Sheet 5

Dual and induced connections.

Exercice 1 - Let ξ be a vector bundle over a manifold M, endowed with a connection ∇.

1. Given $X \in \Gamma(TM)$, $\varphi \in \Gamma(\xi^*)$ and $\sigma \in \Gamma(\xi)$, prove that the formula :
$$(\nabla^*_X \varphi)(\sigma) = X \cdot (\varphi(\sigma)) - \varphi(\nabla_X \sigma)$$
defines a connection ∇^* on ξ^*.
2. Given $X \in \Gamma(TM)$, $u \in \Gamma(\mathrm{End}(\xi))$ and $\sigma \in \Gamma(\xi)$, prove that the formula :
$$(\nabla_X u)(\sigma) = \nabla_X(u(\sigma)) - u(\nabla_X \sigma)$$
defines a connection on $\mathrm{End}(\xi)$.

Local computations.

Exercice 2 - Let ξ be a rank n real vector bundle over a manifold M of dimension d, endowed with a connection ∇. Over some sufficiently small open set Ω of M, one considers n smooth pointwise linearly independent sections $(s_1, \cdots, s_n)$ generating the $C^\infty(\Omega)$-module $\Gamma(\xi_{|\Omega})$, called a *(local) frame* of ξ. Any smooth section of ξ over Ω then writes $s = \sum_{i=1}^n u^i s_i$ for $u^i \in C^\infty(\Omega)$, so that in matrix notation, the coordinates of s can be represented by the column vector
$$S = \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix}.$$

The *Christoffel matrix 1-form* of ∇ in the frame $(s_1, \cdots, s_n)$ is the $n \times n$ matrix of differential 1-forms $\Gamma \in M_n(\Gamma(T^*M))$ defined over Ω by
$$\forall j \in [[1, n]],\ \forall X \in \Gamma(T\Omega)\ :\ \nabla_X s_j = \sum_{i=1}^n \Gamma(X)^i_j s_i\,.$$

Remark : here i is the line index of the matrix $\Gamma(X)$ and j the column one. This notation is used to be coherent with Einstein summation convention, and is also relevant if one decomposes a matrix $M \in M_n(\mathbb{R})$ into $M^i_j\, \epsilon^j \otimes e_i \in (\mathbb{R}^n)^* \otimes \mathbb{R}^n$, where $(\epsilon^j)_{1 \le j \le n}$ is the dual basis of the canonical basis $(e_i)_{1 \le i \le n}$ of $\mathbb{R}^n$.

1. Recall why on the open set Ω, the (one form valued) components of ∇s in the frame $(s_i)_{1 \le i \le N}$ are given by
$$\nabla S = dS + \Gamma S\,,$$
meaning that for any $X \in \Gamma(T\Omega)$, $\nabla_X S = X.S + \Gamma(X)S = \begin{pmatrix} X(u_1) \\ X(u_2) \\ \vdots \\ X(u_n) \end{pmatrix} + \Gamma(X)S.$
2. Given a smooth local change of frame $s_j \mapsto s'_j = P s_j$ for some smooth map $P : \Omega \to \mathrm{GL}(n, \mathbb{R})$, prove that the new Christoffer matrix 1-form Γ' of the connection ∇ in the frame $(s'_1, \cdots, s'_n)$ is given by
$$\Gamma' = P^{-1} dP + P^{-1} \Gamma P\,.$$

3. Let R denotes the curvature of ∇, seen as a $\mathrm{End}(\xi)$ valued differential 2-form, i.e. a section of $\Lambda^2 T^*M \otimes \mathrm{End}(\xi)$. Then over Ω and using the local frame $(s_1, \cdots, s_n)$, R may be seen as a matrix of differential 2-forms. Prove that over Ω, we have

$$R = d\Gamma + \Gamma \wedge \Gamma \,,$$

where $(\Gamma \wedge \Gamma)(X, Y) = \Gamma(X)\Gamma(Y) - \Gamma(Y)\Gamma(X)$ with matrix products.

Trivial vs flat line bundles.

Exercice 3 - Let ξ be a line bundle over a smooth manifold M, endowed with a connection ∇.

1. Using the previous exercice, prove that the curvature R of ∇ is a locally exact (hence a closed) 2-form on M, and globally exact if ξ is trivialisable.

   One says that a vector bundle endowed with a connection is (locally) flat if it admits (local) frames of parallel sections.

2. Prove that a line bundle is locally flat iff its curvature $R = 0$.
3. Give a (concrete) example of a trivial non locally flat line bundle, and another one of a locally flat yet not trivialisable line bundle.

Existence of torsion-free connections.

Exercice 4 - Let M be a smooth manifold with a vector bundle ξ.

1. Prove that there exists a connection ∇ on ξ.
   Hint. You can start from local formulae and try to glue them together.
2. In the case where $\xi = TM$, deduce that TM admits a torsion-free connection. Is it unique ?