Université Paris-Saclay (Orsay) M2 AAG Tuesday, December 1rst 2026, 8:30-11:30

Partial exam of Groups and Geometry

All manifolds are smooth real manifolds, all Lie groups and Lie algebras are real, all vector bundles are real and smooth. Only the clean and neat personal manuscript lectures notes are allowed. The following two problems are independent.

Exercise I Let $n \in \mathbb{N} \smallsetminus \{0\}$ and $k \in \mathbb{Z}$. Recall that the real projective space of dimension n is the manifold $\mathbb{P}_n(\mathbb{R}) = \mathbb{R}^\times \backslash (\mathbb{R}^{n+1} \smallsetminus \{0\})$ of orbits of the multiplicative Lie group $\mathbb{R}^\times = (\mathbb{R} \smallsetminus \{0\}, \times)$ on the manifold $\mathbb{R}^{n+1} \smallsetminus \{0\}$ by the action $(\lambda, (x_0, \dots, x_n)) \mapsto (\lambda\, x_0, \dots, \lambda\, x_n)$. Consider the action of the Lie group $\mathbb{R}^\times$ on the manifold $M = (\mathbb{R}^{n+1} \smallsetminus \{0\}) \times \mathbb{R}$ defined by

$$(\lambda, ((x_0, \dots, x_n), x)) \mapsto ((\lambda\, x_0, \dots, \lambda\, x_n), \lambda^k x)) \,.$$

Let E_k be the quotient topological space $\mathbb{R}^\times \backslash M$, and $p_k : E_k \to \mathbb{P}_n(\mathbb{R})$ the map defined by $[x_0, \dots, x_n; x] = \mathbb{R}^\times((x_0, \dots, x_n), x)) \mapsto [x_0 : \dots : x_n] = \mathbb{R}^\times(x_0, \dots, x_n)$.

(1) Prove that there exists a unique vector bundle ξ_k of rank 1 over $\mathbb{P}_n(\mathbb{R})$, whose projection is the map p_k, and such that for every $(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \smallsetminus \{0\}$, the map from $\mathbb{R}$ to $p_k^{-1}([x_0 : \dots : x_n])$ defined by $x \mapsto [x_0, \dots, x_n; x]$ is a linear isomorphism.

(2) Recall that the Grasmannian manifold $\mathrm{Gr}_1(\mathbb{R}^{n+1})$ of the vector lines of $\mathbb{R}^{n+1}$ identifies with $\mathbb{P}_n(\mathbb{R})$ by the map that, for every $v \in \mathbb{R}^{n+1} \smallsetminus \{0\}$, associates to the vector line $\mathbb{R}\, v$ the projective point $\mathbb{R}^\times v$. Recall that the projection of the tautological vector bundle $\eta = \tau_1(\mathbb{R}^{n+1})$ over $\mathrm{Gr}_1(\mathbb{R}^{n+1})$ is the smooth map $p : (\ell, v) \mapsto \ell$ from $F = \{(\ell, v) \in \mathrm{Gr}_1(\mathbb{R}^{n+1}) \times \mathbb{R}^{n+1} : v \in \ell\}$ to $\mathrm{Gr}_1(\mathbb{R}^{n+1}) = \mathbb{P}_n(\mathbb{R})$. Prove that the map $\phi : E_{-1} \to F$ defined by $[x_0, \dots, x_n; x] \mapsto (\mathbb{R}(x_0, \dots, x_n), x(x_0, \dots, x_n))$ is a vector bundle isomorphism between ξ_{-1} and η.

(3) Let η' be the direct sum vector bundle of the tangent bundle $T\mathbb{P}_n(\mathbb{R})$ of $\mathbb{P}_n(\mathbb{R})$ and of the trivial vector bundle $\xi_{\mathbb{R}}$ with rank 1 over $\mathbb{P}_n(\mathbb{R})$. Prove that the map that, for every $x = [x_0, \dots, x_n] \in \mathbb{P}_n(\mathbb{R})$, associates to $((X_0, \dots, X_n), t) \in T_x\mathbb{P}_n(\mathbb{R}) \times \mathbb{R}$ the vector $[x_0, \dots, x_n; X_0 + tx_0] \oplus \dots \oplus [x_0, \dots, x_n; X_n + tx_n]$, is a vector bundle isomorphism between η' and the direct sum $\bigoplus_{i=0}^n \xi_1$ of $n+1$ copies of ξ_1.

Exercice II Let $I_{1,3} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Let $q : \mathbb{R}^4 \to \mathbb{R}$ be the real quadratic form defined by $x = (x_0, x_1, x_2, x_3) \mapsto q(x) = -{x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2$, whose associated bilinear form is $b : (x, y) \mapsto -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$. Let $\mathrm{O}(1,3) = \{a \in \mathrm{GL}_4(\mathbb{R}) : {}^t a\, I_{1,3}\, a = I_{1,3}\}$, which is a closed Lie subgroup of the linear group $\mathrm{GL}_4(\mathbb{R})$ whose Lie algebra is equal to $\mathfrak{o}(1,3) = \{X \in \mathfrak{gl}_4(\mathbb{R}) : {}^t X\, I_{1,3} + I_{1,3} X = 0\}$. Recall that the linear action of $\mathrm{O}(1,3)$ on $\mathbb{R}^4$ preserves the quadratic form q, that is, for every $a \in \mathrm{O}(1,3)$, we have $q \circ a = q$ (and hence $b \circ (a \times a) = b$).

(1) Two elements $x, y \in \mathbb{R}^4$ are *orthogonal for* q if $b(x, y) = 0$. A *Lorentz basis* of $(\mathbb{R}^4, q)$ is a quadruple $(v_0, v_1, v_2, v_3) \in (\mathbb{R}^4)^4$ of vectors that are pairwise orthogonal for q and satisfy $q(v_0) = -1$, $q(v_1) = 1$, $q(v_2) = 1$ and $q(v_3) = 1$. Prove that the group $\mathrm{O}(1,3)$ acts transitively on the set $\mathscr{B}_{\mathscr{L}}$ of Lorentz basis of $(\mathbb{R}^4, q)$. Deduce that $\mathscr{B}_{\mathscr{L}}$ is a smooth submanifold of $(\mathbb{R}^4)^4$ which is C^∞-diffeomorphic to $\mathrm{O}(1,3)$.

(2) Prove that for every element $a = (a_{i,j})_{0\leqslant i,j\leqslant 3} \in \mathrm{O}(1,3)$, we have the equalities $|\det a| = 1$ and $-{a_{0,0}}^2 + {a_{0,1}}^2 + {a_{0,2}}^2 + {a_{0,3}}^2 = -1$. Let $\mathrm{SO}_0(1,3)$ be the Lie subgroup $\{a \in \mathrm{O}(1,3) : \det a = 1, a_{0,0} > 0\}$ of $\mathrm{O}(1,3)$. Prove that the map defined by $t \mapsto a_t = \begin{pmatrix} \cosh t & \sinh t & 0 & 0 \\ \sinh t & \cosh t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is a one-parameter subgroup of $\mathrm{SO}_0(1,3)$.

(3) For every $\alpha \in \{\pm 1\}$, let $\mathscr{O}_\alpha = \{x \in \mathbb{R}^4 : q(x) = \alpha\}$. We identify $\mathrm{O}(3)$ with a subgroup of $\mathrm{O}(1,3)$ by $c \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & c \end{smallmatrix}\right)$ and $\mathrm{O}(1,2)$ with a subgroup of $\mathrm{O}(1,3)$ by $c \mapsto \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

(a) Prove that $\mathrm{O}(1,3)$ acts transitively on $\mathscr{O}_\alpha$ for every $\alpha \in \{\pm 1\}$. Compute the stabiliser of $(1,0,0,0)$ and of $(0,0,0,1)$ in $\mathrm{O}(1,3)$.

(b) Prove that the quotient homogeneous spaces $\mathrm{O}(1,3)/\mathrm{O}(3)$ and $\mathrm{O}(1,3)/\mathrm{O}(1,2)$ are C^∞-diffeomorphic to $\mathbb{R}^3 \times \{-1,+1\}$ and to $\mathbb{R} \times \mathbb{S}_2$ respectively.

(c) Prove that for every $g \in \mathrm{SO}_0(1,3)$, there exist $\rho, \rho' \in \mathrm{SO}(3)$ and $t \in \mathbb{R}$ such that $g = \rho\, a_t\, \rho'$. Deduce that $\mathrm{SO}_0(1,3)$ is connected.

(4) Consider the real vector space $\mathrm{Herm}_2 = \{X \in \mathscr{M}_2(\mathbb{C}) : X^* = X\}$ (where $X^* = {}^t\overline{X}$ is the adjoint matrix of X), with basis, called the *Pauli basis*,

$$\left(\sigma_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right),\quad \sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right),\quad \sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right),\quad \sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right).$$

(a) Prove that the linear isomorphism $\varphi : \mathrm{Herm}_2 \to \mathbb{R}^4$ that maps the Pauli basis of Herm_2 to the canonical basis of $\mathbb{R}^4$ satisfies $q \circ \varphi = -\det$.

(b) Prove that the map Φ' from $\mathrm{SL}_2(\mathbb{C})$ to the linear group $\mathrm{GL}(\mathrm{Herm}_2)$ defined by $A \mapsto (\Phi'(A) : X \mapsto AXA^*)$ is a Lie group morphism.

(c) Recall that the map from $\mathrm{SU}(2) \times \{X \in \mathscr{M}_2(\mathbb{C}) : X^* = X,\ \mathrm{tr}\, X = 0\}$ to $\mathrm{SL}_2(\mathbb{C})$ defined by $(x, X) \mapsto x \exp(X)$ is a C^∞-diffeomorphism. Prove that the image of the map $\Phi : a \in \mathrm{SL}_2(\mathbb{C}) \mapsto \varphi \circ \Phi'(a) \circ \varphi^{-1} \in \mathrm{GL}(\mathbb{R}^4)$ is contained in $\mathrm{SO}_0(1,3)$. We admit that this image is equal to $\mathrm{SO}_0(1,3)$. Prove that $\Phi : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_0(1,3)$ is a universal cover of Lie groups.

(d) Prove that the real Lie algebras $\mathfrak{sl}_2(\mathbb{C}) = \{X \in \mathscr{M}_2(\mathbb{C}) : \mathrm{tr}\, X = 0\}$ and $\mathfrak{o}(1,3)$ are isomorphic. Prove that the map $\widehat{\Phi} : (a_t)_{t\in\mathbb{R}} \mapsto (\Phi(a_t))_{t\in\mathbb{R}}$ is a bijection from the set of one-parameter subgroups of $\mathrm{SL}_2(\mathbb{C})$ into the one of $\mathrm{SO}_0(1,3)$. For every $\alpha \in \mathbb{C}$, compute $T_{\mathrm{id}}\Phi\left(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix}\right)\right)$.

(5) A one-parameter subgroup of $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SO}_0(1,3)$ is said to be *semi-simple* if it is diagonalisable over $\mathbb{C}$. By admitting that $\widehat{\Phi}$ is a bijection between the sets of semi-simple one-parameter subgroups, prove that every semi-simple one-parameter subgroup of $\mathrm{SO}_0(1,3)$ is, up to conjugation, of the form $t \mapsto \begin{pmatrix} \cosh(\lambda t) & \sinh(\lambda t) & 0 & 0 \\ \sinh(\lambda t) & \cosh(\lambda t) & 0 & 0 \\ 0 & 0 & \cos(\mu t) & -\sin(\mu t) \\ 0 & 0 & \sin(\mu t) & \cos(\mu t) \end{pmatrix}$ where $\lambda, \mu \in \mathbb{R}$.