TD4: Hyperbolic Dynamics (2)

Exercice 1 Let k, N be positive integers and M a compact differential manifold of dimension N. Recall that a fixed point x_0 of a C^1-diffeomorphism ϕ of M is hyperbolic if its tangent map $T_{x_0}\phi \in \mathscr{L}(T_{x_0}M)$ has no (complex) eigenvalue of modulus 1. Let $\mathrm{Diff}^k(M)$ be the Baire space[1] of C^k-diffeomorphisms of M equipped with uniform convergence on the compacts of the maps and their tangent maps up to order k.

(1) For all $p, q \in \mathbb{N}$, construct a C^1-diffeomorphism of the circle preserving orientation having exactly $2p$ fixed points, all hyperbolic, and, if $q \geq 1$, a C^1-diffeomorphism of the circle preserving orientation having exactly $p+q$ fixed points, with p hyperbolic and q non-hyperbolic.

(2) Let ϕ be a C^1-diffeomorphism of the circle preserving orientation having a finite non-zero number of fixed points, all hyperbolic.

- Show that ϕ admits an even number of fixed points, denoted $p_1, \dots, p_{2k}$ in trigonometric order.
- Show that there exists $\varepsilon > 0$ small enough such that the open arcs of the circle A_i of length 2ε centered at p_i are pairwise disjoint, and that if ψ is sufficiently C^1-close to ϕ, then ψ has exactly one fixed point p'_i in each arc of the circle of length 2ε centered at p_i and has no fixed point outside the A_i.
- By fixing a point a_i of the arc of the circle between p_i and p_{i+1} that does not belong to $A_i \cup A_{i+1}$, show that there exists a continuous map $h : \mathbb{S}_1 \to \mathbb{S}_1$ such that, for all $i = 1, \dots, 2k$, we have $h(p_i) = p'_i$, $h(a_i) = a_i$, $h(\phi(a_i)) = \psi(a_i)$, h is affine on the arc of the circle between a_i and $\phi(a_i)$, and $h \circ \phi = \psi \circ h$.
- Deduce that ϕ is structurally stable.[2]

(3) We return to the general framework. Let $\phi \in \mathrm{Diff}^k(M)$ and U an open subset of M with compact closure in which ϕ admits a unique fixed point x_ϕ such that $x_\phi \in U$ and the tangent map of ϕ at this point has no eigenvalue equal to 1 (respectively is hyperbolic). Show that there exists a neighborhood V of ϕ in $\mathrm{Diff}^k(M)$ such that every $\psi \in V$ admits a unique fixed point x_ψ in $\overline{U}$, and such that the tangent map of ψ at this point has no eigenvalue equal to 1 (respectively no eigenvalue of modulus equal to 1). We can successively, if 1 is not an eigenvalue of $d_{x_\phi}\phi$, and if ψ is C^1-close to ϕ:

- reduce to the case where $M = \mathbb{R}^N$ and $x_\phi = 0$;
- show that the map $\widetilde{\psi} : x \mapsto \psi(x) - x$ is a diffeomorphism of an open neighborhood of 0 onto an open neighborhood of 0, and deduce that ψ admits a fixed point x_ψ;
- show that ψ admits at most one fixed point in a neighborhood of 0;

[1] Recall that every smooth differential manifold is assumed to be metrizable and separable (hence σ-compact and with a countable base of neighborhoods), and equipped with a distance inducing its topology. For all smooth differential manifolds M, N and maps $f, g : M \to N$ of class C^k, we define by recursion the iterated tangent spaces by $T^0M = M$ and $T^nM = T(T^{n-1}M)$ if $n \geq 1$, and the iterated tangent maps by $T^0f = f$ and $T^nf = T(T^{n-1}f) : T^nM \to T^nN$ if $1 \leq n \leq k$. By denoting K_i a compact subset of T^iM for all $i = 0, \dots, k$, and $\mathcal{K} = (K_i)_{0 \leq i \leq k}$, let $d_{\mathcal{K}}(f, g) = \sum_{i=0}^{k} \sup_{x \in K_i} d(T^if(x), T^ig(x))$. The C^k *topology* (weak) on the set $C^k(M, N)$ of C^k maps from M to N is the metrizable topology defined by the family of pseudo-distances $(d_{\mathcal{K}})_{\mathcal{K}}$. We show that $\mathrm{Diff}^k(M)$ is open in $C^k(M, M)$, that it is a Baire space, and that the composition of C^k maps is continuous for the C^k topology. In practice, to work with the C^k topology, we look at the uniform convergence, in the compacts of local charts, of the maps and all their partial derivatives up to order k. The same definition applies if $k = \infty$, except that the topology is not necessarily metrizable, and the manifold is not Banach but Fréchet.

[2] For all $k \in (\mathbb{N} - \{0\}) \cup \{\infty\}$, a differentiable dynamical system (X, ϕ) of class C^k is said to be *structurally stable* if there exists a neighborhood V of ϕ in $C^k(X, X)$ for the C^k topology such that every element $\psi \in V$ is topologically conjugate to ϕ. In general, the conjugation is not of class C^k.

• show that if moreover $d_{x_\phi}\phi$ has no eigenvalue of modulus equal to 1, and if ψ is C^1-close to ϕ, then $d_{x_\psi}\psi$ also has no eigenvalue of modulus equal to 1, by considering the maps $x \mapsto \psi(x) - \lambda x$ for all $\lambda \in \mathbb{S}_1$.

(4) Show that the subspace E_1 (respectively E_2) of $\mathrm{Diff}^k(M)$ of C^k-diffeomorphisms of M having a finite number of fixed points and at which the tangent map of ϕ has no eigenvalue equal to 1 (respectively is hyperbolic) is an open dense set. To show density, we can successively

• by applying Sard's theorem[3] to the map $f : x \mapsto x - \psi(x)$, show that for every map $\psi : \Omega \to \mathbb{R}^N$ of class C^1, where we assume that $M = \Omega$ is an open subset of $\mathbb{R}^N$, there exist y arbitrarily close to 0 such that the map $\psi_y : x \mapsto \phi(x) + y$ has, in every compact subset of Ω, a finite number of fixed points, at which the differential of ψ does not have the eigenvalue 1;

• show that every point of M admits a neighborhood U with compact closure such that the set E_U of C^k-diffeomorphisms of M that have only a finite number of fixed points in $\overline{U}$, these belonging to U, and at which the differential of ϕ does not have the eigenvalue 1, is an open dense subset of $\mathrm{Diff}^k(M)$;

• deduce the desired density, using after verification, in the case of hyperbolic fixed points, that for every $\varepsilon > 0$, there exists a function $\varphi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ equal to x outside $B(0, 2\varepsilon)$, equal to $(1-\varepsilon)x$ on $B(0,\varepsilon)$ and converging in C^k topology to the identity as ε tends to 0.

(5) Show that there exists a dense G_δ subset of $\mathrm{Diff}^k(M)$ consisting of elements having, for all $n \in \mathbb{N}$, a finite number of periodic points of period n, all hyperbolic.

(6) Show that if ϕ is a structurally stable C^k-diffeomorphism of M, then ϕ admits, for all $n \in \mathbb{N}$, a finite number of periodic points of period n.

Exercice 2 Let $\mathcal{M} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\phi = \phi_M : x \mod \mathbb{Z}^2 \mapsto \mathcal{M}x \mod \mathbb{Z}^2$ be the diffeomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ defined by $\mathcal{M}$.

(1) Show that $0 \in \mathbb{T}^2$ is a hyperbolic fixed point of ϕ, and determine the stable space E_0^s and unstable space E_0^u in $T_0\mathbb{T}^2 = \mathbb{R}^2$.

(2) Compute the stable manifold $W^s(0)$ and unstable manifold $W^u(0)$ of 0, and show that they are dense.

[3] If $f : M \to N$ is a C^1 map between two smooth differential manifolds, a *critical value* of f is a point $y \in N$ such that there exists a point $x \in M$ such that $f(x) = y$ and $T_x f : T_x M \to T_y N$ is not surjective. A *regular value* is a point of N which is not a critical value. The *Sard's theorem* states that if M and N are two manifolds of dimensions m and n, and if $f : M \to N$ is a map of class C^k where $k > \max\{0, m-n\}$, then the set of regular values of f is dense in N (in fact of full measure for any measure on N absolutely continuous with respect to the Lebesgue measure in each chart.)