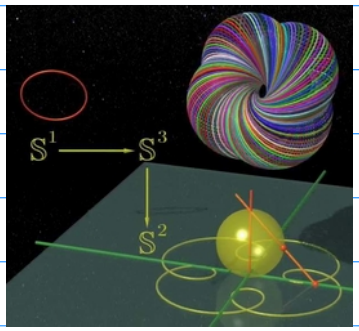
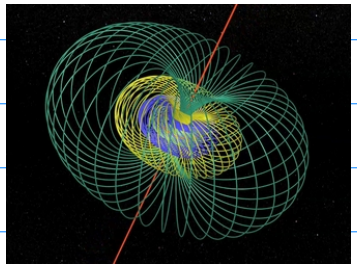
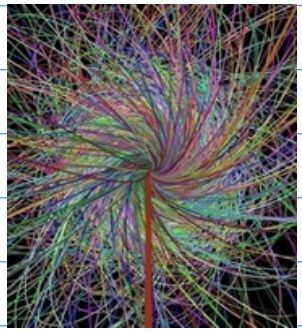
Groups and Geometry course of November 10th, 2026

A list of things assumed to be known

* $f: M \to N$ C^∞ fiber bundle $\Rightarrow$ surjective C^∞ submersion (Ehresmann theorem: converse OK if f proper)
* Theo: If M C^∞ manifold with proper, free, smooth action of G Lie group, then $G\backslash M$ has unique C^∞ manifold structure st $\pi: M \xrightarrow{can} G\backslash M$ C^∞ submersion. Besides π C^∞ fiber bundle, called principal fiber bundle
* A principal fiber bundle is trivialisable $\Leftrightarrow \exists$ C^∞ section

Ex: Hopf fibration $S^{2n+1} \to \mathbb{P}_n(\mathbb{C})$ with fiber S_1. cf Leys-Ghys-Alvarez




* Def: vector bundle ξ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with rank n, projection $p: E \to B$ (base), fiber $\xi_b = (p^{-1}(b), +, \cdot)$, distinguished open set in B, local trivialisation, morphism of vector bundles, $C^\infty(B;\mathbb{K})$-module of sections $\Gamma(\xi)$, trivial/trivialisable vector bundle + trivialisation criterion by $C^\infty(B;\mathbb{K})$-basis of $\Gamma(\xi)$

Ex: tangent bundle $TM \to M$, tangent morphism (Tf, f), $C^\infty(M;\mathbb{R})$-module $\Gamma(TM)$, parallelisable manifold (G Lie group $\Rightarrow$ G parallelisable)

Ex: tautological vector bundle $T_{\mathbb{K}}(V)$ over $Gr_k(V)$

vector subbundle, quotient vector bundle, Kernel + image subbundle for fiberwise constant rank morphism

* Background: multilinear algebra, dual, tensor products, pure tensor, (s,t)-tensor powers, exterior/symmetric powers.
* Def: "linear algebra" construction of vector bundles and sections: direct sum, multilinear bundles, dual, cotangent bundle, power tensor products, (s,t)-tensors, exterior power, symmetric power
* Euclidean metric on vector bundles (existence, duality isomorphism)
* Def: pull-back vector bundles.
* Lem: Every vector bundle with contractible basis is trivialisable

Theo: Classification of vector bundles over compact base.

Coro: Every vector bundle over compact base is subbundle of trivialisable one

3 Connections (Connexion in french)

All manifolds and vector bundles are real, C^∞ and $C^\infty(M) = C^\infty(M;\mathbb{R})$

3.1 Tensoriality lemma and applications

<< Fundamental lemma of differential geometry >>
A. Besse
(N. Bourbaki of Riem. Geom)

Tensoriality lemma

Let $\xi_0, \dots, \xi_m$ vector bundles with base M and $A : \Gamma(\xi_1) \times \dots \times \Gamma(\xi_m) \to \Gamma(\xi_0)$ m-linear/$\mathbb{R}$

(1) A is $C^\infty(M)$-multilinear

What does a section eat? What does it spit?

(2) $\exists!\, \alpha \in \Gamma(\xi_1^* \otimes \dots \otimes \xi_m^* \otimes \xi_0) = \Gamma(\mathcal{L}(\xi_1, \dots, \xi_m; \xi_0))$ s.t. $\forall x \in M$ $\forall \sigma_i \in \Gamma(\xi_i)$, $A(\sigma_1, \dots, \sigma_m)(x) = \alpha(x)(\sigma_1(x), \dots, \sigma_m(x))$

From now, write $A = \alpha$.

Dem: Let $\Phi : \Gamma(\mathcal{L}(\xi_1, \dots, \xi_m; \xi_0)) \to \mathcal{L}_{C^\infty(M)}(\Gamma(\xi_1), \dots, \Gamma(\xi_m); \Gamma(\xi_0))$

$$\alpha \mapsto \Big((\sigma_1, \dots, \sigma_m) \mapsto \big(x \mapsto \alpha(x)(\sigma_1(x), \dots, \sigma_m(x)) \big) \Big)$$

Every element in image is indeed $C^\infty(M)$-multilinear.

Lem 1: If A satisfies (1), then $\forall \sigma_i \in \Gamma(\xi_i)$, the value of $A(\sigma_1, \dots, \sigma_m)$ at $x \in M$ depends only on the values of σ_i at x.

Pf: By induction, $m = 1$. Let $\sigma, \sigma' \in \Gamma(\xi_1)$ $\sigma(x) = \sigma'(x)$

Case 1: If σ, σ' coincide on open neigh U of x, OK.

Let $\varphi \in C^\infty(M)$ with compact support in U, with value 1 near x. Then $\varphi(\sigma - \sigma') = 0$, hence ($C^\infty(M)$-linear)

$$A(\sigma')(x) - A(\sigma)(x) = \varphi(x) A(\sigma')(x) - \varphi(x) A(\sigma)(x) = A(\varphi(\sigma - \sigma'))(x) = 0$$

Case 2: If $\exists s_1, \dots, s_n \in \Gamma(\xi_1)$, $\exists f_1, \dots, f_n \in C^\infty(M)$ s.t. $f_1(x) = \dots = f_n(x) = 0$ and σ' coincides with $\sigma + \sum_{i=1}^n f_i s_i$ near x, OK.

Then $A(\sigma')(x) = A(\sigma + \sum_{i=1}^{n} f_i s_i)(x)$

Case 1 $= A(\sigma(x)) + \sum_{i=1}^{n} f_i(x) A(s_i)(x) = A(\sigma(x))$. ($C^\infty(M)$-linear)

No other case: Let n = rank of ξ_1, U open neigh of x over which ξ_1 is trivialisable, $(s'_1, \dots, s'_n)$ $C^\infty(M)$-basis of $\Gamma(\xi_1|_U)$, $f'_1, \dots, f'_n \in C^\infty(U)$ st $(\sigma' - \sigma)|_U = \sum_{i=1}^{n} f'_i s'_i$. Then $f'_1, \dots, f'_n$ vanish at x. Let φ as in case 1, $f_i = \varphi f'_i \in C^\infty(M)$ $s_i = \varphi s'_i \in \Gamma(\xi_1)$ works for case 2. □

Lemma: $\forall \xi$ vector bundle with base M, $\forall x \in M$, $\forall v \in \xi_x$ $\exists \sigma \in \Gamma(\xi)$ st $\sigma(x) = v$.

Pf: OK on open neigh of x, then multiply by φ. □

Lem 1+2 $\Rightarrow$ (*) bijection

Examples of use: Let ξ vector bundle $/M$

1) exterior differential

$\forall \alpha \in \Gamma(\Lambda^p T^*M)$ differential p-form on M, $d\alpha$ is the unique $(p+1)$-form on M st

$\forall X_0, \dots, X_n \in \Gamma(TM)$,

$$d\alpha(X_0, \dots, X_n) = \sum_{i=0}^{p} (-1)^i X_i\big(\alpha(X_0, \dots, \widehat{X_i}, \dots, X_p)\big) + \sum_{0 \le i < j \le p} (-1)^{i+j} \alpha([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p)$$

($\widehat{\ }$ omit; $d\alpha_x(X_0(x), \dots, X_n(x))$ (ee) tensoriality lemma)

2) exterior product

$\forall \alpha$ diff p-form on M with values in $\mathrm{End}(\xi)$ ($\alpha \in \Gamma(\Lambda^p T^*M \otimes \mathrm{End}(\xi))$, $\mathrm{End}(\xi) = \xi^* \otimes \xi$)

$\forall \beta$ diff q-form on M $\quad = \xi$

$\alpha \wedge \beta$ = unique diff $(p+q)$-form $\quad = \xi$

$\forall X_1, \dots, X_{p+q} \in \Gamma(TM)$,

$\xi = \xi_{\mathbb{R}}$ trivial line bundle over M

$$\alpha\wedge\beta(X_1,\dots,X_{p+q}) = \frac{1}{p!q!}\sum_{\sigma\in\mathfrak{S}_{p+q}} \varepsilon_\sigma\, \alpha(X_{\sigma(1)},\dots,X_{\sigma(p)})\Big(\beta(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)})\Big)$$

ε_σ: signature of σ

$\mathrm{Bij}(\llbracket 1,p+q\rrbracket)$

3) contraction of tensors

Let $s,t\in\mathbb{N}\setminus\{0\}$, $i_0\in\llbracket 1,s\rrbracket$, $j_0\in\llbracket 1,t\rrbracket$

$\exists!$ morphism of $C^\infty(M)$-modules

$$C^{j_0}_{i_0}:\Gamma\big((\Xi^*)^{\otimes s}\otimes\Xi^{\otimes t}\big)\to\Gamma\big((\Xi^*)^{\otimes s-1}\otimes\Xi^{\otimes t-1}\big)$$

such that $\forall \ell_1,\dots,\ell_s\in\Gamma(\Xi^*)$, $\forall\sigma_1,\dots,\sigma_t\in\Gamma(\Xi)$

$$C^{j_0}_{i_0}(\ell_1\otimes\cdots\otimes\ell_s\otimes\sigma_1\otimes\cdots\otimes\sigma_t) = \ell_{i_0}(\sigma_{j_0})\;\ell_1\otimes\cdots\widehat{\ell_{i_0}}\otimes\ell_s\otimes\sigma_1\otimes\cdots\widehat{\sigma_{j_0}}\cdots\otimes\sigma_t$$

$\ell_{i_0}(\sigma_{j_0})\in C^\infty(M)$

3.2 Koszul connections or covariant derivation

Let Ξ vector bundle with $p:E\to M$ projection

Def: A **connection** on Ξ is $\nabla:\Gamma(TM)\times\Gamma(\Xi)\to\Gamma(\Xi)$ $\mathbb{R}$-bilinear and $\forall\sigma\in\Gamma(\Xi)$, $\forall X\in\Gamma(TM)$, $\forall f\in C^\infty(M)$

(1) $\nabla_{fX}\sigma = f\nabla_X\sigma$ $C^\infty(M)$-linearity in X

(2) $\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma$ derivation property on σ (Leibnitz rule)

connection on M = connection on TM

Ex: (a) If Ξ trivial and $\varphi:pr_1:M\times F\to M$, then $\Gamma(\Xi)=C^\infty(M;F)$ and $\nabla^0_X\sigma: x\mapsto d_x\sigma(X(x))$ ($F\simeq\mathbb{R}^n$)

trivial connection

(b) Let ξ, ξ' vector bundles / M, endowed with connections ∇, ∇'. Then by tensoriality lemma, have connections ∇^*, $\nabla \oplus \nabla'$, $\nabla \otimes \nabla'$ on vector bundles ξ^*, $\xi \oplus \xi'$, $\xi \otimes \xi'$ by

$\forall s \in \Gamma(\xi)$, $s' \in \Gamma(\xi')$, $\ell \in \Gamma(\xi^*)$, $X \in \Gamma(TM)$,

$$\left(\nabla^*_X \ell\right)(s) = X\left(\ell(s)\right) - \ell\left(\nabla_X s\right)$$

dual connection

$$\left(\nabla \oplus \nabla'\right)_X (s \oplus s') = \nabla_X s \oplus \nabla_X s'$$

direct sum connection

$$\left(\nabla \otimes \nabla'\right)_X (s \otimes s') = \nabla_X s \otimes s' + s \otimes \nabla_X s'$$

tensor product connection

By induction, the connection ∇ on ξ defines a connection, also denoted by ∇, on $T^q_p \xi = (\xi^*)^{\otimes p} \otimes \xi^{\otimes q}$ called (p,q)-tensor product connection.

Ex: Let $\xi = TM$. If ω alternate $(m,0)$-tensor on M = diff m-form on M, then $\forall X_1, \dots, X_m \in \Gamma(TM)$,

$$\nabla_X \omega(X_1, \dots, X_m) = X\left(\omega(X_1, \dots, X_m)\right) - \sum_{i=1}^{m} \omega(X_1, \dots, \nabla_X X_i, \dots X_m)$$

Let R $(m,1)$-tensor on M, then

$$\nabla_X R(X_1, \dots X_m) = \nabla_X \left(R(X_1, \dots, X_m)\right) - \sum_{i=1}^{m} R(X_1, \dots, \nabla_X X_i, \dots, X_m)$$

$\xi^* \otimes \xi$

Exercice: If $u \in \Gamma(\mathrm{End}\,\xi)$, prove that $\forall X \in \Gamma(TM)$, $\forall s \in \Gamma(\xi)$

$$\left(\nabla_X u\right)(s) = \nabla_X\left(u(s)\right) - u\left(\nabla_X s\right)$$

Let ∇ connection on ξ

Lem (locality lemma of connections): $\forall x \in M$, value of $\nabla_X \sigma$ at x depends only $\forall \sigma \in \Gamma(\xi)$ on value of X at x, values of σ at x, of $T_x \sigma$ at $X(x)$.

Pf: similar to Lem 1 in tensoriality lemma. □

Rem 1: $\forall v \in T_x M$, $\nabla_v \sigma(x) = \nabla_X \sigma(x)$ for any $X \in \Gamma(TM)$ st $X(x) = v$.

Rem 2: $\forall \sigma \in \Gamma(\xi)$: $\nabla\sigma : \Gamma(TM) \to \Gamma(\xi)$, $X \mapsto \nabla_X \xi$ defines by tensoriality lemma an element of $\Gamma(T^*M \otimes \xi)$ = diff 1-form with values in ξ.

Def: $\sigma \in \Gamma(\xi)$ is parallel for ∇ if $\nabla\sigma = 0$

Rem 3: {connections on ξ} is affine space with associated vector space $\Gamma(T^*M \otimes \xi^* \otimes \xi)$: if ∇, ∇' connections, $\nabla' - \nabla : \Gamma(TM) \times \Gamma(\xi) \to \Gamma(\xi)$ are $C^\infty(M)$-bilinear, Hence by tensoriality lemma, $\exists!\ A \in \Gamma(T^*M \otimes \xi^* \otimes \xi)$ diff 1-form with values in $\xi^* \otimes \xi = End(\xi)$ st $\forall X \in \Gamma(TM)$, $\forall \sigma \in \Gamma(\xi)$

$$\nabla'_X \sigma - \nabla_X \sigma = A(X, \sigma)$$

Conversely, $(X, \sigma) \mapsto \nabla_X \sigma + A(X, \sigma)$ is a Koszul connection.

3.4 # Ehresmann connections

geometric interpretation of Koszul connections

Let ξ vector bundle, projection $p: E \to M$.
We have a commutative diagram

$$\begin{array}{ccc} TE & \xrightarrow{Tp} & TM \\ \pi_E \downarrow & & \downarrow \pi_M \\ E & \xrightarrow{p} & M \end{array}$$

$T_v p : T_v E \to T_{p(v)} M$, $V_v = \ker T_v p$

$V = \bigcup_{v \in E} V_v \subset V \cap TE$, $\pi_E|_V : V \to E$

Def: * The **vertical bundle** of ξ is the vector subbundle V of π_E, kernel of vector bundle morphism (Tp, p). OK since p submersion $\Rightarrow Tp$ fiberwise constant rank.

ZZ

Ex: If ξ trivial, $\forall (x,X) \in E = M \times F$, $\overset{v}{\simeq} \mathbb{R}^{\operatorname{rank} E}$

$T_{x,X} E = T_x M \times T_X F \overset{can}{\simeq}_F V_{x,X} = \{0\} \times T_X F$ $\quad H_{x,X} = T_x M \times \{0\}$ (any graph of linear map $M \to F$ will do)

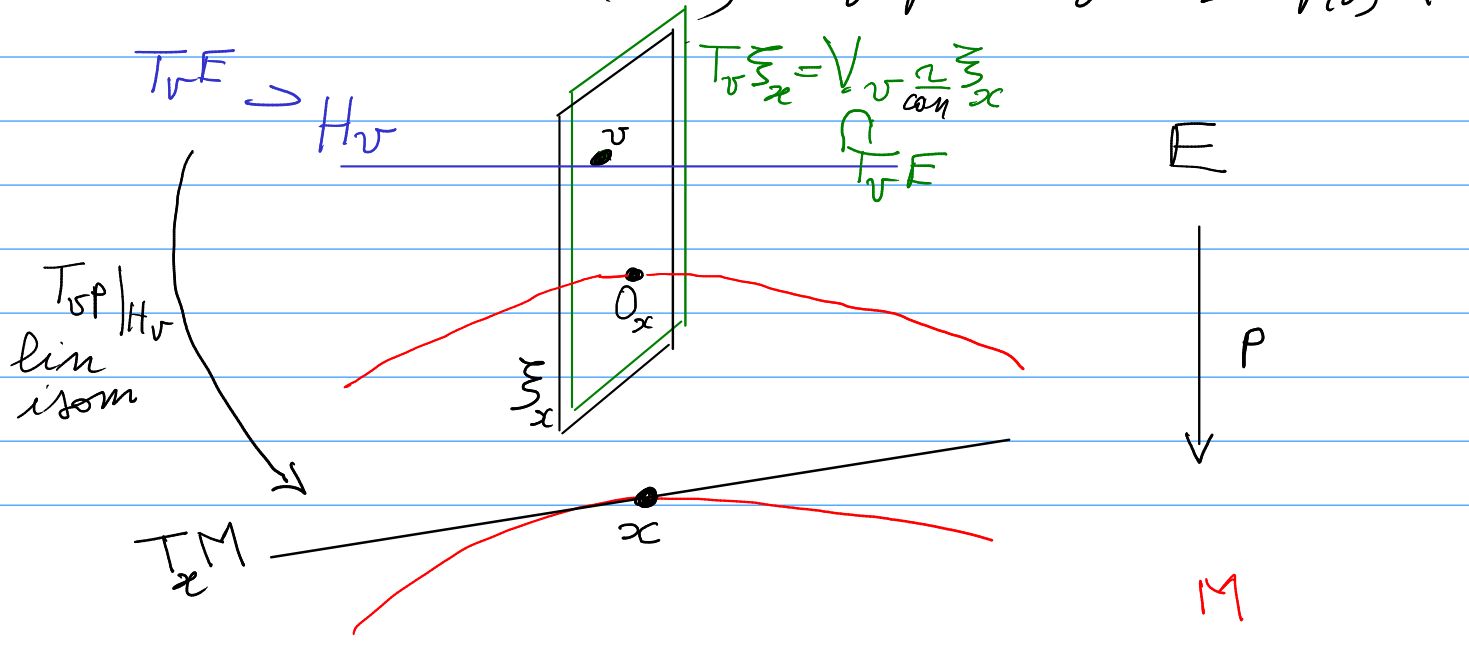
* **A horizontal bundle** (or **Ehresmann connection on ξ**) is a vector subbundle H of π_E supplementary to V: $\boxed{TE = V \oplus H}$

$H \downarrow \pi_E|_H$, E

Rem: $\operatorname{rank} V = \operatorname{rank} \xi$ and $\forall v \in E$

$V_v = T_v \xi_{p(v)} \overset{can}{\simeq} \xi_{p(v)}$ lin isom

$\operatorname{rank} H = \dim M$, $T_v p : H_v \xrightarrow{\sim} T_{p(v)} M$



Def: An Ehresmann connection H of ξ is linear if $\forall \lambda \in \mathbb{R}$, $\forall (v,w) \in E \times_p E = \{(v,w) \in E \times E \mid p(v) = p(w)\}$, $\forall (a,b) \in T_{(v,w)}(E \times_p E)$, if $a \in H_v$ and $b \in H_w$, then $\lambda \cdot a \in H_{\lambda v}$ and $a + b \in H_{v+w}$

($\lambda \cdot$: tangent map of $X \mapsto \lambda X$ on E; $+$: tangent map to $+$ on $E \times_p E$)

Correspondence theorem:

(1) If H linear Ehresmann connection on ξ, if pr_H $TE \to V$ is the fiberwise projection on V parallel to H (via E), then

$$\nabla^H : \Gamma(TM) \times \Gamma(\xi) \to \Gamma(\xi)$$
$$(X, \sigma) \mapsto \left(x \mapsto pr_H \circ T_x\sigma(X(x)) \right)$$

($T_x\sigma(X(x)) \in T_{\sigma(x)}E$; $\Gamma \ni x \mapsto T_x\sigma(X(x)) \in \xi_x = V_{\sigma(x)}$ canonically)

is Koszul connection.

(2) For every Koszul connection ∇ on ξ, $\exists!$ linear Ehresmann connection H on ξ s.t. $\nabla = \nabla^H$.

Dem: (1) $\mathbb{R}$-bilinear: OK since $\forall \sigma, \sigma' \in \Gamma(\xi)$, $\forall \lambda \in \mathbb{R}$ $T(\lambda\sigma) = \lambda \cdot T\sigma$ and $T(\sigma + \sigma') = T\sigma + T\sigma'$ by chain rule. Since H and V linear $\Rightarrow$ $\forall (v,w) \in E \times_p E$, $\forall (a,b) \in T_{(v,w)}E \times_p E$

(exercise)

$$pr_{H_{\lambda v}}(\lambda \cdot a) = \lambda \cdot pr_{H_v}(a) = \lambda\, pr_{H_v}(a)$$

$$pr_{H_{v+w}}(a + b) = pr_{H_v}(a) + pr_{H_w}(b) = pr_{H_v}(a) + pr_{H_w}(b)$$

(seen in $\xi_{p(v)} = \xi_{p(w)}$)

$\nabla^H_X \sigma$ is $C^\infty(M)$-linear in X since $T_x\sigma$ is $\mathbb{R}$-linear and $pr_{H_{\sigma(x)}}$ too.

$\nabla^H_X \sigma$ is derivation in σ since $\forall f \in C^\infty(M)$

$$T_x(f\sigma)(X(x)) = X(f)(x)\,\sigma(x) + f(x) \cdot T_x\sigma(X(x))$$

OK.

(2) One proves $\forall v \in E$, given ∇ Koszul connection

$H_v = \{ a \in T_v E \mid \exists \sigma \in \Gamma(\xi), \exists X \in \Gamma(TM), v = \sigma(p(v)) \}$ works. $T_{p(v)}\sigma(X(x)) = a$, $\nabla_X \sigma(p(v)) = 0$ $\}$ □

Ex 1. If ξ trivial vector bundle, $p: M \times \mathbb{R}^n \to M$

$V_{x,y} = \{0\}_x \times T_y \mathbb{R}^n_{\simeq \mathbb{R}^n}$, $H_{x,y} = T_x M \times \{0\}_y$, $H = \bigcup_{(x,y) \in E} H_{x,y}$

is a linear Ehresmann connection, and

∇^H = trivial connection ∇^0 on ξ, since $\forall \sigma \in \Gamma(\xi)$

$pr_H \circ T_x \sigma(X(x)) = d_x \sigma(X(x))$ $\in C^\infty(M; \mathbb{R}^n)$ standard Euclidean space

Ex 2: Let M k-dim C^∞ submanifold of $\mathbb{R}^n$ and ξ = tangent bundle of M. Then

$TM = \{ (x,u) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \in M, u \in T_x M \}$

and $V = \{ (x,u,v,w) \in (\mathbb{R}^n)^4 \mid x \in M, u \in T_x M, v = 0, w \in T_x M \}$

Claim: $H = \{ (x,u,v,w) \in TTM \mid w \in T_x M^\perp \}$ is a linear Ehresmann connection

Dem: locally, $M = f^{-1}(0)$ where $f: \mathbb{R}^n \to \mathbb{R}^{n-k}$ C^∞ submersion

$TM = \{ (x,u) \in (\mathbb{R}^n)^2 \mid f(x) = 0, d_x f(u) = 0 \} = F^{-1}(0,0)$ where

$TTM = \{ (x,u,v,w) \in (\mathbb{R}^n)^4 \mid f(x) = 0, d_x f(u) = 0, d_x f(v) = 0, d_x f(w) + d^2_x f(u,v) = 0 \}$

$F: (x,u) \mapsto (f(x), d_x f(u))$

$(v,w) \in \ker T_{x,u}(F) = T_{x,u}(TM)$.

if $w \in T_x M^\perp$, this is equivalent to $w = -(d_x f|_{T_x M^\perp})^{-1}(d^2_x f(u,v))$

$\lambda \cdot (x,u,v,w) = (x, \lambda u, v, \lambda w)$

$(x,u,v,w) + (x,u',v,w') = (x, u+u', v, w+w')$.

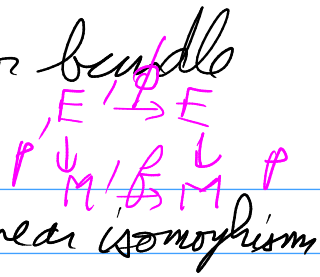
Since $w \in T_x M^\perp$ is linear in w, H linear. □

3.3 Pull-back connections

Let ξ, ξ' vector bundles, with projections

$p: E \to M$, $p': E' \to M'$. Let (ϕ, f) vector bundle morphism from $\xi' \to \xi$ st

$$\forall y \in M', \ \phi_y : \xi'_y \to \xi_{f(y)} \text{ linear isomorphism}$$



Let ∇ connection on ξ

Def: The pull-back connection of ∇ by (ϕ, f) is the connection $\phi^*\nabla$ on ξ' st $\forall X' \in \Gamma(TM')$, $\forall \sigma' \in \Gamma(\xi')$, $\forall x' \in M'$,

$$\boxed{(\phi^*\nabla)_{X'} \sigma'(x') = \phi^{-1} \circ \nabla_X \sigma(f(x'))}$$

where $X \in \Gamma(TM)$ satisfy $X(f(x')) = T_{x'} f(X'(x'))$ and $\sigma \in \Gamma(\xi)$ satisfies $\sigma(f(x')) = \phi \circ \sigma'(x')$ and if $X(f(x')) \neq 0$, then $T_{f(x')} \sigma \left(X(f(x')) \right) = T_{x'}(\phi \circ \sigma')(X'(x'))$

Proof well defined + connection too ugly.

Ex 1 If $M = M'$ and $f = id$, then

$$(\phi^*\nabla)_X \sigma' = \phi^{-1}\left(\nabla_X (\phi \circ \sigma')\right)$$

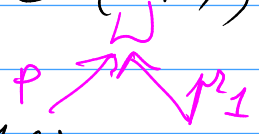
Def: Gauge group of ∇ = subgroup of $Aut(\xi)$ preserving ∇ ($\phi^*\nabla = \nabla$).

Ex 2 If $f: M' \to M$ C^∞ immersion, if $\xi' = f^*\xi$ pull back vector bundle, and $\phi : f^*E \to E$ with $\phi_x = id\ (f^*\xi)_x \to \xi_{f(x)}$, then $\phi^*\nabla =$ unique connection ∇' on ξ' st $\forall x' \in M'$, $\forall X' \in \Gamma(TM')$ $\forall \sigma \in \Gamma(\xi)$,

$$\boxed{\nabla'_{X'} (\sigma \circ f)(x') = \nabla_{T_{x'} f(X'(x'))} \sigma \ (f(x'))}$$

Ex 3 If U open in M, then $\xi|_U$ has unique connection $\nabla|_U$ restricted connection s.t.

$$\forall \sigma \in \Gamma(\xi),\ \forall X \in \Gamma(TM),\quad \boxed{(\nabla|_U)_{(X|_U)}(\sigma|_U) = (\nabla_X \sigma)|_U}$$



Ex 4 Let $\phi : \bar{p}^{-1}(U) \to U \times \mathbb{R}^n$ local trivialisation of ξ over U. On $\xi|_U$, have two connections: $\phi^* \nabla^0$ and $\nabla|_U$. Hence $\Gamma = \nabla|_U - \phi^* \nabla^0 \in \Gamma(T^*U \otimes (\xi|_U)^* \otimes (\xi|_U))$ Christoffel tensor of ∇ for ϕ.

Local chart formula of a connection

Let $(U, x^1, \dots, x^m)$ local chart of M s.t. $\xi|_U$ trivialisable, $m = \dim M$, $n = \operatorname{rank} \xi$

$\left(\partial_i = \frac{\partial}{\partial x^i}\right)_{1 \le i \le m}$ corresponding $C^\infty(U)$-basis of $\Gamma(TU)$

$(e_\alpha)_{1 \le \alpha \le n}$ can basis of $\mathbb{R}^n$. $\phi : \bar{p}^{-1}(U) \to U \times \mathbb{R}^n$

$\left(\varepsilon_\alpha : x \mapsto \phi^{-1}(x, e_\alpha)\right)_{1 \le \alpha \le n}$ corresponding $C^\infty(U)$-basis of $\Gamma(\xi|_U)$

Then $\exists!\ \Gamma_{i\alpha}^{\beta} \in C^\infty(U)$ s.t.

$$\nabla(\partial_i, \varepsilon_\alpha) = \sum_{1 \le \beta \le n} \Gamma_{i\alpha}^{\beta} \varepsilon_\beta$$

Christoffel symbols of ∇ for trivialising local chart so that

$$\nabla\left(\sum_{i=1}^{m} X^i \partial_i\right)\left(\sum_{\alpha=1}^{n} \sigma^\alpha \varepsilon_\alpha\right) =$$

$$\sum_{\substack{1 \le i \le m \\ 1 \le \alpha \le n}} X^i \frac{\partial \sigma^\alpha}{\partial x^i} \varepsilon_\alpha + \sum_{\substack{1 \le i \le m \\ 1 \le \alpha, \beta \le n}} \Gamma_{i\alpha}^{\beta} \sigma^\alpha X^i \varepsilon_\beta$$

3.5 Torsion of a connection on a manifold

Let M manifold and ∇ connection on M

Def: The torsion of ∇ is $T = T^{\nabla} \in \Gamma(T^*M^{\otimes 2} \otimes TM)$ s.t. $\forall X, Y \in \Gamma(TM)$,

$$\boxed{T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]}$$

∇ is torsion-free if $T^{\nabla} = 0$

$X(fY) - X(f)Y$ $\quad$ $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$

Rem:
- well defined by tensoriality lemma — T is $C^\infty(M)$-bilinear in X, Y
- $T(Y,X) = -T(X,Y)$
- local chart expression: Let $(U, (x^1, \dots, x^m))$ local chart of M, if $X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}$, $Y = \sum_{j=1}^{m} Y^j \frac{\partial}{\partial x^j}$,

then

$$\boxed{T(X,Y) = \sum_{1 \le i,j,k \le m} \Gamma_{ij}^k \left(X^i Y^j - Y^i X^j\right) \frac{\partial}{\partial x^k}}$$

where Γ_{ij}^k = Christoffel symbols of ∇ in local chart.

$$\begin{aligned} \mathcal{L}_{[X,fY]} &= \mathcal{L}_X \circ \mathcal{L}_{fY} - \mathcal{L}_{fY} \circ \mathcal{L}_X \\ &= \mathcal{L}_X (f \mathcal{L}_Y) - f \mathcal{L}_Y \circ \mathcal{L}_X \\ &= X(f) Y + \mathcal{L}_{[X,Y]} \end{aligned}$$

- If $U \subset \mathbb{R}^m$ open, then trivial connection ∇^0 on TU is torsion free (Schwarz lemma).

3.6 Curvature of a connection

Let ξ vector bundle, base M, and ∇ connection on ξ.

Def: The curvature of ∇ is $R = R^{\nabla} \in \Gamma\left(\Lambda^2 T^*M \otimes \xi^* \otimes \xi\right)$ differential 2-form with values in $End(\xi)$ s.t. $\forall X, Y \in \Gamma(TM)$, $\forall \sigma \in \Gamma(\xi)$

$$R^{\nabla}(X,Y,\sigma) = \nabla_X \nabla_Y \sigma - \nabla_Y \nabla_X \sigma - \nabla_{[X,Y]}\sigma$$

also denoted $R^{\nabla}_{X,Y}\sigma$ or $R^{\nabla}(X,Y)\sigma$ $\quad \in \Gamma(\xi)$

Rem: * well defined by tensoriality lemma in $\Gamma\left(\Lambda^2 T^*M \otimes End(\xi)\right) = \Gamma\left(\mathcal{L}(TM, TM, \xi; \xi)\right)$. clearly antisymmetric in X, Y. It is a $(2,1)$-tensor on M if $\xi = TM$

* Beware of sign conventions.

Def: The exterior differential associated with ∇ is $d^{\nabla}: \Gamma(\Lambda^* T^*M \otimes \xi) \to$ linear, graded of degree $+1$, $\forall \omega \in \Gamma(\Lambda^p T^*M \otimes \xi)$ $\forall X_0, \dots, X_p \in \Gamma(TM)$,

$$d^{\nabla}\omega(X_0, \dots, X_p) = \sum_{i=0}^{p} (-1)^i \nabla_{X_i}\left(\omega(X_0, \dots, \widehat{X_i}, \dots, X_p)\right)$$
$$+ \sum_{0 \le i < j \le p} (-1)^{i+j} \omega\left([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_p\right)$$

well-defined by tensoriality lemma.

$p=0$ $d^{\nabla}\omega(X) = \nabla_X \omega$, $\quad p=1$ $d^{\nabla}\omega(X,Y) = \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X,Y])$

Prop: (1) $\forall X, Y \in \Gamma(TM)$, $\forall \sigma \in \Gamma(\xi)$ $\quad R^{\nabla}_{X,Y}\sigma = d^{\nabla}(d^{\nabla}\sigma)(X,Y)$