TD 1 : Measure-preserving transformations

Exercice 1 Let X be a metrizable compact space. Let $\varphi_1, \varphi_2 : X \to X$ to continous maps that commute.

Prove that there exists a Borel probability measure on X which is invariant by both φ_1 and φ_2.

Exercice 2 Let $\mathcal{A}$ be a finite alphabet of cardinal at least 2, and μ a probability measure on $\mathcal{A}$. For a in the support of μ, calculate the laws of the first return times on the initial cylinder $[a]$ for both the two-sided and one-sided Bernoulli shift, for the measures $\mu^{\mathbb{Z}}$ and μ^{N} respectively. Check that Kac's theorem holds.

Exercice 3 Let $\alpha \in \mathbb{R}$ be an irrationnal number and T the translation of the circle $\mathbb{R}/\mathbb{Z}$ defined by $T(x) = x + \alpha$. Let $\beta \in]0, 1[$ and $I = [0, \beta]$ be a closed interval (in $\mathbb{R}$, identified to its image in $\mathbb{R}/\mathbb{Z}$).

(1) Let $0 \leq a \leq b < 1$. Show that if $x \in [a, b] \subset I$ and n are such that $T^n x \in I$ then either $T^n a \in I$ or $T^n b \in I$.

(2) Prove that the first return time on I takes at most 3 distinct values.

Exercice 4 (Induced transformation) Let $(X, \mathscr{B}, \mu, \varphi)$ be a probabilized dynamical system with discrete time. Let $A \subset X$ be measurable, with positive measure, and let $\tau_A : X \to (\mathbb{N} \setminus \{0\} \cup \{\infty\})$ be the first return time on A. Note μ_A the induced measure on A : recall that it is defined by $\mu_A(B) = \frac{\mu(B)}{\mu(A)}$ for every B in the induced σ-algebra $\mathscr{B}_A = \{B \cap A \mid B \in \mathscr{B}\}$. The *induced transformation of φ in A* is the map

$$\varphi_A : x \mapsto \begin{cases} \varphi^{\tau_A(x)}(x) & \text{si } \tau_A(x) \neq \infty \\ x & \text{sinon} \end{cases}$$

(1) Prove that φ_A is $\mathscr{B}_A$-mesurable and preserves the induced measure μ_A.

(2) Prove that if $(X, \mathscr{B}, \mu, \varphi)$ is ergodic then $(A, \mathscr{B}_A, \mu_A, \varphi_A)$ also is.

(3) We assume that $X = \bigcup_{n\in\mathbb{N}} \varphi^{-n}(A)$. Prove that if $(A, \mathscr{B}_A, \mu_A, \varphi_A)$ is ergodic then $(X, \mathscr{B}, \mu, \varphi)$ is ergodic.

Exercice 5 (Kakutani transformation) Let $(X, \mathscr{B}, \mu, \varphi)$ be a probabilized dynamical system with discrete time, with φ invertible. Let $\tau : X \to (\mathbb{N} \setminus \{0\} \cup \{\infty\})$ be a measurable map such that $\int_X \tau d\mu < \infty$ (it will be called *height function*). Let

$$X_\tau = \{(x, n) \in X \times \mathbb{N} \mid n < \tau(x)\}$$

endowed with the σ-algebra $\mathscr{B}_\tau$ induced from the product σ-algebra on $X \times \mathbb{N}$. Let $\varphi_\tau : X_\tau \to X_\tau$ defined by

$$(x, n) \mapsto \begin{cases} (x, n+1) & \text{si } n + 1 < \tau(x) \\ (\varphi(x), 0) & \text{sinon} \end{cases}$$

(called the *Kakutani transformation* on X of height τ).

(1) Prove that the Kakutani transformation φ_τ is $\mathscr{B}_\tau$-measurable.

(2) Prove that there exists on the measurable space X_τ a unique probability measure μ_τ, invariant by φ_τ, such that the measure on $X \times \{0\}$ induced from μ_τ is the measure μ.

The space X_τ is called the *Kakutani tower* on X of height τ. The dynamical system $(X_\tau, \mathscr{B}_\tau, \mu_\tau, \varphi_\tau)$ is called the *suspension* of X of height τ.

(3) Make a picture.

(4) Prove that if $(X, \mathscr{B}, \mu, \varphi)$ is ergodic then $(X_\tau, \mathscr{B}_\tau, \mu_\tau, \varphi_\tau)$ also is.

(5) Let $(Y, \mathscr{C}, \nu, \psi)$ be a probabilized invertible dynamical system with discrete time, with ν ergodic, and A be a positive measure subset of Y.

Prove that the measured dynamical system $(Y, \mathscr{C}, \nu, \psi)$ is conjugated to the system obtained as a suspension of the induced dynamical system $(A, \mathscr{C}_A, \nu_A, \psi_A)$ for the height function equal to the first return time τ_A.