TD3 Dynamical Systems, November 10th

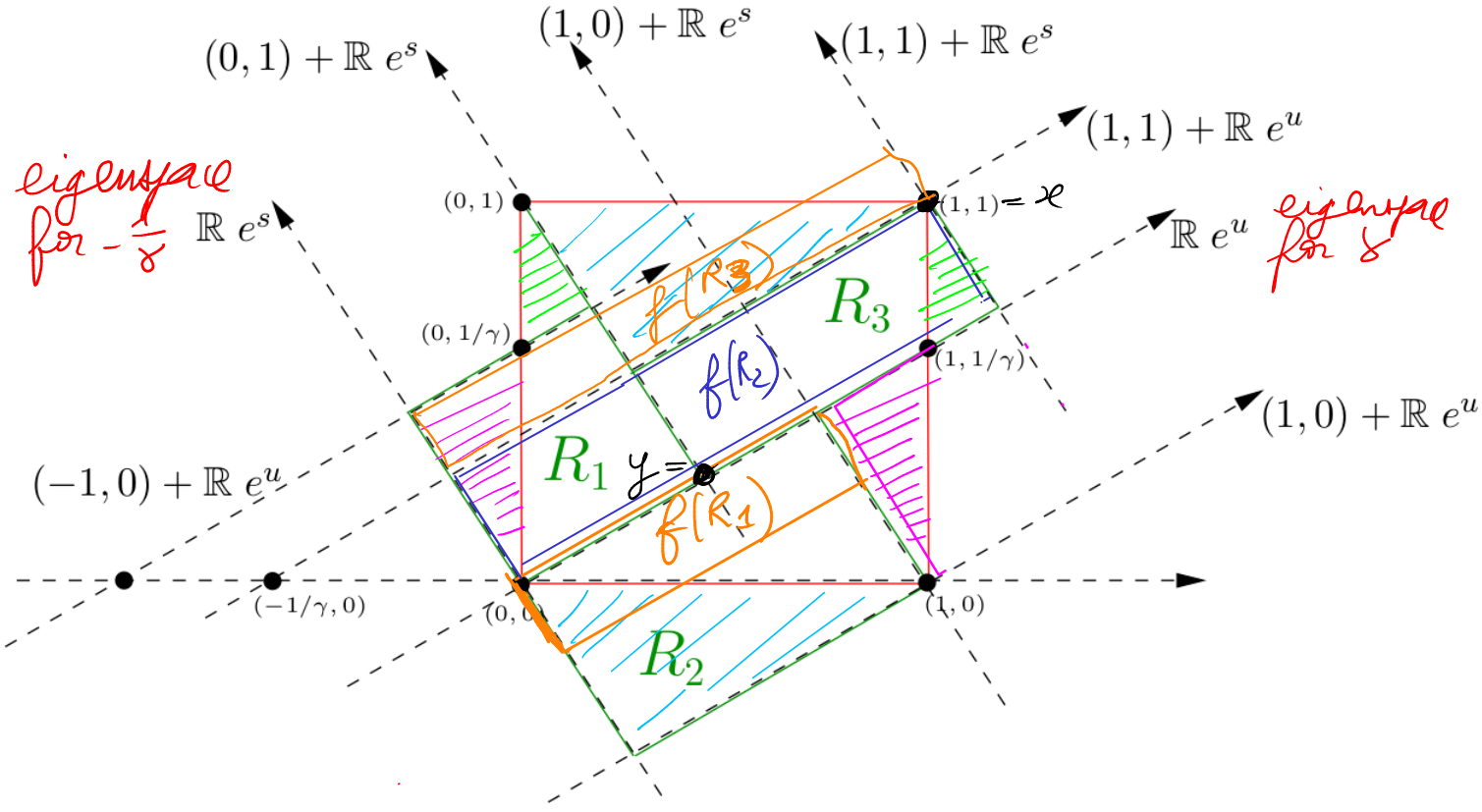
R_1, R_2, R_3 = images by $\pi : \mathbb{R}^2 \to \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ of rectangles of $\mathbb{R}^2$ with sides // unstable/stable directions of $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

golden ratio / nombre d'or $\gamma = \frac{1+\sqrt{5}}{2}$

eigenvalues $\gamma, -\frac{1}{\gamma}$.

eigenvectors $\begin{pmatrix} 1 \\ \frac{1}{\gamma} \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{\gamma} \\ 1 \end{pmatrix}$.

(1)



(2) $i=1$ $f(R_1) \subset R_2$

$i=2$ $f(R_2) \subset R_1 \cup R_3$

$i=3$ $f(R_3) \subset R_1 \cup R_2$

$R_3 + (-1,-1)$

$$t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$R_3 = \text{"convex hull"}$ of $\{f(x), f(y)\} \cup f(R_3)$

$\chi_t(X) = (X+1)(1+X-X^2)$ $\quad X^2 = X+1$

eigenvalues: $-1, \alpha, -\frac{1}{\alpha}$: pairwise distinct, hence t diagonalizable over $\mathbb{R}$

$\forall x \in \mathbb{T}^2 \setminus \pi(\mathbb{R}e_s \cup \mathbb{R}e_u)$ full measure subset of $\mathbb{T}^2$

$\exists ! (\xi_i)_{i\in\mathbb{Z}}$ $\forall i \in \mathbb{Z}$ $f^i(x) \in \overset{\circ}{R}_{\xi_i} \iff x \in f^{-i}\left(\overset{\circ}{R}_{\xi_i}\right)$

itinerary of x (describes the sequence of rectangles visited by f-orbit of x)

$\forall (\xi_i)_{i\in\mathbb{Z}} \in \Sigma'_t \Leftarrow \left(f^{-i}(R_{\xi_i})\right)_{i\in\mathbb{Z}}$

subshift of finite type?

$(\Sigma'_t, \sigma_t = \sigma|_{\Sigma'_t})$

$\forall n \in \mathbb{N}$, let

$K_n = \bigcap_{-n \le j+n} f^{-j}(R_{\xi_j})$ $\quad (K_n)_{n\in\mathbb{N}}$ nonincreasing sequence of compact subsets.

$\bigcap_{n\in\mathbb{N}} K_n \overset{?}{=} \{x\}$ as $\operatorname{diam} K_n \underset{n\to\infty}{\to} 0$ nonempty.

since A contracts $\mathbb{R}e_s$

A^{-1} contracts $\mathbb{R}e_u$

σ_t ↻

$\pi : \Sigma'_t \to \mathbb{T}^2$ ↻ f works.

$(\xi_i)_{i\in\mathbb{Z}} \mapsto x$ as above,

a.e onto by itinerary map on $\mathbb{T}^2$ - image of $(\mathbb{R}e_s \cup \mathbb{R}e_u)$

$\Rightarrow$ onto by continuity + compactness.

$(\xi_i)_{i\in\mathbb{Z}}$ and $(\xi'_i)_{i\in\mathbb{Z}}$ are close if $\exists$ n large such that $\xi_i = \xi'_i$ $\forall i \in [\![-n, n]\!]$ (by definition of the product topology on $\{1,2,3\}^{\mathbb{Z}}$ + $\{1,2,3\}$ discrete.

① $\Rightarrow$ π induces a bijection between

the periodic points of σ_t and f except the ones for f that lie in images of $\mathbb{R}e_s$ and $\mathbb{R}e_u$.
What are the other ones? O is fixed by f, but
$O \in \mathbb{T}^2$
that's the only periodic point in $\pi(\mathbb{R}e_s \cup \mathbb{R}e_u)$ since $\mathbb{R}e_s$, $\mathbb{R}e_u$ have irrational slope and dynamic on them has no fixed point (attracted to O on $\mathbb{R}e_s$ by iteration of A).

NB: $\mathrm{card}(\mathrm{Fix}(f^n)) = |\det(I_2 - A^n)|$ by TD01,
$= (\gamma^n - 1)\left(1 - \left(-\frac{1}{\gamma}\right)^n\right) = \gamma^n - 1 - (-1)^n + \left(-\frac{1}{\gamma}\right)^n$
eigenvalues of A are γ, $-\frac{1}{\gamma}$, distinct, hence A diagonalizable over $\mathbb{R}$.
A^n conjugated to $\begin{pmatrix} \gamma^n & 0 \\ 0 & \left(-\frac{1}{\gamma}\right)^n \end{pmatrix}$
$I - A^n$ // $\begin{pmatrix} 1-\gamma^n & 0 \\ 0 & 1-\left(\frac{1}{\gamma}\right)^n \end{pmatrix}$

d) Artin-Mazur zeta function:

For σ_t: $\forall n \in \mathbb{N}\setminus\{0\}$, $\mathrm{Card}\ \sigma_t^n = \mathrm{trace}(t^n)$
// $(-1)^n + \gamma^n + \left(-\frac{1}{\gamma}\right)^n$

$$\zeta_{\sigma_t}(z) = \exp\left(\sum_{n=1}^{+\infty} \frac{z^n}{n} \mathrm{card}\,\mathrm{Fix}(\sigma_t)^n\right)$$
$$= \exp\left(\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} z^n + \sum_{n=1}^{+\infty} \frac{\gamma^n}{n} z^n + \sum_{n=1}^{+\infty} \frac{\left(-\frac{1}{\gamma}\right)^n}{n} z^n\right)$$

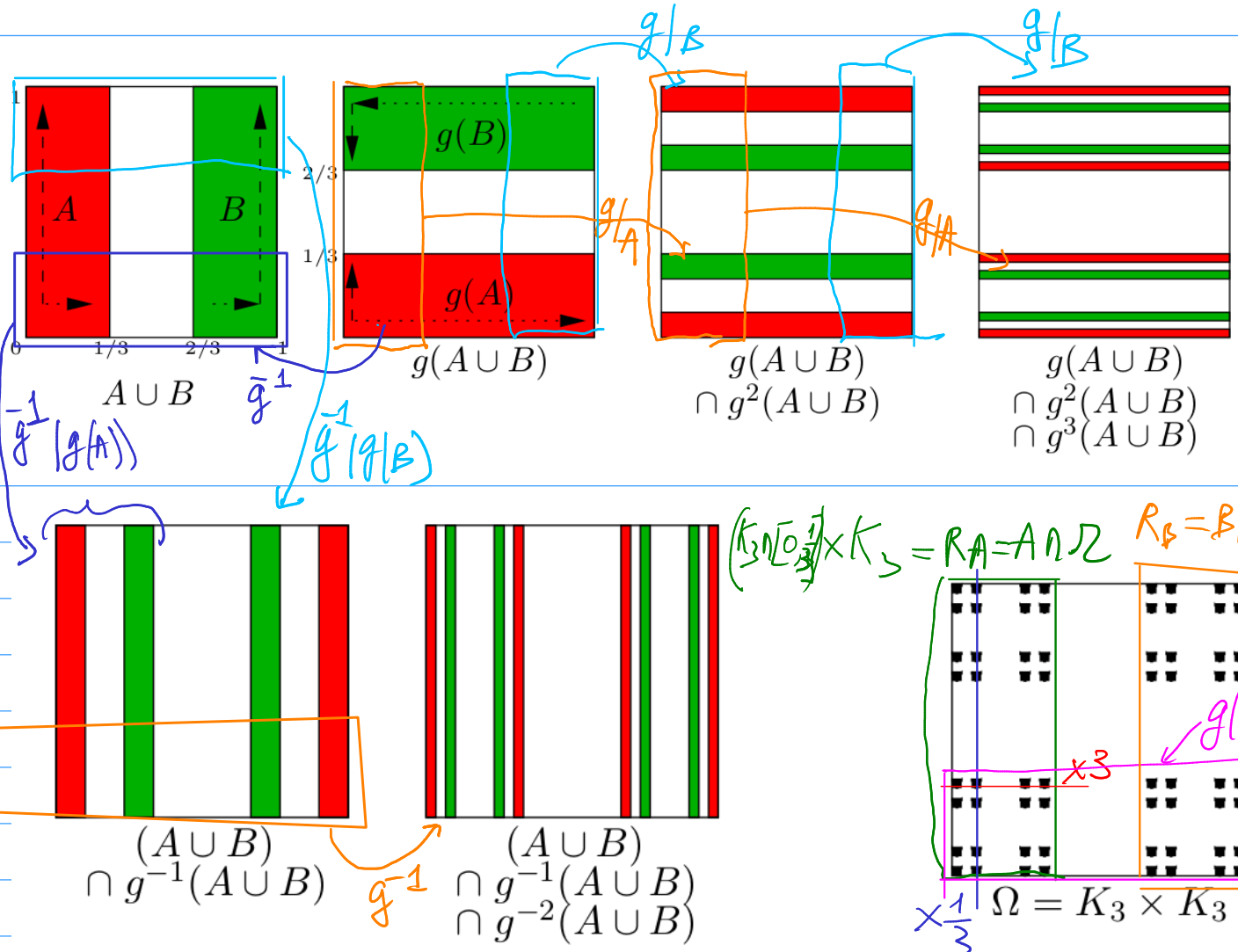
$\left(\sum_{n=1}^{+\infty} \frac{(az)^n}{n}\right)' = \sum_{n=0}^{+\infty} (az)^n$ $\qquad \sum_{n=0}^{\infty} (az)^n = \frac{1}{1-az}$

$$= \exp\left(-\ln(1+z) - \ln(1-\gamma z) - \ln\left(1+\frac{1}{\gamma}z\right)\right)$$
$$= \frac{1}{(1+z)\left(1+\frac{1}{\gamma}z\right)(1-\gamma z)}$$ rational function.

$\sum_{n=0}^{+\infty} \omega^n = \frac{1}{1-\omega}$ $\quad \sum_{n=1}^{+\infty} \frac{\omega^n}{n} = -\ln(1-\omega)$.

Same thing for S_f.

(2)



$(K_3 \cap [0,\frac{1}{3}]) \times K_3 = R_A = A \cap \Omega$ $\quad R_B = B \cap \Omega$

$$\Omega^+ = \bigcap_{1 \le k < +\infty} g^k(A \cup B) = \bigcap_{n \in \mathbb{N}} \bigcap_{1 \le k \le n} g^k(A \cup B)$$

$= [0,1] \times K_3$ triadic Cantor set

$[0,1] \times \bigcup_{2^n}$ vertical intervals of length $(\frac{1}{3})^n$

$$\Omega^- = \bigcap_{-\infty < k \le 0} g^k(A \cup B) = K_3 \times [0,1]$$

$$\Omega = \Omega^- \cap \Omega^+ = K_3 \times K_3$$

b) Strategy to find coding of dynamics of g on Σ_2 by σ on Σ_2: again use itinary map of g for partition $\{R_A, R_B\}$ of Ω.

$\forall i \in \mathbb{Z}$, $\forall x \in \Omega$, $\exists!\ \xi_i \in \{A, B\}$ s.t. $g^i(x) \in R_{\xi_i}$

$\xi_i(x)$

$\xi_i = A$ if $g^i(x) \in R_A$

$\xi_i = B$ if $g^i(x) \in R_B$.

g σ $\mathbb{Z}$.

$\Omega \longrightarrow \{A, B\}^{\mathbb{Z}}$

$x \longmapsto (\xi_i(x))_{i \in \mathbb{Z}}$

conjugates dynamics

continuous, bijective, between compact sets, hence homeo