Cours de Théorie ergodique du 9 novembre 2026

IV Measured linear dynamics of tori

Let $N \in \mathbb{N} \setminus \{0\}$, $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$ N-dim tori (metrisable compact abelian topological group)

$\theta = (\theta_1, \dots, \theta_N)$ N-tuple of real numbers mod 1

$\forall \theta \in \mathbb{T}^N$ let $\tau_\theta : \mathbb{T}^N \to \mathbb{T}^N$, $x \mapsto x + \theta$ translation by θ

$d\theta$ = probability Haar measure on $\mathbb{T}^N$

$\forall f \in C^0(\mathbb{T}^N)$, $\int_{\mathbb{T}^N} f \, d\theta = \int_{[0,1]^N} f(x)\,dx$ ($x \bmod \mathbb{Z}^N$)

1) Fourier analysis reminder

$\forall m = (m_1, \dots, m_N) \in \mathbb{Z}^N$, $\forall \mu \in \mathrm{Prob}(\mathbb{T}^N)$, $\forall f \in C^0(\mathbb{T}^N)$, let

$e_m : \theta \mapsto e^{2\pi i \langle m, \theta \rangle}$, $\langle m, \theta \rangle = \sum_{i=1}^N m_i \theta_i$ trigonometric function

$c_m(\mu) = \int_{\mathbb{T}^N} e_{-m} \, d\mu$

$$c_m(f) = \int_{\mathbb{T}^N} e_{-m} f \, d\theta$$

m-th Fourier coefficient

Prop: (1) $\mathrm{Prob}(\mathbb{T}^N) \to \mathbb{C}^{\mathbb{Z}^N}$, $\mu \mapsto (c_m(\mu))_{m \in \mathbb{Z}^N}$ is injective, continuous, homeo onto its image

(2) $\forall f \in C^{\lfloor \frac{N}{2} \rfloor + 1}(\mathbb{T}^N)$

$$f(\theta) = \sum_{m \in \mathbb{Z}^N} c_m(f) \, e_m(\theta)$$

Fourier series of f — inverse Fourier transformation (normal convergence, uniform convergence)

(3) $\forall f \in C^1(\mathbb{T}^N)$, $\forall m \in \mathbb{Z}^N$, $\forall i \in [\![1, N]\!]$

$$c_m(\partial_i f) = 2\pi m_i \, c_m(f)$$

(4) $\forall m, m' \in \mathbb{Z}^N$ $\int_{\mathbb{T}^N} e_m \overline{e_{m'}} \, d\theta = \begin{cases} 1 & \text{if } m = m' \\ 0 & \text{otherwise} \end{cases}$ ($e_m \overline{e_{m'}} = e_{m-m'}$)

orthogonality formula

---

2) Unique ergodicity

Prop: Let X nonempty compact metrisable topo space and $\phi : X \to X$ cont. Are equivalent

(1) The topo dyn sys (X, ϕ) is uniquely ergodic ($\exists!$ ergodic ϕ-invariant meas on X)

(2) $\exists!$ ϕ-invariant proba on X

(3) $\forall f \in C^0(X)$, $\exists L(f) \in \mathbb{C}$ $\forall x \in X$, ZZ rare event!

$$S_n f = \frac{1}{n} \sum_{k=0}^{n-1} f \circ \phi^k(x) \xrightarrow[n \to +\infty]{} L(f)$$

Dem: (1) $\Leftrightarrow$ (2) already seen Kryloo-Bogolubov + Krein-Milman

(3) $\Rightarrow$ (1)

Existence ↑

Uniqueness $\forall \mu, \mu'$ ergodic proba, by Birkhoff

$\forall f \in C^0(X) \; \forall x \in X \; S_n f(x) \to L(f)$

$\forall \mu\text{-ae } x \in X \quad S_n f(x) \to \mu(f)$

$\forall \mu'\text{-ae } x \in X \quad S_n f(x) \to \mu'(f)$

Hence $\mu(f) = L(f) = \mu'(f) \Rightarrow \mu = \mu'$.

(2) $\Rightarrow$ (3) Let μ ϕ-invariant proba, Let $x \in X$ $f \in C^0(X)$. Assume that $(S_n f(x))_{n \in \mathbb{N}}$ does not converge to $\mu(f)$. Since this sequence is bounded, $\exists \; S_{n_k} f(x) \xrightarrow[k \to +\infty]{} \ell_{f,x} \neq \mu(f)$

By compactness, up to extraction,

$$\frac{1}{n_k} \sum_{i=0}^{n_k - 1} \Delta_{\phi^i(x)} \xrightarrow[k \to +\infty]{*} \mu' \quad \phi\text{-inv proba}$$

(Dirac)

As $\mu'(f) = \ell_{f,x} \neq \mu(f) \Rightarrow \mu' \neq \mu$ contradicts uniqueness in (2). □

3 Unique ergodicity of torus translations

Prop: $\forall \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N$ one' equivalent $\alpha = \dot{a}$ its image in $\mathbb{T}^N$

(1) τ_α is uniquely ergodic } rare event

(2) τ_α is ergodic for $d\theta$ }

(3) $\exists$ dense orbit for τ_α } rare event!

(4) $\forall$ orbit of τ_α is dense }

(5) τ_α is minimal (the only nonempty closed invariant subset of $\mathbb{T}^N$ is $\mathbb{T}^N$)

(6) $1, \alpha_1, \dots, \alpha_N$ are linearly independent / $\mathbb{Q}$

Dem: 1 ⇒ 2 since $d\theta$ is invariant

2 ⇒ 3 Let $\mathcal{C}$ countable basis of nonempty open subsets of $\mathbb{T}^N$. $\forall B \in \mathcal{C}$, let $A_B = \bigcap_{p \in \mathbb{N}} \bigcup_{k \geq p} \tau_\alpha^{-k}(B)$

is invariant and $d\theta(A_B) \geq d\theta(B) > 0$ since $d\theta$ has full support. By ergodicity $d\theta(B) = 1$. Hence $A = \bigcap_{B \in \mathcal{C}} A_B$ has full measure and the orbit of every point of A meets every $B \in \mathcal{C}$

meas $\geq$ meas B

3 ⇒ 4 $\forall \theta, \theta' \in \mathbb{T}^N$ $\tau_{\theta'-\theta}$ maps orbits of τ_θ on the ones of $\tau_{\theta'}$ since translation commutes.

4 ⇒ 5 closed + dense ⇒ everything.

5 ⇒ 6 Else $\exists m \in \mathbb{Z}^N \setminus \{0\}$ $\langle m, \alpha \rangle = 0 \pmod{\mathbb{Z}}$. Hence $\{\theta \in \mathbb{T}^N \mid \langle m, \theta \rangle = 0\}$ closed, nonempty, invariant by τ_α, not everything; contradiction

6 ⇒ 1 Let μ proba on $\mathbb{T}^N$ τ_α-invariant.

(fly Fourier analysis!)

$$c_m(\mu) = \mu(e_{-m}) = \mu(e_{-m} \circ \tau_\alpha)$$

$e^{2\pi i \langle m, \alpha \rangle} e_{-m}$

$$= e^{-2\pi i \langle m, \alpha \rangle} c_m(\mu)$$

$\neq 1$ if $m \neq 0$.

Hence $\forall m \in \mathbb{Z}^N \setminus \{0\}$ $c_m(\mu) = 0 = c_m(d\theta)$ and $c_0(\mu) = 1 = c_0(d\theta) \Rightarrow$ by injectivity, $\mu = d\theta$. □

4 Mixing properties of linear automorphisms of tori

Let $M = (m_{ij})_{1 \leq i,j \leq N} \in GL_N(\mathbb{Z})$ and

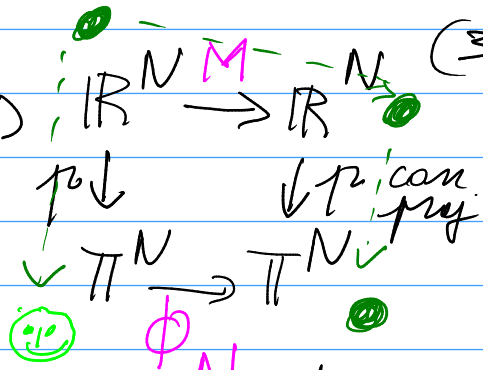
$$\varphi_M : \mathbb{T}^N \to \mathbb{T}^N$$

$$\theta = (\theta_i)_{1 \leq i \leq N} \longmapsto \left(\sum_{j=1}^{N} m_{ij} \theta_j \right)_{1 \leq i \leq N}$$

linear endomorphism

Lem: If $\det M \neq 0$, then (1) ϕ_M is surjective
(2) ϕ_M preserves $d\theta$
(3) $\forall \theta \in \mathbb{T}^N$ Card $\phi_M^{-1}(\theta) = |\det M|$

Dem: (1)



commutes and p onto and M onto

(2) $\forall \theta \in \mathbb{T}^N$, $\phi_M \circ \tau_\theta = \tau_{\phi_M(\theta)} \circ \phi_M$
and $(\tau_{\phi_M(\theta)})_* d\theta = d\theta$, Hence $(\phi_M)_* d\theta = d\theta$ by Haar uniqueness

(3) ϕ_M group morphism and M injective
$\Rightarrow \phi_M^{-1}(0)$ discrete, hence finite and $\forall \theta \in \mathbb{T}^N$,
Card $\phi_M^{-1}(\theta) =$ Card $\phi_M^{-1}(0) := k$.
If $\varepsilon > 0$ small enough, if $B_\varepsilon = p\left(]-\varepsilon,+\varepsilon[^N\right)$
then $\phi_M^{-1}\left(\phi_M(B_\varepsilon)\right) = \bigsqcup_{\theta \in \phi_M^{-1}(0)} (\theta + B_\varepsilon)$
by linearity.
Hence change of variable $d\theta(\phi_M(B_\varepsilon)) = d\theta\left(\phi_M^{-1}\left(\phi_M(B_\varepsilon)\right)\right)$
$= |\det M|\, d\theta(B_\varepsilon)$
$= k\, d\theta(B_\varepsilon)$
$\Rightarrow k = |\det M|$

Theo: $\forall M \in dB_N(\mathbb{Z})$ with $\det M \neq 0$, are equivalent
(1) ϕ_M mixing for $d\theta$
(2) ϕ_M ergodic for $d\theta$
} rare event
(3) $\forall n \in \mathbb{N}$ $Sp_{\mathbb{C}}(M) \cap \mu_n = \emptyset$ (no complex eigenvalue of M is a root of unity).
$\Leftrightarrow \forall n \in \mathbb{N}$ $\det(M^n - id) \neq 0$.

Moreover: if $\forall \lambda \in Sp_c(M)$ $|\lambda| > 1$, if $k = \left[\frac{N+1}{2}\right]$, then $(\mathbb{T}^N, d\theta, \phi_M)$ is exponentially mixing on $E_{\infty,k}$:

$\exists K, C > 0$, $\forall f, g \in C^k(\mathbb{T}^N)$,

$$|c_n(f,g)| \leq C \|f\|_{\infty,k} \|g\|_{\infty,k} e^{-Kn}$$

Dem: (by Fourier analysis)

Rem: $\forall m \in \mathbb{Z}^N$ $\forall n \in \mathbb{N}$ $e_m \circ \phi_M^n = e_{m_n}$

$e^{2\pi i \langle m, \phi_M^n \theta \rangle} = e^{2\pi i \langle {}^t M^n m, \theta \rangle}$

$M^n \theta$

where $m_n = {}^t M^n m$

(1) $\Rightarrow$ (2) OK

(2) $\Rightarrow$ (3) If $\exists n \geq 1$ s.t. $\det(M^n - id) = 0$.

Let $m_0 \in \mathbb{Z}^N \setminus \{0\}$ s.t. ${}^t M^n m_0 = m_0$

first in $\mathbb{Q}^N \setminus \{0\}$, get rid of denominator — OK

Let $m_i = {}^t M^i m_0$ for $i \in [\![0, n-1]\!]$

Then $f = \sum_{i=0}^{n-1} e_{m_i} \in C^0(\mathbb{T}^N)$ is ϕ_M invariant (by circular argument), non constant, hence ϕ_M is not ergodic

(3) $\Rightarrow$ (1) Since $(e_m)_{m \in \mathbb{Z}^N}$ is Hilbert basis of $L^2(\mathbb{T}^N, d\theta)$, hence only need to prove that $c_n(f,g) \xrightarrow[n \to \infty]{} 0$ for $f = e_m$, $g = e_\ell$ $\forall m, \ell \in \mathbb{Z}^N$.

OK if $m = 0$, as f constant and $c_n(f,g) = 0$

Assume $m \neq 0$. Then $\int_{\mathbb{T}^N} f \, d\theta = 0$

But (3) $\Rightarrow {}^tM$ has no periodic point in $\mathbb{Z}^N \setminus \{0\}$, that is,

$\exists n_0 \in \mathbb{N}, \forall n \geq n_0, \quad m_n = {}^tM^n m \neq -\ell$

Hence $c_n(f,g) = \int_{\mathbb{T}^N} \underbrace{(e_m \circ \phi_M^n)}_{= e_{m_n + \ell}} e_\ell \, d\theta = 0$

$m_n + \ell \neq 0$ if $n \geq n_0$

$\longrightarrow_{n \to +\infty} 0$, OK.

Moreover As $c_n(f,g) = 0$ if f constant and $\|f - c(f)\|_{\infty,k} \leq 2\|f\|_{\infty,k}$, may assume that $c_0(f) = 0$

As $k = \left\lceil \frac{N+1}{2} \right\rceil \geq \left\lfloor \frac{N}{2} \right\rfloor + 1$, may apply inverse Fourier transformation on f, g.

$$c_n(f,g) = \int_{\mathbb{T}^N} (f \circ \phi_M^n) g \, d\theta$$

$$= \int_{\mathbb{T}^N} \left(\sum'_{m \in \mathbb{Z}^N} c_m(f) e_m \circ \phi_M^n \right) \left(\sum_{m' \in \mathbb{Z}^N} c_{m'}(g) e_{m'} \right) d\theta$$

$\sum'$: $m \neq 0$

$=$

exchange $\int$, $\sum'$ by uniform convergence

by orthogonality formulas

$$= \sum'_{m \in \mathbb{Z}^N} c_m(f) \, c_{-m_n}(g)$$

$m \neq 0$

By induction + formula $c_m(\partial_j f) = 2\pi i \, j \, c_m(f)$

$\forall m' \in \mathbb{Z}^N, \forall h \in C^\ell(\mathbb{T}^N), \quad |c_{m'}(h)| \leq \frac{1}{\|m'\|_\infty^\ell} \|h\|_\infty$ (*)

where $\|(m'_1, \dots, m'_N)\|_\infty = \max_{1 \leq j \leq N} |m'_j|$

If $\lambda_1, \dots, \lambda_N$ eigenvalues of M, then

$\lambda = \inf_{1<j\in N} |\lambda_j| > 1$ by assumption, hence $\exists c' > 0$

$\forall m \in \mathbb{Z}^N \setminus \{0\}$ $\forall n \in \mathbb{N}$

trigonalise M over $\mathbb{C}$

$\|m_n\|_\infty = \|{}^t M^n m\|_\infty \geq \|m\|_\infty (c' \lambda^n)$

Hence $|c_n(f,g)| \leq \sum_{m \in \mathbb{Z}^N \setminus \{0\}} |c_m(f)|\,|c_{m_n}(g)|$

twice (*)

$\leq \sum_{m \in \mathbb{Z}^N \setminus \{0\}} \frac{1}{\|m\|_\infty^k \|m_n\|_\infty^k} \|f\|_{\infty,k} \|g\|_{\infty,k}$

$\leq \sum_{p \in \mathbb{N} \setminus \{0\}} \frac{2N (2p+1)^{N-1}}{p^k c'^k p^k \lambda^{nk}} \|f\|_{\infty,k} \|g\|_{\infty,k}$ (with $\|m\|_\infty = p$)

$\leq c'' e^{-\kappa n} \|f\|_{\infty,k} \|g\|_{\infty,k}$ $\square$

with $\kappa = k \ln \lambda > 0$

since $k = \left\lceil \frac{N+1}{2} \right\rceil \Rightarrow 2k \geqslant N+1$.

V

Measured entropy

'!' metric entropy

measuring complexity of proba dyn sys

Origin Shannon entropy in information theory revised by Kolmogorov – Sinai

It's a number $h_\mu(\phi) \geq 0$, conjugacy invariant of proba dyn sys $(X, \mathcal{B}, \mu, \phi)$ that <<measures the disorder created by ϕ on X>>

Blueprint of construction

$H_\mu(\alpha)$ entropy of partition α of X.

$h_\mu(\phi,\alpha)$ = growth speed of $H_\mu\left(\bigvee_{i=0}^{n-1} \phi^{-1}\alpha\right)$

$h_\mu(\phi) = \sup_\alpha h_\mu(\phi,\alpha)$.