TD 2 : Geodesic Flow, Horocyclic Flow

Exercice 1 Let Γ be a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$.

(1) Prove the following commutation property of the geodesic and horocyclic flows on $\Gamma\backslash T^1\mathbb{H}^2_{\mathbb{R}}$: for all $t, s \in \mathbb{R}$, we have

$$\mathfrak{g}^t \circ \mathfrak{h}^s \circ \mathfrak{g}^{-t} = \mathfrak{h}^{s\,e^{-t}} . \tag{1}$$

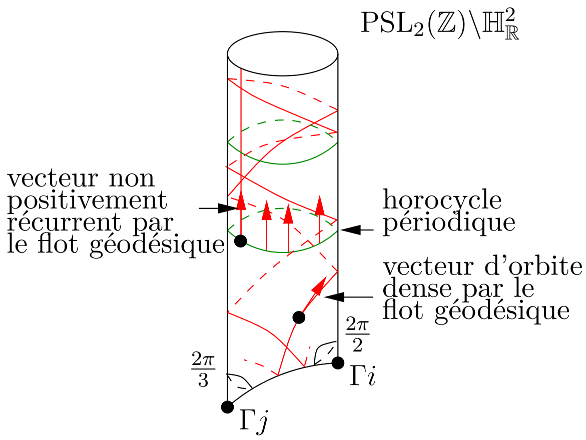
(2) Show that the family $(\overline{\mathfrak{h}}^s)_{s\in\mathbb{R}}$ is a one-parameter group of C^∞-diffeomorphisms of $T^1\mathbb{H}^2_{\mathbb{R}}$, called the *unstable horocyclic flow* on the real hyperbolic plane $\mathbb{H}^2_{\mathbb{R}}$, which commutes with the action of G on $T^1\mathbb{H}^2_{\mathbb{R}}$, preserves the Liouville measure on $T^1\mathbb{H}^2_{\mathbb{R}}$, and satisfies, for all $g \in G$ and $s, t \in \mathbb{R}$,

$$\mathfrak{g}^t \circ \overline{\mathfrak{h}}^s \circ \mathfrak{g}^{-t} = \overline{\mathfrak{h}}^{s\,e^t} \quad \text{and} \quad \overline{\mathfrak{h}}^s(\Phi(g)) = \Phi(g\,u_s^-)$$

where $u_s^- = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$.

(3) Let Γ be a lattice of $\mathrm{PSL}_2(\mathbb{R})$. We will show that the geodesic flow on $Y = \Gamma\backslash T^1\mathbb{H}^2_{\mathbb{R}}$ is ergodic with respect to m_{Liou}. To do this, consider a function $f \in \mathbb{L}^2(Y, m_{\mathrm{Liou}})$ invariant under the geodesic flow (i.e., such that $f \circ \mathfrak{g}^t = f$ in $\mathbb{L}^2(Y, m_{\mathrm{Liou}})$ for all $t \in \mathbb{R}$). Show that $\|f \circ \mathfrak{h}^s - f\|_{\mathbb{L}^2} = 0$ and $\|f \circ \overline{\mathfrak{h}}^s - f\|_{\mathbb{L}^2} = 0$ for all $s \in \mathbb{R}$ using formula (1), and deduce that f is invariant under the right translation action of G on Y identified with $\Gamma\backslash G$ via Φ^{-1}.

Exercice 2 Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ be the modular group, $M = \Gamma\backslash\mathbb{H}^2_{\mathbb{R}}$ the *modular curve*, and $T^1M = \Gamma\backslash T^1\mathbb{H}^2_{\mathbb{R}}$. Let $\mathscr{F} = \{z \in \mathbb{C} \ : \ |z| \geq 1,\ |\mathrm{Re}\ z| \leq \frac{1}{2}\}$ denote the usual (weak) fundamental domain of the modular group Γ on $\mathbb{H}^2_{\mathbb{R}}$. Let $S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.



(1) Show that if two distinct points z and z' in $\mathscr{F}$ belong to the same orbit under Γ, then either $\mathrm{Re}\ z = \pm\frac{1}{2}$ and $z = z' \pm 1$, or $|z| = 1$ and $z' = -\frac{1}{z}$. Show that the stabilizer $\Gamma_z = \{\gamma \in \Gamma \ : \ \gamma \cdot z = z\}$ of a point $z \in \mathscr{F}$ in Γ is trivial (reduced to $\{\mathrm{id}\}$) except in the following three cases :

- $z = i$, in which case $\Gamma_z = \{\mathrm{id}, S\}$,
- $z = \omega = e^{\frac{2i\pi}{3}}$, in which case $\Gamma_z = \{\mathrm{id}, ST, (ST)^2\}$,
- $z = -\overline{\omega} = e^{\frac{i\pi}{3}} = \omega + 1$, in which case $\Gamma_z = \{\mathrm{id}, TS, (TS)^2\}$.

(2) We now focus on the periodic orbits of the geodesic flow on T^1M.

(a) Explain why such a periodic orbit is determined by a vector $v \in T^1\mathbb{H}^2_{\mathbb{R}}$ such that there exist $t \in \mathbb{R}$ and $\gamma \in \Gamma$ with $\overline{\mathfrak{g}}_t v = \gamma v$. Fix such a v and let ℓ denote the geodesic in $\mathbb{H}^2_{\mathbb{R}}$ associated with v, and $z_1, z_2 \in \mathbb{R} \cup \{\infty\}$ the endpoints of ℓ.

(b) Explain why we cannot have $z_1 = \infty$ or $z_1 \in \mathbb{Q}$. Verify that z_1 and z_2 are conjugate quadratic irrationals (they are the two roots of the same polynomial of degree 2 with integer coefficients).

(c) Conversely, assume z_1 and z_2 are conjugate quadratic irrationals, and let ℓ be the geodesic in $\mathbb{H}^2_{\mathbb{R}}$ joining z_1 and z_2. Using the continued fraction expansion of z_1, show that the image of ℓ is a periodic geodesic in M. *Recall that the continued fraction expansion of a quadratic irrational is eventually periodic.*

(3) Determine the periodic orbits of the horocyclic flow on T^1M.

(4) Show that the union of the periodic orbits of the geodesic flow is dense in T^1M. *Recall that if $\alpha = [\,\overline{a_0, a_1, \ldots, a_{m-1}, a_m}\,]$ is a quadratic irrational in $\mathbb{R}$ whose continued fraction expansion is purely periodic with period $a_0, a_1, \ldots, a_{m-1}, a_m$ (where $a_0 = \lfloor \alpha \rfloor \geq 1$), and if α^σ is the Galois conjugate of α (the other root of a quadratic polynomial with integer coefficients having α as a root), then the continued fraction expansion of $-\frac{1}{\alpha^\sigma}$ is $[\,\overline{a_m, a_{m-1}, \ldots, a_1, a_0}\,]$, periodic with the reverse period $a_m, a_{m-1}, \ldots, a_1, a_0$.*

(5) Show that for any $v \in T^1\mathbb{H}^2_{\mathbb{R}}$, the limit set $\omega(\Gamma v)$ of $\Gamma v \in T^1M$ under the geodesic flow is non-empty if and only if the endpoint at infinity of the geodesic ray in $\mathbb{H}^2_{\mathbb{R}}$ defined by v does not belong to $\mathbb{Q} \cup \{\infty\}$.