TD 3 : Mixing

Exercice 1 Let $(X, \mathcal{B}, \mu, \phi)$ and $(X', \mathcal{B}', \mu', \phi')$ be two discrete-time measured dynamical systems, with μ and μ' being probability measures. Let $h : X \to X'$ be a semi-conjugacy between (X, ϕ) and (X', ϕ'). Among the following properties, which are satisfied by (X', ϕ') if they are satisfied by (X, ϕ), possibly adding assumptions on h ?

- being ergodic,
- being mixing,
- being mixing of order k for a given $k \in \mathbb{N} - \{0, 1\}$ (see the exercise below for the definition),
- being ψ-mixing on a dense vector subspace.

Exercice 2 Let $(X, \mathscr{A}, \mu, \phi)$ be a discrete-time measured dynamical system, with μ being a probability measure. For $k \in \mathbb{N} - \{0, 1\}$, we say that $(X, \mathscr{A}, \mu, \phi)$ is *mixing of order* k if for all measurable sets $A_1, \dots, A_k$ in X, and for all functions $\tau_1, \dots, \tau_k$ from $\mathbb{N}$ to $\mathbb{N}$ such that $\lim_{n\to\infty} \tau_{i+1}(n) - \tau_i(n) = +\infty$ for $i = 1, ..., k-1$, we have

$$\mu\left(\bigcap_{i=1}^{k} \phi^{-\tau_i(n)}(A_i)\right) \underset{n\to\infty}{\longrightarrow} \prod_{i=1}^{k} \mu(A_i) \ .$$

We say that $(X, \mathscr{A}, \mu, \phi)$ satisfies the property of *multiple mixing* if it is mixing of order k for all $k \in \mathbb{N} - \{0, 1\}$.

(1) Show that $(X, \mathscr{A}, \mu, \phi)$ is mixing of order k if and only if for all $f_1, \dots, f_k \in \mathbb{L}^k(X, \mu)$ and all functions $\tau_1, \dots, \tau_k$ as above, we have

$$\int_X \left(\prod_{i=1}^{k} f_i \circ \phi^{\tau_i(t)}\right) d\mu \underset{t\to\infty}{\longrightarrow} \prod_{i=1}^{k} \int_X f_i \, d\mu \ .$$

(2) Let $\mathscr{A}$ be a finite alphabet with at least 2 symbols, endowed with a probability measure μ. Show that the Bernoulli system on $\mathscr{A}$, endowed with the product measure $\mu^{\mathbb{N}}$ in the unilateral case and the product measure $\mu^{\mathbb{Z}}$ in the bilateral case, satisfies the property of multiple mixing.

(3) Let p be in $\mathbb{N} - \{0, 1\}$. Show that the function $\phi_p : x \mapsto p\, x$ from the circle $\mathbb{R}/\mathbb{Z}$ to itself satisfies the property of multiple mixing for the Lebesgue measure on the circle.

(4) Let N in $\mathbb{N} - \{0\}$ and M an invertible $N \times N$ matrix with integer coefficients, all of whose eigenvalues have modulus greater than 1. Show that the function $\phi_M : x \bmod \mathbb{Z}^N \mapsto Mx \bmod \mathbb{Z}^N$ from the torus $\mathbb{R}^N/\mathbb{Z}^N$ to itself satisfies the property of multiple mixing for the Haar measure on the torus.

Exercice 3 Let $N \in \mathbb{N} - \{0\}$ and $M \in \mathcal{M}_N(\mathbb{Z})$ be an integer $N \times N$ matrix that does not have any complex eigenvalue with modulus less than or equal to 1. Let $\pi : y \mapsto \dot{y}$ be the canonical projection from $\mathbb{R}^N$ onto the torus $\mathbb{T}^N$. Let $\phi_M : \mathbb{T}^N \to \mathbb{T}^N$ be the smooth application $\theta = (\theta_i)_{1\leq i\leq N} \mapsto \left(\sum_{1\leq j\leq N} m_{ij}\theta_j\right)_{1\leq i\leq N}$, defined by taking the quotient modulo $\mathbb{Z}^N$ of the linear function $M : \mathbb{R}^N \to \mathbb{R}^N$.

Let $C^0(\mathbb{T}^N)$ be the Banach space of continuous real functions on $\mathbb{T}^N$, equipped with the uniform norm $\| \, \|_\infty$. Let $L_M : C^0(\mathbb{T}^N) \to C^0(\mathbb{T}^N)$ be the continuous linear operator (with norm at most 1), called the *transfer operator*, defined for all $f \in C^0(\mathbb{T}^N)$ and $x \in \mathbb{T}^N$ by

$$L_M f : x \mapsto \frac{1}{\mathrm{card}(\phi_M^{-1}(x))} \sum_{z\in\phi_M^{-1}(x)} f(z) \ .$$

(1) Show that $M\mathbb{Z}^N$ is a subgroup of $\mathbb{Z}^N$ of index $|\det(M)|$. What is the cardinality of $\phi_M^{-1}(x)$?

(2) For all $f \in C^0(\mathbb{T}^N)$ and $\dot{y} \in \mathbb{T}^N$, show that

$$Lf(\dot{y}) = \frac{1}{|\det M|} \sum_{[i]\in\mathbb{Z}^N/M\mathbb{Z}^N} f \circ \pi(M^{-1}(y+i)) \ .$$

(3) Show that if we denote $\lambda = d\theta$ the Haar measure on $\mathbb{T}^N$, we have, for all $f, g \in C^0(\mathbb{T}^N)$,

$$\int_{\mathbb{T}^N} (f \circ \phi_M) \ g \ d\lambda = \int_{\mathbb{T}^N} f \ (L_M \, g) \ d\lambda \ .$$

(4) We equip $\mathcal{L}(\mathbb{R}^N)$ with the usual operator norm. Show that there exist $c > 0$ and $\lambda > 1$ such that for all $f \in C^1(\mathbb{T}^N)$ and $n \in \mathbb{N}$, we have

$$\left| \ L_M^n f(x) - \int_{\mathbb{T}^N} f \ d\lambda \ \right| \leq \frac{c}{\lambda^n} \sup_{x\in\mathbb{T}^N} \|d_x f\|$$

(5) Deduce a new proof of the mixing property of ϕ_M for the Haar measure on the torus.

(6) Deduce that the transformation ϕ_M is exponentially mixing for the Haar measure on the torus, in the vector space $C^1(\mathbb{T}^N)$ equipped with the Sobolev norm $W^{1,\infty}$, defined by $\|f\|_{1,\infty} = \max\{\|f\|_\infty, \|df\|_\infty\}$, where $\|df\|_\infty = \sup_{x\in\mathbb{T}^N} \|d_x f\|$.

Exercice 4 Let $\mathscr{A}$ be a finite alphabet with cardinality at least 2, equipped with a probability measure ν, and let $(X = \mathscr{A}^{\mathbb{Z}}, \mu = \nu^{\mathbb{Z}}, \phi)$ be the associated (bilateral) Bernoulli shift.

(1) For $a \in \mathscr{A}$, we denote $[a]$ the cylinder of sequences $(x_n)_{n\in\mathbb{Z}}$ such that $x_0 = a$. Let $f = \mathbb{1}_{[a]} - \mu([a])$ (so that $f \in \mathbb{L}_0^2(\mu)$). Show that the spectral measure of the function f is a multiple of the Haar measure on the circle.

(2) For $b \neq a \in \mathscr{A}$, let $C = [a, b]$ be the cylinder of sequences $(x_n)_{n\in\mathbb{Z}}$ such that $x_0 = a$ and $x_1 = b$. Let $f = \mathbb{1}_C - \mu(C)$. Show that there exist reals α, β such that the spectral measure of the function f is of the form $(\alpha \cos(2\pi\theta) + \beta)d\theta$. Determine these values.