TD 4 : Entropy, subshifts

Exercice 1 Let $(X, \mathscr{B}, \mu, \phi)$ and $(Y, \mathscr{C}, \nu, \psi)$ be two probability measure-preserving dynamical systems at discrete time.

(1) Consider the product dynamical system $(X \times Y, \mathscr{B} \otimes \mathscr{C}, \mu \otimes \nu, \phi \times \psi)$.
Prove that
$$h_{\mu\otimes\nu}(\phi \times \psi) = h_\mu(\phi) + h_\nu(\psi).$$

(2) Now assume that Y is a factor of X. Prove that $h_\nu(\psi) \leq h_\mu(\phi)$.

Exercice 2 (Pinsker Factor) Let $(X, \mathscr{B}, \mu, \phi)$ be a probability measure-preserving dynamical system. Denote $\Pi(\phi) = \{B \in \mathscr{B} \mid h_\mu(\phi, \{B, B^c\}) = 0\}$ (where by convention if $B = \varnothing$ or $B = X$ we set $\{B, B^c\} = \{X\}$).

(1) Prove that $\Pi(\phi)$ is a ϕ-invariant σ-algebra.

(2) Prove that $(X, \Pi(\phi), \mu, \phi)$ is a factor of X with zero entropy, and it is maximal for this property (in the sense that if $(Y, \mathscr{C}, \nu, \psi)$ is a factor of X with zero entropy, and $h : X \to Y$ is a semi-conjugation, then h is $\Pi(\phi)$-measurable).

Exercice 3 Let $X = \mathbb{S}^1$ be the unit circle equipped with the Lebesgue measure μ, and let ϕ be the translation $x \mapsto x + \theta$ for some $\theta \in \mathbb{S}^1$. Compute the entropy $h_\mu(\phi)$.

Exercice 4 Let $\mathscr{A} = \{a, 1, 2, b\}$. Consider the subshift Σ defined as the set of sequences $(x_n) \in \mathscr{A}^{\mathbb{Z}}$ such that for all $n \in \mathbb{Z}$:

- if $x_n = a$ then $x_{n+1} = 1$,
- if $x_n = 1$ then $x_{n+1} = 1$ or $x_{n+1} = b$,
- if $x_n = b$ then $x_{n+1} = 2$,
- if $x_n = 2$ then $x_{n+1} = a$.

Equip Σ with the σ-algebra $\mathscr{B}$ generated by cylinders, a Markov measure μ, and the shift map ϕ.

(1) Verify that $(\Sigma, \mathscr{B}, \mu, \phi)$ is mixing.

(2) Let $A \subset \Sigma$ be the set $\{(x_n) \mid x_0 = a \text{ or } b\}$. Prove that the induced dynamical system $(A, \mathscr{B}_A, \mu_A, \phi_A)$ is not mixing.

Exercice 5 Let $\Omega \subset \{0, 1\}^{\mathbb{N}}$ be the set of sequences $(\omega_i)_{i\geq 0}$ such that $\omega_i\omega_{i+1} = 0$ for all i. We equip Ω with the shift θ.

(1) For $p \in (0, 1)$, show that there exists a Markov measure ν_p with support equal to Ω, and whose associated incidence matrix gives probability p to remain in state 0 from state 0. Then compute the entropy $h_{\nu_p}(\theta)$.

(2) Show that there exists p_0 such that $h_{\nu_{p_0}}(\theta)$ is maximal, and calculate this maximum. From now on, we denote $\nu = \nu_{p_0}$.

(3) Let $I = [0, 1]$, and let β be the golden ratio (satisfying $\beta > 1$ and $\beta^2 = \beta + 1$). Define the map $\psi : \Omega \to I$ by
$$\psi((\omega_k)_{k\geq 0}) = \sum_{k=0}^{\infty} \frac{\omega_k}{\beta^{k+1}}$$
and let $T : I \to I$ be the map $x \mapsto \{\beta x\}$ (where $\{z\}$ is the fractional part of z).

(a) Show that $\psi : \Omega \to I$ is well-defined, then that it is surjective and continuous. Describe the image measure $\mu = \psi_* \nu$.

(b) Verify that the dynamical systems (Ω, θ, ν) and (I, T, μ) are conjugate. What is the value of $h_\mu(T)$?