# TD 2 : Ergodicity, Birkhoff's Theorem, Kingman and Oseledets' Theorems

**Exercise 1** Let $(X, \mathscr{B}, \mu, \varphi)$ be a probabilized dynamical system.
Prove that $\varphi$ is ergodic if and only if for every $A, B \in \mathscr{B}$ we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(\varphi^{-k}(A) \cap B) = \mu(A)\mu(B)$$

**Exercise 2** Let $x \in \{1, \ldots, 9\}$. What is the frequence of $x$ as a first digit in the decimal representation of $2^n$ ?

**Exercise 3** Let $(X, \mathscr{B}, \varphi)$ be a measurable dynamical system, and $\mu$ and $\nu$ be two invariant probability measures.

(1) Prove that if $\nu$ is ergodic and $\mu$ is absolutely continuous with respect to $\nu$ then $\nu = \mu$.

(2) Prove that if $\mu$ and $\nu$ are ergodic then they are either equal or mutually singular.

**Exercise 4** A probabilized, discrete-time dynamical system $(X, \mathscr{B}, \mu, \varphi)$ is called *totally ergodic* if for every $n \in \mathbb{N} \setminus \{0\}$ the dynamical system $(X, \mathscr{B}, \mu, \varphi^n)$ is ergodic.

(1) Give an example of a probabilized, discrete-time dynamical system which is ergodic but not totally ergodic.

(2) Prove that the dynamical system $(X, \mathscr{B}, \mu, \varphi)$ is totally ergodic if and only if it doesn't admit any non-trivial finite factor (i.e such that the support of the measure is not a singleton).

**Exercise 5** (Skew Products) Let $(X, \mathscr{B}, \mu, \varphi)$ be a discrete-time measured dynamical system. Let $G$ be a locally compact abelian topological group (whose operation is written additively), $\mathscr{B}_G$ its Borel $\sigma$-algebra, and $\mu_G$ a Haar measure on $G$. Let $f : X \to G$ be a measurable map. Consider the map $\varphi_f : X \times G \to X \times G$ defined by

$$\varphi_f : (x, g) \mapsto (\varphi(x), g + f(x))$$

(1) Show that $\varphi_f$ is measurable with respect to the product $\sigma$-algebra $\mathscr{B} \times \mathscr{B}_G$ and preserves the product measure $\mu \otimes \mu_G$. The quadruple $(X \times G, \mathscr{B} \times \mathscr{B}_G, \mu \otimes \mu_G, \varphi_f)$ is called the *skew product dynamical system* of $(X, \mathscr{B}, \mu, \varphi)$ by $f$.

(2) (**Anzai's Theorem**) If $\varphi$ is ergodic and $G$ is the circle $\mathbb{R}/\mathbb{Z}$, show, for instance by using Fourier expansion with respect to the second variable, that the skew product dynamical system is not ergodic if and only if there exists $p \in \mathbb{Z} \setminus \{0\}$ such that the function $pf : X \to G$ is a *coboundary*, i.e., there exists a measurable function $\theta : X \to G$ such that almost everywhere

$$pf = \theta \circ \varphi - \theta.$$

**Exercise 6** (Lower bound for Lyapunov exponents) Let $X = \mathbb{R}/(2\pi\mathbb{Z})$ be equipped with the (normalized) Lebesgue measure, and $\varphi : X \to X$ defined by $\varphi(x) = x + 2\pi\alpha$ with $\alpha \notin \mathbb{Q}$. For $x \in X$, we denote by $R_x = \begin{pmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{pmatrix}$ the rotation by angle $x$.

Let $c > 1$, and define $H = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$. We then define $M : X \to \mathrm{SL}_2(\mathbb{R})$ by $M(x) = HR_x$. Let

$$M_n(x) = M(\varphi^{n-1}(x))M(\varphi^{n-2}(x))\dots M(\varphi(x))M(x)$$

be the associated cocycle ; we denote by $\lambda_{\max}$ its largest Lyapunov exponent. The goal of this exercise is to show that

$$\lambda_{\max} \geq \ln\left(\frac{c + c^{-1}}{2}\right).$$

(1) Explain why

$$\lambda_{\max} = \lim_{n\to+\infty} \frac{1}{2\pi n} \int_0^{2\pi} \ln \|HR_{x+2\pi(n-1)\alpha} \cdots HR_x\|_\infty dx$$

(where $\|A\|_\infty$ denotes the maximum of the absolute values of the coefficients of the matrix $A$).

(2) For $z \in \mathbb{C}$, we define

$$\widetilde{R}(z) = \begin{pmatrix} (z^2+1)/2 & (-z^2+1)/2i \\ (z^2-1)/2i & (z^2+1)/2 \end{pmatrix}.$$

Express $R_x$ in terms of $\widetilde{R}(e^{ix})$. Deduce that

$$\lambda_{\max} = \lim_{n\to+\infty} \frac{1}{2\pi n} \int_0^{2\pi} f_n(e^{ix})dx,$$

where the functions $f_n : \mathbb{C} \to \mathbb{R}$ are subharmonic functions to be made explicit. [1]

(3) Deduce that

$$\lambda_{\max} \geq \lim_{n\to+\infty} \frac{1}{n} \ln \|(H\widetilde{R}(0))^n\|_\infty$$

and conclude.

1. Recall that a function $f : \mathbb{C} \to \mathbb{R}$ is *subharmonic* if it is upper semicontinuous and the value of the function at any point $z_0$ is less than or equal to the average of $f$ over circles around $z_0$ : in other words,

$$f(z_0) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{ix})dx$$

for any $r$.

You may use the following two results :
— If $u : \mathbb{C} \to \mathbb{C}$ is holomorphic and non-zero, then $z \mapsto \ln|u(z)|$ is subharmonic.
— A maximum of subharmonic functions is subharmonic.