Sheet 6

A canonical connection on Lie groups.

Exercice 1 - Let G be a (real) Lie group with Lie algebra $\mathfrak{g}$.

1. Prove that there exists a unique connection ∇ on TG such that, for any left-invariant vector fields X and Y on G, we have $\nabla_X Y = \frac{1}{2}[X, Y]$.
2. Prove that ∇ is torsion free.
3. Compute $R(X, Y)Z$ for all left-invariant vector fields X, Y and Z.
4. Prove that, for every X in $\mathfrak{g}$ and every $g \in G$, the map $t \mapsto g \exp_G(tX)$ is a geodesic for ∇. Prove that all geodesics have this shape.

The parallel transport determines the connection.

Exercice 2 - Let M be a smooth manifold with a vector bundle ξ over M equipped with a connection ∇. Given $x \in M$, $X \in \Gamma(TM)$ and a section $\sigma \in \Gamma(\xi)$, fix some smooth curve $c :]-\varepsilon, \varepsilon[\to M$ such that $c(0) = x$ and $\dot{c}(0) = X(x)$.
Denote by τ_t^0 the parallel transport along c from $\xi_{c(t)}$ to $\xi_{c(0)}$. Prove that

$$\nabla_X \sigma(x) = \lim_{t \to 0,\, t \neq 0} \frac{\tau_t^0(\sigma(c(t))) - \sigma(x)}{t}\,.$$

Hint. One may use a parallel frame of ξ along c if ε is small enough.

Flat tori and Hopf manifolds.

Exercice 3 - We equip $\mathbb{R}^n$ with the flat linear connection (this is the name given to the trivial connection on $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$), denoted by ∇. Let Λ be a subgroup of translations of $\mathbb{R}^n$, generated by n linearly independent translations $\tau_1, \ldots, \tau_n$. Let M be the quotient of $\mathbb{R}^n$ by the action of Λ. It is a manifold C^∞-diffeomorphic to the torus $\mathbb{T}^n$, and we denote by $\pi : \mathbb{R}^n \to M$ the canonical projection.

1. Recall that the geodesics of ∇ are the straight lines of $\mathbb{R}^n$ with affine parametrization.
2. Let X be a vector field on M. Consider the pull-back vector field $\tilde{X} = \pi^* X$ on $\mathbb{R}^n$ (defined by $\tilde{X}(x) = (T_x\pi)^{-1}(X(\pi(x))$ for every $x \in \mathbb{R}^n$).

   Prove that $\tilde{X}$ is invariant under the translations in Λ. Conversely, prove that every vector field $\tilde{X}$ on $\mathbb{R}^n$ that is invariant under the translations of Λ defines, by projection, a vector field X on M, which we denote by $\pi_*(\tilde{X})$.
3. Let X and Y be two vector fields on M. Define $\overline{\nabla}_X Y := \pi_*(\nabla_{\tilde{X}} \tilde{Y})$. Show that $\overline{\nabla}$ is a connection on M.
4. Prove that the curvature of $\overline{\nabla}$ vanishes.
5. Relate the geodesics of $\overline{\nabla}$ to those of ∇. Check in particular that they are parametrized by all of $\mathbb{R}$. One says that the connections ∇ and $\overline{\nabla}$ are *complete*.

We now consider h, the homothety of ratio $\frac{1}{2}$ on $\mathbb{R}^n$, and Γ the subgroup generated by h.

6. Prove that the action of Γ on $\mathbb{R}^n \setminus \{0\}$ is free and proper, and that the quotient manifold N is diffeomorphic to the product $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.

7. Prove, by an argument analogous to the one above, that ∇ induces a connection $\overline{\nabla}$ on N. Show that the curvature of $\overline{\nabla}$ is zero.

8. Prove that some geodesics of $\overline{\nabla}$ are not complete, that is, their maximal interval of parametrisation is not $\mathbb{R}$.

Holonomy and flat bundles.

Exercice 4 - Let M be a manifold endowed with a flat vector bundle ξ. That means that ξ is endowed with a connection ∇ with zero curvature.
We consider a smooth family (homotopy) of curves $(s,t) \in [0,1]^2 \mapsto c_s(t) \in M$ and a section $\sigma \in \Gamma(\xi)$ such that $t \mapsto \sigma(c_s(t))$ is parallel along the curve $t \mapsto c_s(t)$ for each $s \in [0,1]$.

1. Prove that if $s \mapsto \sigma(c_s(0))$ is parallel along $s \mapsto c_s(0)$, then so is $s \mapsto \sigma(c_s(1))$.

We recall that the holonomy map at x associates to any smooth closed loop c in M starting from x an isomorphism of ξ_x, defined by the parallel transport of vectors of ξ along c.
Let $\pi_1(M,x)$ be the fundamental group of M at the base point x, i.e. the group of homotopy classes of closed loops in M starting at x.

2. Deduce from the previous question that the holonomy map factorises to a group morphism $\rho : \pi_1(M,x) \to \mathrm{GL}(\xi_x)$.

Practical workshop: paper, scissors.

Exercice 5 - In the Euclidean plane, consider the open set

$$M = \Big( ]0,3[\times]0,1[\Big) \cup \Big( ]0,1[\times]0,3[\Big).$$

Let A denote the open square $]0,1[\times]2,3[$ and let φ be the unique composition of a rotation of angle π and a translation that maps A onto the square $]2,3[\times]0,1[$. Let M be the quotient of M_1 by the equivalence relation generated by $a \sim \varphi(a)$ for a in A.

1. Construct M using paper, scissors and adhesive tape (or glue).

2. Prove that the trivial connection on the plane defines a flat connection on TM.

3. Using the previous exercice, describe the parallel transport along the smooth loops of M.

4. Is M diffeomorphic to the interior of a Möbius strip?

An interpretation of the torsion.

Exercice 6 - Let ∇ be a connection on the tangent bundle of a manifold M.

1. Assume that, in a neighbourhood of a point $p \in M$, there exists a local chart satisfying $(\nabla_{\partial_i}\partial_j)(p) = 0$ for all i and j. Prove that the torsion of ∇ vanishes at p.

We now study the converse. This will be a particular case of the following result for general connections on TM. Given some $x \in M$, pick an open neighbourhood Ω of 0 in T_xM and U of $x \in M$ such that the exponential map $\exp_x^\nabla : \Omega \to U$ is a C^∞-diffeomorphism. We identify then locally M to Ω using it, and endow Ω with the induced connection, still called ∇, coming from M.

2. Given $v \in T_xM$, extended as a constant vector field on Ω, prove that $(\nabla_v v)(0) = 0$.

3. Deduce that for all u and $v \in T_xM$, extended as constant vector fields in Ω, one has at 0

$$\nabla_u v = \tfrac{1}{2}\mathrm{Tor}^\nabla(u,v)\,.$$