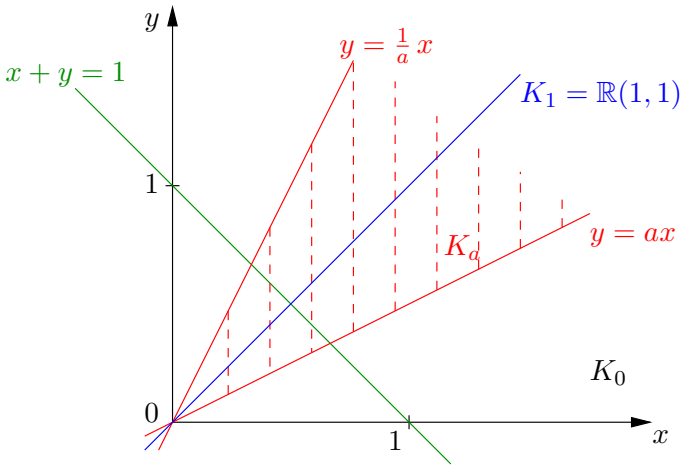
The crossratio



Since z, w are noncollinear, we have $\xi(t_1) \in \mathbb{R}x \smallsetminus \{0\}$, $\xi(t_2) \in \mathbb{R}y \smallsetminus \{0\}$, $\xi(t_3) \in \mathbb{R}u \smallsetminus \{0\}$ and $\xi(t_4) \in \mathbb{R}v \smallsetminus \{0\}$. Since the crossratio $[x, y\,; u, v]$ is projectively invariant, we may assume that $x = \xi(t_1)$, $y = \xi(t_2)$, $u = \xi(t_3)$ and $v = \xi(t_4)$. Since x, y, u, v are pairwise noncollinear, the points $\xi(t_1)$, $\xi(t_2)$, $\xi(t_3)$ and $\xi(t_4)$ are pairwise distinct. Hence t_1, t_2, t_3, t_4 are pairwise distinct since the maps $\xi : \mathbb{R} \to \ell$ is bijective. Since $(t_4 - t_3)w = \xi(t_4) - \xi(t_3) = v - u$, we have $w = \frac{v-u}{t_4-t_3}$. Since $u = t_3 w + z$, we have $z = u - t_3 \frac{v-u}{t_4-t_3} = \frac{t_4 u - t_3 v}{t_4-t_3}$. Hence if $a = \frac{t_4-t_1}{t_4-t_3}$, $b = \frac{t_1-t_3}{t_4-t_3}$, $c = \frac{t_4-t_2}{t_4-t_3}$ and $d = \frac{t_2-t_3}{t_4-t_3}$, we have $x = t_1 w + z = \frac{t_1 v - t_1 u + t_4 u - t_3 v}{t_4-t_3} = au + bv$ and $y = t_2 w + z = \frac{t_2 v - t_2 u + t_4 u - t_3 v}{t_4-t_3} = cu + dv$. Therefore, by the definition of the crossratio, we have

$$[x, y\,; u, v] = \frac{a\ d}{b\ c} = \frac{(t_4 - t_1)(t_3 - t_2)}{(t_4 - t_2)(t_3 - t_1)}\,.$$

Positive matrices.



(1) (a) Since $a > 0$, for all $x, y \in K_a^* \subset]0, +\infty[^n$, we have

$$\beta(x, y) = \inf\{t > 0 : tx - y \in K_0\} = \max_{1 \le i \le n} \frac{y_i}{x_i}\,.$$

Hence by the definition of the Hilbert distance d_{K_0} of K_0, we have

$$d_{K_0}(x,y) = \ln(\beta(x,y)\beta(y,x)) = \max_{1\le i,j\le n} \ln \frac{y_i}{x_i}\frac{x_j}{y_j} = \max_{1\le i,j\le n} \ln \frac{x_j}{x_i}\frac{y_i}{y_j} \le \ln \frac{1}{a^2} = -2\ln a\,.$$

(b) Note that $K_1 = \mathbb{R}(1,\dots,1)$ has empty interior. If $a < 1$, let $\epsilon = \frac{1-a}{1+a} > 0$, and note that $1-\epsilon = \frac{2a}{1+a} \ge 0$. Then for every $t, t' \in]1-\epsilon, 1+\epsilon[$, we have $t \ge 1-\epsilon = a(1+\epsilon) \ge a\,t'$, so that the nonempty open set $]1-\epsilon,1+\epsilon[^n$ is contained in K_a and K_a is inner regular.

For all $x = (x_1,\dots,x_n) \in K_a \subset K_0 = (\mathbb{R}_+)^n$ and $i \in [\![1,n]\!]$, we have $x_i \le \sum_{j=1}^n x_j$. Hence with $\ell : \mathbb{R}^n$ the (continuous) linear form $(x_1,\dots,x_n) \mapsto \sum_{j=1}^n x_j$, we have

$$\|x\|_\infty = \max_{1\le i\le n} |x_i| = \max_{1\le i\le n} x_i \le \sum_{j=1}^n x_j = \ell(x)\,,$$

so that K_a is outer regular.

(c) Note that the strictly positive matrix M preserves $K_0 = (\mathbb{R}_+)^n$. Let $a = \min_{1\le i,j,k\le n} \frac{m_{ik}}{m_{jk}}$, which is positive. Then for all $x = (x_1,\dots,x_n) \in K_0 = (\mathbb{R}_+)^n$ and $i,j \in [\![1,n]\!]$, we have the inequality $\sum_{k=1}^n m_{ik}\,x_k \ge a \sum_{k=1}^n m_{jk}\,x_k$. Therefore we have the inclusion $MK_0 \subset K_a$. Let us consider $\Delta = \operatorname{diam}_{K_0}(MK_0) \le \operatorname{diam}_{K_0}(K_a) \le -2\ln a$. By the Perron-Frobenius theorem, we know that the eigenvalue λ_1 is positive. By the spectral gap theorem, we know that we have $\frac{|\lambda_2|}{\lambda_1} \le \tanh\frac{\Delta}{4}$. Since $\tanh x = \frac{e^{2x}-1}{e^{2x}+1}$, we hence have

$$\frac{|\lambda_2|}{\lambda_1} \le \tanh\Big(-\frac{\ln a}{2}\Big) = \frac{1-a}{1+a} \le \max_{1\le i,j,k\le n} \frac{1-\frac{m_{ik}}{m_{jk}}}{1+\frac{m_{ik}}{m_{jk}}} = \max_{1\le i,j,k\le n} \frac{m_{jk}-m_{ik}}{m_{jk}+m_{ik}}\,.$$

The eigenvalues of the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ are $\lambda_1 = 3$ and $\lambda_2 = 1$. Hence $\frac{|\lambda_2|}{\lambda_1} = \frac{1}{3} = \frac{2-1}{2+1}$, and the equality holds in the above inequality.

(2) (a) For all $x = (x_1,\dots,x_n), y = (y_1,\dots,y_n) \in (\mathbb{R}_+)^n$, if $x_k \le y_k$ for every $k \in [\![1,n]\!]$, then for every $i \in [\![1,n]\!]$, we have

$$(Mx)_i = \sum_{k=1}^n m_{ik}\,x_k \le \sum_{k=1}^n m_{ik}\,y_k = (My)_i\,.$$

(b) By the Perron-Frobenius theorem, we know that there exists an eigenvector x of the strictly positive matrix M for the eigenvalue λ_1 all of whose coordinates are positive. Hence $x \in (\mathbb{R}_+)^n \smallsetminus \{0\}$ and if $Mx \preceq rx$ (respectively $Mx \succeq rx$), then $\lambda\,x \preceq rx$ (respectively $\lambda\,x \succeq rx$), so that $\lambda \le r$ (respectively $\lambda_1 \ge r$).

(c) Let $x \in (\mathbb{R}_+)^n \smallsetminus \{0\}$ and $\rho_- = \min_{1\le i\le n\,:\,x_i\ne 0} \frac{(Mx)_i}{x_i}$, so that $(Mx)_i \ge \rho_- x_i$ for actually every $i \in [\![1,n]\!]$. Let w be an eigenvector of ${}^tM$ for λ_1 all of whose coordinates are positive. For every $i \in [\![1,n]\!]$, by the matrix multiplication, we have

$$\lambda_1\,{}^twx = {}^t(\lambda_1 w)x = {}^t({}^tMw)x = {}^tw(Mx) \ge \rho_-\,{}^twx\,.$$

Since all components of w (respectively x) are positive (respectively non negative), we have ${}^t wx > 0$, hence $\lambda_1 \geq \rho_-$. By taking the upperbound, we hence have $\lambda_1 \geq \sup_{x \in K_0 \smallsetminus \{0\}} \min_{1 \leq i \leq n : x_i \neq 0} \frac{(Mx)_i}{x_i}$, and equality holds by taking for x an eigenvector of M for λ_1.

Similarly, let $x \in (\mathbb{R}_+^*)^n$ and $\rho_+ = \max_{1 \leq i \leq n} \frac{(Mx)_i}{x_i}$, so that for every $i \in [\![1, n]\!]$, we have the inequalities $(Mx)_i \leq \rho_+ x_i$ and

$$\lambda_1 \, {}^t wx = {}^t(\lambda_1 w)x = {}^t({}^t Mw)x = {}^t w(Mx) \leq \rho_+ \, {}^t wx \, .$$

Hence $\lambda_1 \leq \rho_+$ and by taking the lowerbound, we have $\lambda_1 \leq \inf_{x \in (\mathbb{R}_+^*)^n} \max_{1 \leq i \leq n} \frac{(Mx)_i}{x_i}$, and equality holds by taking for x an eigenvector of M for λ_1.

(d) The matrix A is not strictly positive (its square is), but the proofs of the previous questions still work. We have $Ax = \begin{pmatrix} 22 \\ 11 \\ 6 \end{pmatrix}$, hence we have by (c) that $\frac{11}{2} \leq \lambda_1 \leq 6$.

A family of operators associated with the Gauss map

Recall that for every $\phi \in X$, we denote $\|\phi\|'_{\mathrm{Lip},d} = \sup_{y, y' \in I \, : \, y \neq y'} \frac{d(\phi(y), \phi(y'))}{d(y, y')}$ and the Banach norm on X is $\|\phi\|_{\mathrm{Lip},d} = \|\phi\|_\infty + \|\phi\|'_{\mathrm{Lip},d}$.

(1) First note that the distance d induces the usual topology on I. For all $y \in I$ and $\phi \in X$, since I is compact and ϕ is continuous, the sum defining sum $L_s\phi(y)$ is absolutely convergent, and we have $|L_s\phi(y)| \leq \zeta(2)\|s\|_\infty \|\phi\|_\infty$, hence

$$\|L_s\phi\|_\infty \leq \zeta(2) \, \|s\|_\infty \, \|\phi\|_\infty \, .$$

In particular, $L_s : C^0(I, \mathbb{C}) \to C^0(I, \mathbb{C})$ is a bounded linear operator.

By comparing the map $u \mapsto |\ln(1+u)|$ (which sends 0 to 0) with linear maps, there exists a constant $c \geq 1$ such that for every $u \in \left[-\frac{1}{2}, 1\right]$, we have $\frac{1}{c}|u| \leq |\ln(1+u)| \leq c\,|u|$. Since the map $t \mapsto -\frac{t}{1+t}$ is nonincreasing on I, for all $x, x' \in [0,1]$, we have

$$-\frac{1}{2} \leq -\frac{x'}{1+x'} \leq \frac{x-x'}{1+x'} \leq \frac{x}{1+x'} \leq x \leq 1 \, .$$

Hence for all $x, x' \in [0,1]$, we have

$$\frac{1}{2c}|x-x'| \leq \frac{1}{c}\frac{|x-x'|}{1+x'} \leq \left|\ln\left(1+\frac{x-x'}{1+x'}\right)\right| = \left|\ln\frac{1+x}{1+x'}\right| \leq c\,\frac{|x-x'|}{1+x'} \leq c\,|x-x'| \, .$$

Therefore the distance d on I is equivalent to the usual distance d', the two distances have the same Lipschitz functions, and the two Lipschitz constants for d and for d' of any Lipschitz function differ by at most c^2:

$$\frac{1}{c^2} \, \| \ \|'_{\mathrm{Lip},\,d'} \leq \| \ \|'_{\mathrm{Lip},\,d} \leq c^2 \, \| \ \|'_{\mathrm{Lip},\,d'} \, .$$

For all $y, y' \in I$ and $\phi \in X$, we have

$$\begin{aligned}
\big|\, L_s\phi(y) - L_s\phi(y') \,\big| &\le \sum_{k=1}^{+\infty} \|s\|_\infty \ \Big|\, \frac{\phi\big(\frac{1}{k+y}\big)}{(k+y)^2} - \frac{\phi\big(\frac{1}{k+y'}\big)}{(k+y')^2} \Big| \\
&\le \sum_{k=1}^{+\infty} \|s\|_\infty \ \Big(\Big|\, \frac{\phi\big(\frac{1}{k+y}\big)}{(k+y)^2} - \frac{\phi\big(\frac{1}{k+y'}\big)}{(k+y)^2} \Big| + \Big|\, \frac{\phi\big(\frac{1}{k+y'}\big)}{(k+y)^2} - \frac{\phi\big(\frac{1}{k+y'}\big)}{(k+y')^2} \Big| \Big) \\
&\le \sum_{k=1}^{+\infty} \|s\|_\infty \Big(\frac{1}{k^2} \Big|\, \phi\Big(\frac{1}{k+y}\Big) - \phi\Big(\frac{1}{k+y'}\Big) \Big| + \|\phi\|_\infty \Big| \frac{1}{(k+y)^2} - \frac{1}{(k+y')^2} \Big| \Big) \\
&\le \sum_{k=1}^{+\infty} \|s\|_\infty \Big(\frac{1}{k^2}\ \|\phi\|'_{\mathrm{Lip},\,d'} \Big|\, \frac{1}{k+y} - \frac{1}{k+y'} \,\Big| + \|\phi\|_\infty\, \frac{2}{k} \Big|\, \frac{1}{k+y} - \frac{1}{k+y'} \,\Big| \Big) \\
&\le \|s\|_\infty \big(\zeta(4)\ \|\phi\|'_{\mathrm{Lip},\,d'} + 2\,\zeta(5)\ \|\phi\|_\infty \big)\ |\, y - y' \,|\,.
\end{aligned}$$

Hence $\|L_s\phi\|'_{\mathrm{Lip},\,d'} \le \|s\|_\infty \big(\zeta(4)\ \|\phi\|'_{\mathrm{Lip},\,d'} + 2\,\zeta(5)\ \|\phi\|_\infty \big)$, so that

$$\|L_s\phi\|'_{\mathrm{Lip},\,d} \le \|s\|_\infty \big(c^4 \zeta(4)\ \ \|\phi\|'_{\mathrm{Lip},\,d} + 2\,c^2\,\zeta(5)\ \|\phi\|_\infty \big)\,.$$

Therefore, we have $\|L_s\phi\|_{\mathrm{Lip},\,d} \le (\zeta(2) + \ c^4\,\zeta(4) + 2\,c^2\,\zeta(5))\ \|s\|_\infty\ \|\phi\|_{\mathrm{Lip},\,d}$, so that L_s is a bounded linear operator from X to X, with operator norm at most

$$\|L_s\| \le \big(\zeta(2) + \ c^4\,\zeta(4) + 2\,c^2\,\zeta(5) \big)\ \|s\|_\infty\,.$$

(2) For every $\phi \in X$, notice that $L_s\phi \ge 0$ if $\phi \ge 0$, since $s \ge 0$. Furthermore, for every $a > 0$, if $\phi \in K_a^*$, let $y_0 \in I$ be such that $\phi(y_0) \ne 0$. Then for every $y \in I$, we have $\phi(y) \ge \phi(y_0) e^{-a\,d(y,y_0)} > 0$. Hence if there exists $k \in \mathbb{N} \smallsetminus \{0\}$ such that $s_k > 0$, then we have $L_s\phi(y) \ge s_k\ \frac{1}{(k+y)^2}\ \phi\big(\frac{1}{k+y}\big) > 0$.

For every $k \in \mathbb{N} \smallsetminus \{0\}$, let $f_k, g : I \to \mathbb{R}$ be the smooth maps $y \mapsto \ln \dfrac{k+1+y}{k+y}$ and $y \mapsto \ln(1+y)$ respectively. For every $y \in I$, since $k \ge 1$, we have

$$\frac{|f_k'(y)|}{g'(y)} = \frac{1+y}{(k+1+y)(k+y)} \le \frac{1}{k+1+y} \le \frac{1}{2}\,.$$

Hence for all $y, y' \in I$, we have

$$\begin{aligned}
d\Big(\frac{1}{k+y}, \frac{1}{k+y'}\Big) = |\, f_k(y) - f_k(y') \,| &= \Big|\ \int_y^{y'} f_k'(t)\,dt \,\Big| \le \Big|\ \int_y^{y'} |f_k'(t)|\,dt\ \Big| \\
&\le \frac{1}{2} \Big|\ \int_y^{y'} g'(t)\,dt\ \Big| = \frac{1}{2} |\, g(y) - g(y') \,| = \frac{1}{2}\ d(y,y')\,.
\end{aligned}$$

Similarly, if $f_k : y \mapsto \ln(k+y)$ and $g : y \mapsto \ln(1+y)$, then $0 < f_k'(y) = \frac{1}{k+y} \le \frac{1}{1+y} = g'(y)$, hence

$$\begin{aligned}
\frac{\frac{1}{(k+y)^2}}{\frac{1}{(k+y')^2}} &= \exp\big(2\,\big(\ln(k+y') - \ln(k+y) \big) \big) \le \exp\Big(2 \Big|\ \int_y^{y'} f_k'(t)\,dt \,\Big| \Big) \\
&\le \exp\Big(2 \Big|\ \int_y^{y'} g'(t)\,dt \,\Big| \Big) = \exp\big(2\,\big|\ln(1+y') - \ln(1+y)\big|\ \big) = e^{2\,d(y,\,y')}\,.
\end{aligned}$$

Thus for every $a > 0$, we have $L_s(K_a) \subset K_{2+\frac{a}{2}}$. Taking $a = 5$ and $\sigma = \frac{9}{10}$ gives the result.

(3) Let us prove *Liverani's cone diameter formula* :

$$\Delta = \mathrm{diam}_{K_a}(K_{\sigma a}) \leq 2 \ln \frac{1+\sigma}{1-\sigma} + 2\,\sigma\, a\, \ln 2\,. \tag{0.1}$$

Let $\phi, \phi' \in K_{\sigma a}$ and $y, y' \in I$ with $y \neq y'$. Since $\sigma < 1$, we have

$$\phi'(y) \leq \phi'(y')\, e^{\sigma\, a\, d(y,\, y')} < \phi'(y')\, e^{a\, d(y,y')}\,.$$

For every $t > 0$, we have the inequalities $0 \leq t\,\phi'(y) - \phi(y) \leq (t\,\phi'(y') - \phi(y'))e^{a\, d(y,\, y')}$ if and only if we have the inequalities $t \geq \frac{\phi(y)}{\phi'(y)}$ and $t(\phi'(y')\, e^{a\, d(y,\, y')} - \phi'(y)) \geq \phi(y')\, e^{a\, d(y,\, y')} - \phi(y)$, that is, if and only if we have

$$t \geq \min\Big\{ \frac{\phi(y')\, e^{a\, d(y,\, y')} - \phi(y)}{\phi'(y')\, e^{a\, d(y,\, y')} - \phi'(y)} = \frac{\phi(y')}{\phi'(y')}\, \frac{e^{a\, d(y,\, y')} - \frac{\phi(y)}{\phi(y')}}{e^{a\, d(y,\, y')} - \frac{\phi'(y)}{\phi'(y')}},\quad \frac{\phi(y)}{\phi'(y)} \Big\}\,.$$

Since $\phi, \phi' \in K_{\sigma a}$, we have

$$\frac{e^{a\, d(y,\, y')} - \frac{\phi(y)}{\phi(y')}}{e^{a\, d(y,\, y')} - \frac{\phi'(y)}{\phi'(y')}} \leq \frac{e^{a\, d(y,\, y')} - e^{-\sigma\, a\, d(y,\, y')}}{e^{a\, d(y,\, y')} - e^{\sigma\, a\, d(y,\, y')}} = \frac{\phi(y')}{\phi'(y')}\, \frac{1 - e^{-(1+\sigma)\, a\, d(y,\, y')}}{1 - e^{-(1-\sigma)\, a\, d(y,\, y')}}\,.$$

Consider the function $f :]0, +\infty[\to \mathbb{R}$ defined by $f : s \mapsto \dfrac{1 - e^{-(1+\sigma)\, s}}{1 - e^{-(1-\sigma)\, s}}$. Its derivative $f'(s)$ is a fraction whose denominator is positive and whose numerator is, since $\sigma \in]0, 1[$,

$$\begin{aligned} g(s) &= (1+\sigma)\, e^{-(1+\sigma)\, s}(1 - e^{-(1-\sigma)\, s}) - (1-\sigma)\, e^{-(1-\sigma)\, s}(1 - e^{-(1+\sigma)\, s}) \\ &= e^{-2\, s}\big((1+\sigma)\, e^{(1-\sigma)\, s} - (1-\sigma)\, e^{(1+\sigma)\, s} - 2\,\sigma\big)\,. \end{aligned}$$

Let $h = e^{2\, s}\, g(s)$. We have $h(0) = 0$ and $h'(s) = (1-\sigma^2)\,(e^{(1-\sigma)\, s} - e^{(1+\sigma)\, s}) < 0$. Hence $g(s) < 0$, so that f is strictly decreasing on $]0, +\infty[$, so that it is bounded from above by $\lim\limits_{s \to 0^+} f(s) = \dfrac{1+\sigma}{1-\sigma} > 1$. Therefore, since $\frac{1+\sigma}{1-\sigma} \geq 1$, we have

$$\begin{aligned} \beta(\phi, \phi') &= \inf\{t > 0 : t\phi - \phi' \in K_a\} \\ &= \inf\{t > 0 : 0 \leq t\,\phi'(y) - \phi(y) \leq (t\,\phi'(y') - \phi(y'))e^{a\, d(y,\, y')}\} \\ &\leq \frac{1+\sigma}{1-\sigma}\, \sup_{y' \in I} \frac{\phi(y')}{\phi'(y')}\,. \end{aligned}$$

By the definition of the Hilbert projective distance of K_a, again since $\phi, \phi' \in K_{\sigma a}$, and since the diameter of the metric space (I, d) is $\max\limits_{x,\, x' \in I} \ln \dfrac{1+x}{1+x'} = \ln 2$, we hence have

$$\begin{aligned} d_{K_a}(\phi, \phi') = \ln\big(\beta(\phi, \phi')\, \beta(\phi', \phi)\big) &\leq \ln\Big(\Big(\frac{1+\sigma}{1-\sigma}\Big)^2\, \sup_{y, y' \in I} \frac{\phi(y')}{\phi'(y')}\, \frac{\phi'(y)}{\phi(y)}\Big) \\ &= \ln\Big(\Big(\frac{1+\sigma}{1-\sigma}\Big)^2\, \sup_{y, y' \in I} \frac{\phi(y')}{\phi(y)}\, \frac{\phi'(y)}{\phi'(y')}\Big) \\ &\leq \ln\Big(\Big(\frac{1+\sigma}{1-\sigma}\Big)^2\, \sup_{y, y' \in I} (e^{\sigma\, a\, d(y,y')})^2\Big) = 2 \ln \frac{1+\sigma}{1-\sigma} + 2\,\sigma\, a\, \ln 2\,. \end{aligned}$$

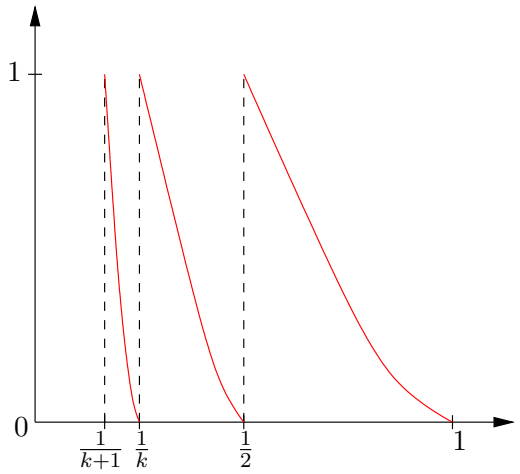
This proves formula (0.1). By the spectral gap theorem, for every spectral value λ' of L_s different from the (simple, positive) eigenvalue of L_s with maximal absolute value λ, since $\tanh s = \frac{e^{2s}-1}{e^{2s}+1}$ for every $s \in \mathbb{R}$, we hence have

$$\frac{|\lambda'|}{\lambda} \leq \tanh \frac{\Delta}{4} = \frac{2^{\sigma\, a}(1+\sigma) - (1-\sigma)}{2^{\sigma\, a}(1+\sigma) + (1-\sigma)}\,.$$

(4) The map h is smooth, hence Lipschitz on the compact interval I. By the telescopic sum process, we have

$$\begin{aligned} L_1 h(y) &= \frac{1}{\ln 2} \sum_{k=1}^{+\infty} \frac{1}{(k+y)^2}\, h\Big(\frac{1}{k+y}\Big) = \frac{1}{\ln 2} \sum_{k=1}^{+\infty} \frac{1}{(k+y)^2}\, \frac{k+y}{k+1+y} \\ &= \frac{1}{\ln 2} \sum_{k=1}^{+\infty} \Big(\frac{1}{k+y} - \frac{1}{k+1+y} \Big) = h(y)\,. \end{aligned}$$

Hence h is an eigenvector of L_1 for the eigenvalue 1.



(5) Let dt be the Lebesgue measure on $I = [0,1]$. Note that $\int_0^1 \frac{1}{1+t}\, dt = \ln 2$, hence $d\mu(t) = h(t)dt$ is a probability measure on I, which is absolutely continous with respect to dt. By induction, for all $n \in \mathbb{N} \smallsetminus \{0\}$ and $\phi, \phi' \in X$, by subdividing the interval $I = [0,1]$ in the almost-everywhere disjoint union $\bigcup_{k \in \mathbb{N} \smallsetminus \{0\}} \Big[\frac{1}{k+1}, \frac{1}{k}\Big]$, and using the change of variable $y \in [0,1] \mapsto \frac{1}{k+y} \in \Big[\frac{1}{k+1}, \frac{1}{k}\Big]$ on each of them, we have,

$$\int_0^1 (\phi \circ \psi^n)\, \phi'\, dt = \int_0^1 \phi\, L_1^n(\phi')\, dt\,.$$

Applying this with $n = 1$ and $\phi' = h$, and by the density of X in $C^0(I)$, this proves that the measure μ is invariant under the Gauss map ψ. The uniqueness and mixing property follows by writting $X = \mathbb{C}\, h \oplus X'$ where X' is a closed hyperplane of X, invariant under L_1, on which the restriction of L_1 has a spectral radius < 1.