TD 1 : Topological dynamical systems, circle homeomorphisms

Exercice 1 Let (X, ϕ) and (X', ϕ') be two topological dynamical systems, and let $h : X \to X'$ be a semi-conjugacy between (X, ϕ) and (X', ϕ'). Among the following properties, which are satisfied by (X', ϕ') if they are satisfied by (X, ϕ) ? Give counterexamples when this is not the case.

- having its periodic points dense,
- having a positively recurrent point,
- having no wandering point,
- being positively transitive (and transitive in the invertible case),
- being topologically mixing,
- being positively minimal (and minimal in the invertible case).

Exercice 2 Let X be a topological space with a countable base of open sets, $\mu \in \mathrm{Prob}(X)$ with support equal to X, and $\phi : X \to X$ a continuous map that preserves μ. Show that if ϕ is μ-ergodic, then μ-almost every positive orbit is dense in X, i.e.,

$$\overline{\{\phi^n(x) \mid n \in \mathbb{N}\}} = X$$

for μ-almost every x.

Exercice 3 Let $p \in \mathbb{Z} - \{-1, 0, 1\}$, $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, and $\phi_p : \mathbb{T}^1 \to \mathbb{T}^1$ the continuous map defined by $x \mapsto p\,x$.

(1) Show that the discrete-time topological dynamical system $(\mathbb{T}^1, \phi_p)$ is non-invertible, that it is conjugate to the topological dynamical system $(\mathbb{S}_1, z \mapsto z^p)$, that the set $\mathrm{per}(\phi_p)$ of periodic points of ϕ_p is dense in the circle $\mathbb{T}^1$, that the topological dynamical system $(\mathbb{T}^1, \phi_p)$ is non-wandering, positively transitive, and topologically mixing.

(2) Assume $p \geq 2$. Let $\Theta : \{0, \dots, p-1\}^{\mathbb{N}} \to \mathbb{T}^1$ be the map defined by

$$(x_i)_{i \in \mathbb{N}} \mapsto \sum_{i \in \mathbb{N}} \frac{x_i}{p^{i+1}} \mod \mathbb{Z}\ .$$

Show that Θ is a semi-conjugacy between the one-sided Bernoulli shift with alphabet $\{0, \dots, p-1\}$ and the topological dynamical system $(\mathbb{T}^1, \phi_p)$, which is called a coding of $(\mathbb{T}^1, \phi_p)$. Retrieve the previous properties.

Exercice 4 Let (X, d) be a metric space. A continuous map $f : X \to X$ is said to be *strictly expanding with factor* $\lambda > 1$ if there exists $\epsilon > 0$ such that for all $x, y \in X$, if $d(x, y) \leq \epsilon$, then

$$d(f(x), f(y)) \geq \lambda\, d(x, y)\ .$$

A continuous map $f : X \to X$ is said to be *strictly expanding* if there exists $\lambda > 1$ such that f is strictly expanding with factor λ.

The goal of the exercise is to classify, up to conjugacy, strictly expanding maps on the circle. We consider the circle as the metric space $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$, with the induced distance[1] from $\mathbb{R}$.

(1) Let $p \in \mathbb{Z} - \{-1, 0, 1\}$. Show that the map $\phi_p : \mathbb{T}^1 \to \mathbb{T}^1$ defined by $x \mapsto p\,x$ is strictly expanding.

1. Letting $\dot{x} \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ be the image of $x \in \mathbb{R}$ under the canonical projection, the induced distance is defined by $d(\dot{x}, \dot{y}) = \min_{k \in \mathbb{Z}} |x - y + k|$ for all $x, y \in \mathbb{R}$. Note that if $x, y \in \mathbb{R}$ satisfy $|x - y| < \frac{1}{2}$, then $d(\dot{x}, \dot{y}) = |x - y|$.

(2) Show that the degree p of a strictly expanding map $f : \mathbb{T}^1 \to \mathbb{T}^1$ is different from $-1, 0, 1$.

From now on, we fix a strictly expanding map $f : \mathbb{T}^1 \to \mathbb{T}^1$ of degree $p \geq 2$, and we want to show that f is topologically conjugate to ϕ_p.

(3) Show that f has a fixed point. Show that we can assume that f admits a lift $\widetilde{f}$ (which we fix for the rest of the exercise) such that $\widetilde{f}(0) = 0$. Show that there exists a finite, strictly increasing sequence of points $a_0 = 0 < a_1 < \cdots < a_{p-1} < a_p = 1$ such that $\widetilde{f}(a_i) = i$ for $i = 0, \ldots, p$.

(4) Let E be the set of increasing continuous maps $h : [0,1] \to [0,1]$ such that $h(0) = 0$ and $h(1) = 1$, equipped with the distance induced by the uniform norm

$$d(h_1, h_2) = \max_{t \in [0,1]} |h_1(t) - h_2(t)| \ .$$

Show that E is complete. Let $L : E \to E$ be the map $h \mapsto Lh$ defined by setting, for all $h \in E$ and $t \in [0,1]$,

$$Lh : t \mapsto \begin{cases} \frac{1}{p}\, h(\widetilde{f}(t) - i) + \frac{i}{p} \mod 1 & \text{if } a_i \leq t < a_{i+1} \text{ and } i = 0, \ldots, p-1 \\ 1 & \text{if } t = 1 \ . \end{cases}$$

Show that L is a $\frac{1}{p}$-contracting map [2] on E.

(5) Conclude.

Exercice 5 Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be an orientation-preserving homeomorphism with a rational rotation number α.

(1) Show that all periodic orbits have the same period.

(2) Show that all periodic orbits have the same cyclic order on the circle as any orbit of the rotation R_α.

(3) If the rotation number of f is p/q with p and q being coprime integers, show that f is topologically conjugate to the rotation $R_{p/q}$ if and only if there exists a lift $\widetilde{f}$ of f such that $\widetilde{f}^{\,q}$ is the map $t \mapsto t + p$.

2. Let $c \in [0, 1[$. A map $f : X \to Y$ between two metric spaces is *c-contracting* if for all $x, y \in X$, we have $d(f(x), f(y)) \leq c\, d(x, y)$.