

The Hopf fibration.

Exercise 1 - Consider the projective action of the subgroup $SU(2)$ of $GL(2, \mathbb{C})$ on the complex projective line $\mathbb{CP}^1 = \mathbf{P}(\mathbb{C}^2)$.

1. Prove that it induces a circle bundle (fibration by circles).
2. Is this fiber bundle trivialisable?

A quotient of the Heisenberg group.

Exercise 2 - In the Heisenberg group $\text{Heis}(3) = \left\{ M = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}$, we consider the subgroup Γ that consists in such matrices with x, y and z integers.

1. Considering the action of the center of $\text{Heis}(3)$ on itself, prove that $M = \text{Heis}(3)/\Gamma$ has a smooth circle bundle structure over some manifold and identify it.
2. Is this fiber bundle trivialisable?

Exercise 3 - $L(E, F)$.

Let M be a smooth manifold and E and F be two vector bundles over M . Describe the transitions maps between two local trivializations for the vector bundle $L(E, F)$.

Triviality, or not, of some vector bundles.

Exercise 4 -

1. Let γ_n be the tautological vector bundle over the projective space $\mathbb{RP}^n = \mathbf{P}(\mathbb{R}^{n+1})$ (with $n \geq 1$). Recall that its total space is $\{(x, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v \in x\}$.
 - (a) Prove that if s is a nowhere-vanishing continuous section of γ_n , then there exists a continuous map $t : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}^*$ such that $s(\mathbb{R}v) = t(v)v$.
 - (b) Deduce that the vector bundle γ_n is not trivial.
2. Prove that the tangent bundle of $\mathbb{S}^3$ is trivialisable.

Hint. Think about Lie groups.
What about the tangent bundle of $\mathbb{S}^2$?
3. Let ξ be a real line bundle. Prove that the vector bundle $\text{End}(\xi)$ of endomorphisms of ξ is always trivial.

Pull back of vector bundles.

Exercise 5 -

1. Let ξ be the tangent bundle to $\mathbb{S}^2$ and $f : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ be the equatorial map $\theta \mapsto (\cos \theta, \sin \theta, 0)$. Show that $f^*\xi$ is a trivial vector bundle.

2. (Bonus) Let $E = \{(e^{i\theta}, w) \in \mathbb{S}^1 \times \mathbb{C}, w \in \mathbb{R}e^{i\theta/2}\}$. Let $p : E \rightarrow \mathbb{S}^1$ be the projection to the first factor.
 - (a) Prove that p is the projection map of a line bundle over $\mathbb{S}^1$.
 - (b) Identify E .
 - (c) Let $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be defined by $f(z) = z^2$. Describe f^*E .

On the frame bundle of $\mathbb{S}^2$.

Exercise 6 - One considers the action of $\mathrm{SL}(3, \mathbb{R})$ on the sphere $\mathbb{S}^2$ induced from the linear action of $\mathrm{SL}(3, \mathbb{R})$ on the set of half-lines in $\mathbb{R}^3$. Considering the usual orientation of $\mathbb{S}^2 = \partial\mathbb{B}^3$, let $\mathcal{R}^+(\mathbb{S}^2)$ denote the fiber bundle of positive (direct) frames.

1. Prove that $\mathrm{SL}(3, \mathbb{R})$ acts smoothly and transitively on $\mathcal{R}^+(\mathbb{S}^2)$.
2. (Bonus) Deduce a computation of the fundamental group of $\mathcal{R}^+(\mathbb{S}^2)$.
Hint. One may use the fact that a fibration $F \rightarrow E \rightarrow B$ with connected fibers induces an exact sequence $\pi^1(F) \rightarrow \pi^1(E) \rightarrow \pi^1(B) \rightarrow 0$.
3. (Bonus) Deduce by this way that the tangent bundle of $\mathbb{S}^2$ has no non vanishing section.